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SYMBOLIC DYNAMICAL SYSTEMS AND ENDOMORPHISMS ON C*-ALGEBRAS

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1. INTRODUCTION

This article is a survey of the author’s recent preprint entitled ”Actions of symbolic dynamical systems on C*-algebras”, that is written based on the talk at RIMS, Jan. 2004. Details are given in the preprint.

In [CK], J. Cuntz and W. Krieger have founded a close relationship between symbolic dynamics and C*-algebras (cf.[C2]). They constructed purely infinite simple C*-algebras from irreducible topological Markov shifts. They have proved that their stabilization with gauge action is invariant under topological conjugacy of topological Markov shifts, so that K-theoretic invariants of the C*-algebras with gauge actions yield invariants of topological Markov shifts. The invariants are the dimension group introduced by W. Krieger [Kr] and the Bowen-Franks group [BF]. They play a crucial role in the classification theory of topological Markov shifts.

R. F. Williams has classified topological Markov shifts in terms of an algebraic relation of underlying matrices [Wi]. The algebraic relation is called a strong shift equivalence. M. Nasu generalized Williams’s classification result to sofic shifts, that are subshifts coming from finite labeled graphs [N].

In [Ma], the author introduced a notion of λ-graph system, whose matrix version is called symbolix matrix system. A λ-graph system is a generalization of a finite labeled graph and presents a subshift. Conversely any subshift is presented by a λ-graph system, and the topological conjugacy classes of the subshifts are exactly corresponding to the strong shift equivalence classes of the symbolic matrix systems of the canonical λ-graph systems. He constructed C*-algebras from λ-graph systems [Ma3] as a generalization of the above Cuntz-Krieger algebras. It has been proved that the outer conjugacy class of the stabilized gauge action is invariant under strong shift equivalence of the symbolic matrix system of the λ-graph system [Ma4]. Hence K-theoretic invariants of the C*-algebras with gauge actions constructed from λ-graph systems yield invariants of topological conjugacy classes of subshifts.

In this survey article, we will study and generalize the above discussions in purely C*-algebra setting. We will introduce a notion of C*-symbolic dynamical system, that is a finite family \( \{ \rho_\alpha \}_{\alpha \in \Sigma} \) of endomorphisms of a unital C*-algebra \( \mathcal{A} \) indexed by symbols \( \Sigma \) satisfying the condition \( \sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1 \). A finite labeled graph gives rise to a C*-symbolic dynamical system \( (\mathcal{A}, \rho, \Sigma) \) such that \( \mathcal{A} \) is commutative
and finite dimensional. Conversely, if the $C^*$-algebra $A$ is commutative and finite dimensional, the $C^*$-symbolic dynamical system comes from a finite labeled graph. A $\lambda$-graph system gives rise to a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that $A$ is commutative and AF. Conversely, if the $C^*$-algebra $A$ is commutative and AF, the $C^*$-symbolic dynamical system comes from a $\lambda$-graph system ([Theorem 3.4]). We may prove that equivalence classes of the predecessor-separated $\lambda$-graph systems exactly correspond to the isomorphism classes of the predecessor-separated $C^*$-symbolic dynamical systems of the commutative AF-algebras ([Corollary 3.7]).

A $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ yields a nontrivial subshift $\Lambda_{(A, \rho, \Sigma)}$ over $\Sigma$ and a Hilbert $C^*$-right $A$-module $\mathcal{H}_A^\rho$. For $\alpha_1, \ldots, \alpha_k \in \Sigma$, a word $(\alpha_1, \ldots, \alpha_k)$ is admissible for the subshift if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$. The Hilbert $C^*$-right $A$-module $\mathcal{H}_A^\rho$ has an orthogonal finite basis $\{u_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi_\rho : A \to \mathcal{L}(\mathcal{H}_A^\rho)$. It is called a Hilbert $C^*$-symbolic bimodule over $A$, and written as $(\phi_\rho, \mathcal{H}_A^\rho, \{u_\alpha\}_{\alpha \in \Sigma})$.

We will consider $C^*$-algebras constructed from the Hilbert $C^*$-symbolic bimodules $(\phi_\rho, \mathcal{H}_A^\rho, \{u_\alpha\}_{\alpha \in \Sigma})$. A general construction of $C^*$-algebras from Hilbert $C^*$-bimodules has been established by M. Pimsner [Pim] (see [Ka] for the case of von Neumann algebras). The $C^*$-algebras are called Cuntz-Pimsner algebras. Its ideal structure and simplicity conditions have been studied by Kajiwara-Pinzari-Watatani [KWP] and Muhly-Solel [MS] (see also [KW], [Sch]). The constructed $C^*$-algebra from the Hilbert $C^*$-symbolic bimodule $(\phi_\rho, \mathcal{H}_A^\rho, \{u_\alpha\}_{\alpha \in \Sigma})$ is denoted by $A \rtimes_\rho \Lambda$, where $\Lambda$ is the subshift $\Lambda_{(A, \rho, \Sigma)}$ associated with the $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$. We call the algebra $A \rtimes_\rho \Lambda$ the $C^*$-symbolic crossed product of $A$ by the subshift $\Lambda$. As in [Pim] (cf. [KWP]), the gauge action, denoted by $\hat{\rho}$, on the algebra $A \rtimes_\rho \Lambda$ of the torus $T = \{z \in \mathbb{C} \mid |z| = 1\}$ is defined as a generalization of that of the Cuntz-Krieger algebras. We remark that Pimsner showed the following fact [Pim]: For every Hilbert $C^*$-bimodule $E$ over a $C^*$-algebra $A$, if $A$ is commutative and finite dimensional, and if $E$ is projective and finitely generated, the associated $C^*$-algebra is a Cuntz-Krieger algebra. We present the following theorem.

**Theorem A (Theorem 5.2).** Let $(A, \rho, \Sigma)$ be a $C^*$-symbolic dynamical system and $\Lambda$ be the associated subshift $\Lambda_{(A, \rho, \Sigma)}$. Assume that $A$ is commutative.

(i) If $A = C$, the subshift $\Lambda$ is the full shift $\Sigma^\mathbb{Z}$, and the $C^*$-algebra $A \rtimes_\rho \Lambda$ is the Cuntz algebra $O_{|\Sigma|}$ of order $|\Sigma|$.

(ii) If $A$ is finite dimensional, the subshift $\Lambda$ is a sofic shift $\Lambda G$ presented by a left-resolving labeled graph $G$, and the $C^*$-algebra $A \rtimes_\rho \Lambda$ is a Cuntz-Krieger algebra $O_G$ associated with the labeled graph. Conversely, for any sofic shift, that is presented by a left-resolving labeled graph $G$, there exists a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that the associated subshift is the sofic shift, the algebra $A$ is finite dimensional, and the algebra $A \rtimes_\rho \Lambda$ is the Cuntz-Krieger algebra $O_G$ associated with the labeled graph.

(iii) If $A$ is an AF-algebra, there uniquely exists a $\lambda$-graph system $\Sigma$ up to equivalence such that the subshift $\Lambda$ is presented by $\Sigma$ and the $C^*$-algebra $A \rtimes_\rho \Lambda$ is the $C^*$-algebra $O_\Sigma$ associated with the $\lambda$-graph system $\Sigma$. Conversely, for any subshift, that is presented by a left-resolving $\lambda$-graph system $\Sigma$, there exists a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that the associated
subshift is the subshift presented by $\mathcal{L}$, the algebra $\mathcal{A}$ is a commutative AF-algebra, and the algebra $\mathcal{A} \times_\rho \Lambda$ is the $C^*$-algebra $\mathcal{O}_\mathcal{L}$ associated with the $\lambda$-graph system $\mathcal{L}$.

We will introduce notions of strong shift equivalence and shift equivalence of $C^*$-symbolic dynamical systems, that are generalizations of those of square nonnegative matrices defined by Williams [Wi], of finite symbolic square matrices defined by Nasu [N] and Boyle-Krieger [BK] and of symbolic matrix systems defined by [Ma]. They are generalizations of conjugacy of single automorphisms of $C^*$-algebras. Strong shift equivalence and shift equivalence of Hilbert $C^*$-symbolic bimodules are introduced. We know that two $C^*$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent (resp. shift equivalent) if and only if their associated Hilbert $C^*$-symbolic bimodules $(\phi_\rho, \mathcal{H}_\mathcal{A}^\rho, \{u_\alpha\}_{\alpha \in \Sigma})$ and $(\phi_{\rho'}, \mathcal{H}_\mathcal{A}'^\rho, \{u'_\alpha\}_{\alpha \in \Sigma'})$ are strong shift equivalent (resp. shift equivalent). A notion of strong shift equivalence of $C^*$-symbolic crossed products with gauge actions is introduced. We finally obtain the following theorem.

**Theorem B (Theorem 7.5).** Let $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ be two $C^*$-symbolic dynamical systems. Let $\Lambda$ and $\Lambda'$ be their associated subshifts $\Lambda_{(A, \rho, \Sigma)}$ and $\Lambda_{(A', \rho', \Sigma')}$ respectively. If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are strong shift equivalent, then

(i) the subshifts $\Lambda$ and $\Lambda'$ are topologically conjugate,

(ii) the $C^*$-symbolic crossed products $(\mathcal{A} \times_\rho \Lambda, \hat{\rho}, \mathcal{T})$ and $(\mathcal{A}' \times_{\rho'} \Lambda', \hat{\rho}', \mathcal{T})$ with gauge actions are strong shift equivalent, and

(iii) the stabilized gauge actions $(\mathcal{A} \times_\rho \Lambda \otimes \mathcal{K}, \hat{\rho} \otimes \text{id}, \mathcal{T})$ and $(\mathcal{A}' \times_{\rho'} \Lambda' \otimes \mathcal{K}, \hat{\rho}' \otimes \text{id}, \mathcal{T})$ are co-cycle conjugate, where $\mathcal{K}$ denotes the $C^*$-algebra of all compact operators on a separable infinite dimensional Hilbert space.

The result (iii) is a generalization of the main result of [Ma4] (cf.[CK:3.8. Theorem]).

We define the $K$-groups $K_*(\mathcal{A}, \rho, \Sigma)$, the Bowen-Franks groups $BF^*(\mathcal{A}, \rho, \Sigma)$ and the dimension groups $D_*(\mathcal{A}, \rho, \Sigma)$ for $(\mathcal{A}, \rho, \Sigma)$ by setting for $* = 0, 1$

$$K_*(\mathcal{A}, \rho, \Sigma) = K_*(\mathcal{A} \times_\rho \Lambda), \quad BF^*(\mathcal{A}, \rho, \Sigma) = \text{Ext}_*(\mathcal{A} \times_\rho \Lambda),$$

$$D_*(\mathcal{A}, \rho, \Sigma) = (K_*(\mathcal{A} \times_\rho \Lambda) \times_\hat{\rho} \mathcal{T}), \hat{\rho}_*$$

where $\hat{\rho}_*$ is the automorphism of $K_*(\mathcal{A} \times_\rho \Lambda \times_\hat{\rho} \mathcal{T})$ induced from the dual action $\hat{\rho}$ of the gauge action $\rho$. The dimension groups and the Bowen-Franks groups are generalizations of those groups for a finite square nonnegative matrix, that is regarded as a finite labeled graph for which labels are edges itself (cf.[BF], [Kr], [LM]). Then Theorem B implies that all the abelian groups $K_*(\mathcal{A}, \rho, \Sigma)$, $BF^*(\mathcal{A}, \rho, \Sigma)$ and $D_*(\mathcal{A}, \rho, \Sigma)$ are invariant under strong shift equivalence of $C^*$-symbolic dynamical systems (Proposition 7.6).

2. $\lambda$-GRAPH SYSTEMS AND ITS $C^*$-ALGEBRAS

Let $\Sigma$ be a finite set with its discrete topology. We call it an alphabet. Each element of $\Sigma$ is called a symbol or a label. Let $\Sigma^\mathbb{Z}$ be the infinite product spaces
\[ \prod_{i \in \mathbb{Z}} \Sigma_i, \text{ where } \Sigma_i = \Sigma, \text{ endowed with the product topology. The transformation } \sigma \text{ on } \Sigma^\mathbb{Z} \text{ given by } (\sigma(x_i))_{i \in \mathbb{Z}} = (x_{i+1})_{i \in \mathbb{Z}} \text{ is called the (full) shift. Let } \Lambda \text{ be a shift invariant closed subset of } \Sigma^\mathbb{Z} \text{ i.e. } \sigma(\Lambda) = \Lambda. \text{ The topological dynamical system } (\Lambda, \sigma|_{\Lambda}) \text{ is called a subshift. We write the subshift as } \Lambda \text{ for brevity. A finite sequence } \mu = (\mu_1, \ldots, \mu_k) \text{ of elements } \mu_j \in \Sigma \text{ is called a word of length } |\mu| = k. \]

For a subshift \( \Lambda \), we denote by \( \Lambda^l \) the set of all admissible words of length \( l \) of \( \Lambda \). By a symbolic matrix \( B \) over \( \Sigma \) we mean a finite matrix with entries in finite formal sums of elements of \( \Sigma \). A square symbolic matrix \( B \) naturally gives rise to a finite labeled directed graph which we denote by \( \mathcal{G}_B \). The labeled directed graph defines a subshift over \( \Sigma \) which consists of all infinite labeled sequences following the labeled directed edges in \( \mathcal{G}_B \). Such a subshift is called a sofic shift presented by \( \mathcal{G}_B \) and written as \( \Lambda_{\mathcal{G}_B} \) ([Fi],[Kr2],[Kr3],[We], cf. [Kit],[LM]). Throughout this paper, a labeled graph means a labeled directed graph with finite vertices and finite directed edges such as every vertex has at least one in-coming edge and at least one out-going edge.

Let \( B \) and \( B' \) be symbolic matrices over \( \Sigma \) and \( \Sigma' \) respectively. Let \( \phi \) be a bijection from a subset of \( \Sigma \) onto a subset of \( \Sigma' \), that is called a specification. Following M. Nasu in [N1],[N2], we say that \( B \) and \( B' \) are specified equivalent under specification \( \phi \) if \( B' \) can be obtained from \( B \) by replacing every symbol \( \sigma \) appearing in \( B \) by \( \phi(\sigma) \). We write it as \( B \cong B' \). Let \( \mathbb{Z}_+ \) be the set of all nonnegative integers.

Recall that a \( \lambda \)-graph system \( \mathcal{L} = (V, E, \lambda, \iota) \) over \( \Sigma \) is a directed Bratteli diagram with a vertex set \( V = \bigcup_{i \in \mathbb{Z}_+} V_i \), an edge set \( E = \bigcup_{l \in \mathbb{Z}_+} E_{l,l+1} \), and a map \( \lambda : E \to \Sigma \), and that is supplied with a sequence of surjective maps \( \iota = (\iota_l) : V_{l+1} \to V_l \) for \( l \in \mathbb{Z}_+ \). Here the vertex sets \( V_l \) are disjoint sets. An edge \( e \) in \( E_{l,l+1} \) has its source vertex \( s(e) \) in \( V_l \), its terminal vertex \( t(e) \) in \( V_{l+1} \) and its label \( \lambda(e) \) in \( \Sigma \). Every vertex in \( V \) has successors and every vertex in \( V \), except \( V_0 \), has predecessors. It is then required that for \( u \in V_{l-1} \) and \( v \in V_{l+1} \), there exists a bijective correspondence between the edge set \( \{ e \in E_{l,l+1} | t(e) = v, \iota(s(e)) = u \} \) and the edge set \( \{ e \in E_{l-1,l} | s(e) = u, \iota(t(e)) = v \} \) that preserves labels. The required property is called the local property.

Two \( \lambda \)-graph systems \( \mathcal{L} = (V, E, \lambda, \iota) \) over \( \Sigma \) and \( \mathcal{L}' = (V', E', \lambda', \iota') \) over \( \Sigma' \) are said to be isomorphic if there exist bijections \( \Phi_V : V \to V' \), \( \Phi_E : E_{l,l+1} \to E'_{l,l+1} \) and a specification \( \phi : \Sigma \to \Sigma' \) such that \( \Phi_V(s(e)) = s(\Phi_E(e)), \Phi_V(t(e)) = t(\Phi_E(e)) \) and \( \lambda'(\Phi_E(e)) = \phi(\lambda(e)) \) for \( e \in E \), and \( \iota'(\Phi_V(v)) = \Phi_V(\iota(v)) \) for \( v \in V \).

A symbolic matrix system over \( \Sigma \) consists of a sequence of pairs of rectangular matrices \( (\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+ \). The matrices \( \mathcal{M}_{l,l+1} \) have their entries in formal sums of \( \Sigma \) and the matrices \( I_{l,l+1} \) have their entries in \{0,1\}. The matrices \( \mathcal{M}_{l,l+1} \) and \( I_{l,l+1} \) have the same size for each \( l \in \mathbb{Z}_+ \) and satisfy the following relations

\[
I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \quad l \in \mathbb{Z}_+.
\]

The matrices \( I_{l,l+1}, l \in \mathbb{Z}_+ \) have one 1 in each column and at least one 1 in each row. We denote it by \( (\mathcal{M}, I) \). A \( \lambda \)-graph system naturally arises from a symbolic matrix system \( (\mathcal{M}, I) \). The edges from a vertex \( v_{il} \in V_l \) to a vertex \( v_{j{l+1}} \) are given by the \((i,j)\)-component \( \mathcal{M}_{l,l+1}(i,j) \) of the matrix \( \mathcal{M}_{l,l+1} \). The matrix \( I_{l,l+1} \) defines a surjection \( \iota_{l,l+1} \) from \( V_{l+1} \) to \( V_l \) for each \( l \in \mathbb{Z}_+ \).
Two symbolic matrix systems \((\mathcal{M}, I)\) over \(\Sigma\) and \((\mathcal{M}', I')\) over \(\Sigma'\) are said to be isomorphic if there exists a specification \(\phi\) from \(\Sigma\) to \(\Sigma'\) and an \(m(l) \times m(l)\)-square permutation matrix \(P_l\) for each \(l \in \mathbb{N}\) such that

\[
P_l \mathcal{M}_{i,l+1} \simeq \mathcal{M}'_{i,l+1} P_{l+1}, \quad P_l I_{i,l+1} = I'_{i,l+1} P_{l+1} \quad \text{for} \quad l \in \mathbb{Z}_+.
\]

There exists a bijective correspondence between the set of all isomorphism classes of symbolic matrix systems and the set of all isomorphism classes of \(\lambda\)-graph systems.

Let \(\mathcal{G} = (G, \lambda)\) be a labeled graph with finite directed graph \(G\) and labeling \(\lambda\). Let \(\{v_1, \ldots, v_n\}\) be the vertex set of \(G\). Put \(V_l = \{v_1, \ldots, v_n\}\) for all \(l \in \mathbb{Z}_+\). We regard the sets \(V_l, l \in \mathbb{Z}_+\) as disjoint sets. Define \(\iota : V_{l+1} \to V_l\) by \(\iota(v_i) = v_i\) for \(i = 1, \ldots, n\). Write labeled edges from \(V_l\) to \(V_{l+1}\) for \(l \in \mathbb{N}\) following the directed graph \(G\) with labeling \(\lambda\). The resulting labeled Bratteli diagram with \(\iota\) becomes a \(\lambda\)-graph system. A labeled graph and also a \(\lambda\)-graph system are said to be left-resolving if different edges with the same label have different terminals. Hence a labeled graph defines a \(\lambda\)-graph system such that if the labeled graph is left-resolving, so is the \(\lambda\)-graph system. We call the resulting \(\lambda\)-graph system the associated \(\lambda\)-graph system with the labeled graph graph. We note that any sofic shift may be presented by left-resolving labeled graph ([Kr2],[Kr3],[We]).

A \(\lambda\)-graph system \(\mathcal{L}\) gives rise to a subshift \(\Lambda_{\mathcal{L}}\) on the sequence space of labels appearing in the labeled Bratteli diagram. We say that \(\mathcal{L}\) presents the subshift \(\Lambda_{\mathcal{L}}\).

A canonical method to construct a \(\lambda\)-graph system from an arbitrary subshift \(\Lambda\) has been introduced in [Ma]. The \(\lambda\)-graph system and its symbolic matrix system are said to be canonical for the subshift and written as \(\mathcal{L}^\Lambda\) and \((\mathcal{M}^\Lambda, I^\Lambda)\) respectively.

Let \(\mathcal{L} = (V, E, \lambda, \iota)\) be a \(\lambda\)-graph system over \(\Sigma\). For a vertex \(v \in V_l\), we denote by \(\Gamma_{\mathcal{L},l}(v)\) the set of all label sequences of length \(l\) in \(\Sigma\) that start at vertices of \(V_0\) and terminate at \(v\). We say that \(\mathcal{L}\) is predecessor-separated if for \(u, v \in V_l\) the condition \(\Gamma_{\mathcal{L},l}(u) = \Gamma_{\mathcal{L},l}(v)\) implies \(u = v\). The canonical \(\lambda\)-graph systems are left-resolving and predecessor-separated.

We will introduce an equivalence relation of predecessor-separated \(\lambda\)-graph systems. Let \((\mathcal{M}, I)\) and \((\mathcal{M}', I')\) be the symbolic matrix systems over \(\Sigma\) and \(\Sigma'\) respectively. We denote by \(m(l)\) the row size of the matrix \(\mathcal{M}_{i,l+1}\) and by \(m'(l)\) that of \(\mathcal{M}'_{i,l+1}\) respectively. We say that \((\mathcal{M}, I)\) and \((\mathcal{M}', I')\) are equivalent if there exist \(N \in \mathbb{Z}_+\) and a bijection \(\pi : \Sigma \to \Sigma'\) such that for each \(l \in \mathbb{Z}_+\), there exist an \(m(l) \times m'(N + l)\) matrix \(H_l\) over \(\{0,1\}\) and an \(m'(l) \times m(N + l)\) matrix \(K_l\) over \(\{0,1\}\) satisfying the following equations:

\[
\mathcal{M}_{i,l+1} H_{i+1} \sim H_{i} H_{i+N, i+N+1}, \quad \mathcal{M}'_{i,l+1} K_{l+1} \sim K_{i} K_{i+N, i+N+1},
\]

\[
I_{i,l+1} H_{i+1} = H_{i} I_{i+N, i+N+1}, \quad I'_{i,l+1} K_{l+1} = K_{i} I_{i+N, i+N+1}
\]

and

\[
H_{i} K_{N+1} = I_{i,2N+1}, \quad K_{i} H_{N+1} = I'_{i,2N+1}.
\]

We write this equivalence relation as \((\mathcal{M}, I) \cong (\mathcal{M}', I')\). Two \(\lambda\)-graph systems are called equivalent if their respect symbolic matrix systems are equivalent.
In the rest of this section, we briefly review the $C^*$-algebra $\mathcal{O}_\mathcal{L}$ associated with $\lambda$-graph system $\mathcal{L}$. The $C^*$-algebras have been originally constructed in [Ma3] as groupoid $C^*$-algebras of certain r-discrete groupoids constructed from continuous graphs in the sense of Deaconu ([De],[De2],[De3],cf.[Re]) obtained by the $\lambda$-graph systems. They are realized as universal unique $C^*$-algebras as in the following way.

For a $\lambda$-graph system $\mathcal{L} = (V, E, \lambda, \iota)$ over $\Sigma$, let $\{v_1^l, \ldots, v_{m(l)}^l\}$ be the vertex set $V_l$. We put

(2.2) 
$$ A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise}, \end{cases} $$

(2.3) 
$$ I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } l_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise} \end{cases} $$

for $i = 1, 2, \ldots, m(l), j = 1, 2, \ldots, m(l+1), \alpha \in \Sigma$.

**Lemma 2.1 ([Ma3; Theorem A]).** The $C^*$-algebra $\mathcal{O}_\mathcal{L}$ is the universal concrete $C^*$-algebra generated by partial isometries $S_\alpha, \alpha \in \Sigma$ and projections $E_i^l, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+$ satisfying the following relations called $(\mathcal{L})$:

(2.4) 
$$ \sum_{\beta \in E} S_\beta S_\beta^* = 1, $$

(2.5) 
$$ \sum_{k=1}^{m(l)} E_k^l = 1, \quad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) E_j^{l+1}, $$

(2.6) 
$$ S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) E_j^{l+1}, $$

for $i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+, \alpha \in \Sigma$.

If $\mathcal{L}$ satisfies condition $(I)$, a generalized condition of condition (I) for a finite square matrix with entries in $\{0, 1\}$ defined in [CK], the algebra $\mathcal{O}_\mathcal{L}$ is the unique $C^*$-algebra subject to the above relations $(\mathcal{L})$. Furthermore, if $\mathcal{L}$ is irreducible, the $C^*$-algebra $\mathcal{O}_\mathcal{L}$ is simple and purely infinite ([Ma3],[Ma5]). The gauge action $\alpha^\mathcal{L}$ on $\mathcal{O}_\mathcal{L}$ is defined by an action of $T = \{z \in \mathbb{C} \mid |z| = 1\}$ such that $\alpha^\mathcal{L}_z(S_\alpha) = z S_\alpha, \alpha^\mathcal{L}_z(E_i^l) = E_i^l$ for $\alpha \in \Sigma, i = 1, 2, \ldots, m(l), l \in \mathbb{Z}_+$.

3. $C^*$-SYMBOLIC DYNAMICAL SYSTEMS

Let $\mathcal{A}$ be a unital $C^*$-algebra. Throughout this paper, an endomorphism of $\mathcal{A}$ means a $*$-endomorphism of $\mathcal{A}$ that does not necessarily preserve the unit $1_\mathcal{A}$ of $\mathcal{A}$. The unit $1_\mathcal{A}$ is denoted by $1$ unless we specify. We denote by $\text{End}(\mathcal{A})$ the set of all endomorphisms of $\mathcal{A}$. Let $\Sigma$ be a finite set. A finite family of endomorphisms $\rho_\alpha \in \text{End}(\mathcal{A}), \alpha \in \Sigma$ is said to be essential if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and $\Sigma_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$. 


It is said to be \textit{faithful} if for any nonzero $x \in A$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$.

\textbf{Definition.} A $C^*$-symbolic dynamical system is a triplet $(A, \rho, \Sigma)$ consisting of a unital $C^*$-algebra $A$ and a finite family of endomorphisms $\rho_\alpha$ of $A$ indexed by $\alpha \in \Sigma$, that is essential and faithful.

Two $C^*$-symbolic dynamical systems $(A, \rho, \Sigma)$ and $(A', \rho', \Sigma')$ are said to be isomorphic if there exist an isomorphism $\Phi : A \to A'$ and a bijection $\pi : \Sigma \to \Sigma'$ such that $\Phi \circ \rho_\alpha = \rho'_{\pi(\alpha)} \circ \Phi$ for all $\alpha \in \Sigma$.

\textbf{Proposition 3.1.} For a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$, there uniquely exists a subshift $\Lambda_{(A, \rho, \Sigma)}$ over $\Sigma$ such that a word $\alpha_1 \cdots \alpha_k$ of $\Sigma$ is admissible for the subshift if and only if $(\rho_\alpha \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$.

Suppose that $A$ is a commutative $C^*$-algebra $C(\Omega)$ of all continuous functions on a compact Hausdorff space $\Omega$. An endomorphism of $A$ bijectively corresponds to a continuous map from a clopen set of $\Omega$ to $\Omega$. Hence a $C^*$-symbolic dynamical system $(C(\Omega), \rho, \Sigma)$ bijectively corresponds to a family $\{f_\alpha, E_\alpha\}_{\alpha \in \Sigma}$ of clopen sets $E_\alpha \subset \Omega$ and continuous maps $f_\alpha : E_\alpha \to \Omega$, $\alpha \in \Sigma$ such that

$$\bigcup_{\alpha \in \Sigma} E_\alpha = \Omega \quad \text{and} \quad \bigcup_{\alpha \in \Sigma} f_\alpha(E_\alpha) = \Omega.$$  

We will study this situation in more graphical examples for a while.

For a left-resolving labeled graph $\mathcal{G} = (G, \lambda)$, let $v_1, \ldots, v_n$ be its vertex set. Consider the $n$-dimensional commutative $C^*$-algebra $A_\mathcal{G} = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$ where each minimal projection $E_i$ corresponds to the vertex $v_i$ for $i = 1, \ldots, n$. Then we may define an $n \times n$ matrix $A(i, \alpha, j)$ with entries in $\{0, 1\}$ by

$$A_\mathcal{G}(i, \alpha, j) = \begin{cases} 1 & \text{if there exists an edge } e \text{ from } v_i \text{ to } v_j \text{ with } \lambda(e) = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, \ldots, n$. We set

$$\rho_\mathcal{G}(E_i) = \sum_{j=1}^{n} A_\mathcal{G}(i, \alpha, j)E_j, \quad i = 1, \ldots, n, \alpha \in \Sigma.$$ 

Then $\rho_\alpha, \alpha \in \Sigma$ define endomorphisms of $A_\mathcal{G}$ such that $(A_\mathcal{G}, \rho_\mathcal{G}, \Sigma)$ is a $C^*$-symbolic dynamical system.

Conversely, let $(A, \rho, \Sigma)$ be a $C^*$-symbolic dynamical system such that $A$ is $n$-dimensional and commutative. Take $E_1, \ldots, E_n$ the orthogonal minimal projections of $A$ such that $A = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$. Define an $n \times n$ matrix $[A(i, \alpha, j)]_{i,j=1,\ldots,n}$ for $\alpha \in \Sigma$ by setting

$$A(i, \alpha, j) = \begin{cases} 1 & \text{if } \rho_\alpha(E_i) \geq E_j, \\ 0 & \text{otherwise} \end{cases}$$

so that one has

$$\rho_\alpha(E_i) = \sum_{j=1}^{n} A(i, \alpha, j)E_j, \quad i = 1, \ldots, n, \alpha \in \Sigma.$$ 

Let $v_1, \ldots, v_n$ be the vertex set corresponding to the projections $E_1, \ldots, E_n$. Define a directed labeled edge $e$ such as the source vertex $s(e) = v_i$, the terminal vertex $t(e) = v_j$ and the label $\lambda(e) = \alpha$ if $A(i, \alpha, j) = 1$. Then we have a left-resolving labeled graph $\mathcal{G}$ which presents the subshift $\Lambda_{(A, \rho, \Sigma)}$. Hence we have
Proposition 3.2. For a left-resolving labeled graph $G$, there exists a $C^*$-symbolic dynamical system $(A_G, \rho^G, \Sigma)$ such that the algebra $A_G$ is commutative and finite dimensional, and the associated subshift $\Lambda_G$ is the sofic shift $\Lambda_G$ presented by $G$. Conversely, for a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$, if $A$ is commutative and finite dimensional, there exists a left-resolving labeled graph $G$ such that $A = A_G$ and the associated subshift $\Lambda_G$ is the sofic shift $\Lambda_G$ presented by $G$.

Let us apply the above discussions to general subshifts and $\lambda$-graph systems. For a $\lambda$-graph system $\mathcal{L} = (V, E, \lambda, \iota)$ over $\Sigma$, let $(\mathcal{M}, I)$ be its corresponding symbolic matrix system. Let $A_{l,l+1}$ be the matrices defined by (2.2). We equip $V_l$ with discrete topology. We denote by $\Omega_{\mathcal{L}}$ the topological space of the projective limit

$$V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \cdots,$$

that is a compact, totally disconnected, second countable topological space. We regard the algebra of all continuous functions on $V_l$ as the direct sum

$$C(V_l) = \mathbb{C}E_1^l \oplus \mathbb{C}E_2^l \oplus \cdots \oplus \mathbb{C}E_{m(l)}^l,$$

where the vertices $v_i^l \in V_l, i = 1, \ldots, m(l)$ correspond to the minimal projections $E_i^l \in V_l, i = 1, \ldots, m(l)$. We denote $C(V_l)$ by $A_{\mathcal{L},l}$. Let $A_{\mathcal{L}}$ be the commutative $C^*$-algebra of all continuous functions on $\Omega_{\mathcal{L}}$, that is the inductive limit algebra

$$A_{\mathcal{L},0} \xrightarrow{I_{0,1}} A_{\mathcal{L},1} \xrightarrow{I_{1,2}} A_{\mathcal{L},2} \xrightarrow{I_{2,3}} A_{\mathcal{L},3} \xrightarrow{I_{3,4}} \cdots.$$

Hence $A_{\mathcal{L}}$ is a unital commutative AF-algebra. For a symbol $\alpha \in \Sigma$ we set

$$\rho^\mathcal{L}_\alpha(E_i^l) = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha,j)E_{j}^{l+1} \quad \text{for } i = 1, 2, \ldots, m(l).$$

By the commutation relation (2.1), $\rho^\mathcal{L}_\alpha$ defines an endomorphism of $A_{\mathcal{L}}$. Since each vertex $v_i^l \in V_l$ except $l = 0$ has an in-coming edge, the family $\{\rho^\mathcal{L}_\alpha\}_{\alpha \in \Sigma}$ is essential. It is also faithful because each vertex $v_i^l \in V_l$ has an out-going edge. Thus we have

Proposition 3.3. For a $\lambda$-graph system $\mathcal{L}$ over $\Sigma$, there exists a $C^*$-symbolic dynamical system $(A_{\mathcal{L}}, \rho^\mathcal{L}, \Sigma)$ such that the $C^*$-algebra $A_{\mathcal{L}}$ is commutative and AF, and the associated subshift $\Lambda_{(A_{\mathcal{L}}, \rho^\mathcal{L}, \Sigma)}$ coincides with the subshift $\Lambda_{\mathcal{L}}$ presented by $\mathcal{L}$.

Conversely

Theorem 3.4. Let $(A, \rho, \Sigma)$ be a $C^*$-symbolic dynamical system. If the algebra $A$ is commutative and AF, there exists a $\lambda$-graph system $\mathcal{L}$ over $\Sigma$ such that the associated $C^*$-symbolic dynamical system $(A_{\mathcal{L}}, \rho^\mathcal{L}, \Sigma)$ is isomorphic to $(A, \rho, \Sigma)$.

A $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ is said to be predecessor-separated if the projections $\{\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1}(1) \mid \alpha_1, \ldots, \alpha_k \in \Sigma, k \in \mathbb{N}\}$ generate the $C^*$-algebra $A$. 
Proposition 3.5.
(i) If a $\lambda$-graph system $\mathcal{L}$ is predecessor-separated, the associated $C^*$-symbolic dynamical system $(A_{\mathcal{L}}, \rho^\mathcal{L}, \Sigma)$ is predecessor-separated.
(ii) Suppose that an algebra $A$ is unital, commutative and AF. If a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ is predecessor-separated, there exists a predecessor-separated $\lambda$-graph system $\mathcal{L}$ over $\Sigma$ such that the associated $C^*$-symbolic dynamical system $(A_{\mathcal{L}}, \rho^\mathcal{L}, \Sigma)$ is isomorphic to $(A, \rho, \Sigma)$.

Proposition 3.6. Let $\mathcal{L}$ and $\mathcal{L}'$ be predecessor-separated $\lambda$-graph systems over $\Sigma$ and $\Sigma$ respectively. Then $(A_{\mathcal{L}}, \rho^\mathcal{L}, \Sigma)$ is isomorphic to $(A_{\mathcal{L}'}, \rho^{\mathcal{L}'}, \Sigma')$ if and only if $\mathcal{L}$ and $\mathcal{L}'$ are equivalent. In this case, the presented subshifts $\Lambda_{\mathcal{L}}$ and $\Lambda_{\mathcal{L}'}$ are identified through a symbolic conjugacy.

Therefore we have

Corollary 3.7. The equivalence classes of the predecessor-separated $\lambda$-graph systems are identified with the isomorphism classes of the predecessor-separated $C^*$-symbolic dynamical systems of the commutative AF-algebras.

We formulate here an action of a subshift to a $C^*$-algebra. We say that a subshift $\Lambda$ acts on a $C^*$-algebra $A$ if there exists a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that the associated subshift $\Lambda_{(A, \rho, \Sigma)}$ coincides with $\Lambda$.

4. HILBERT $C^*$-SYMBOLIC BIMODULES

In this section we will construct a Hilbert $C^*$-bimodule from a $C^*$-symbolic dynamical system. Let $(A, \rho, \Sigma)$ be a $C^*$-symbolic dynamical system. We put the projections $P_\alpha = \rho_\alpha(1)$ in $A$ for $\alpha \in \Sigma$. Let $\{e_\alpha\}_{\alpha \in \Sigma}$ denote the standard basis of the $|\Sigma|$-dimensional vector space $\mathbb{C}^{|\Sigma|}$, where $|\Sigma|$ denotes the cardinal number of the set $\Sigma$. Set

$$\mathcal{H}^\rho_A := \sum_{\alpha \in \Sigma} \mathbb{C} e_\alpha \otimes P_\alpha A.$$

Define a right $A$-action and an $A$-valued inner product on $\mathcal{H}^\rho_A$ by setting

$$(e_\alpha \otimes P_\alpha x) y := e_\alpha \otimes P_\alpha x y,$$

$$\langle e_\alpha \otimes P_\alpha x | e_\beta \otimes P_\beta y \rangle := \begin{cases} x^* P_\alpha y & \text{if } \alpha = \beta, \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha, \beta \in \Sigma$ and $x, y \in A$. Then $\mathcal{H}^\rho_A$ forms a Hilbert $C^*$-right $A$-module. We put

$$u_\alpha := e_\alpha \otimes P_\alpha, \quad \alpha \in \Sigma.$$

Lemma 4.1. The family $u_\alpha, \alpha \in \Sigma$ forms an orthogonal finite basis of $\mathcal{H}^\rho_A$ in the sense of [KPW] such that

$$\sum_{\alpha \in \Sigma} \langle u_\alpha | u_\alpha \rangle \geq 1.$$
We say that a finite basis of a Hilbert $C^*$-module is essential if the basis satisfies the condition (4.1). We will next define a diagonal left action $\phi_\rho$ of $\mathcal{A}$ to the set of all adjointable bounded $\mathcal{A}$-module maps $L(\mathcal{H}_\mathcal{A})$ on $\mathcal{H}_\mathcal{A}$ as follows:

$$
\phi_\rho(a)u_\alpha x := u_\alpha \rho_\alpha(a)x, \quad a, x \in \mathcal{A}, \alpha \in \Sigma.
$$

The above definition is well-defined. If $u_\alpha x = u_\alpha y$, then $P_\alpha x = P_\alpha y$ so that $\rho_\alpha(a1)x = \rho_\alpha(a1)y$ for $a \in \mathcal{A}$. Hence one has that $u_\alpha \rho_\alpha(a)x = u_\alpha \rho_\alpha(a)y$. Since the family \( \{\rho_\alpha\}_{\alpha \in \Sigma} \) is faithful, the left action $\phi_\rho$ of $\mathcal{A}$ on $\mathcal{H}_\mathcal{A}$ is faithful, that is, the element $\phi_\rho(x)$ is nonzero for any nonzero $x \in \mathcal{A}$. Therefore we have

**Proposition 4.2.** For a $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$, there exists a Hilbert $C^*$-right $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi_\rho : \mathcal{A} \to L(\mathcal{H}_\mathcal{A})$ such that

\[
\begin{align*}
\phi_\rho(a)u_\alpha =& u_\alpha \rho_\alpha(a), \\
(\langle u_\alpha | u_\alpha \rangle = &\rho_\alpha(1), \quad a \in \mathcal{A}, \alpha \in \Sigma.
\end{align*}
\]

We note that the above two conditions imply

\[
(\langle u_\alpha | \phi_\rho(a)u_\alpha \rangle = \rho_\alpha(a), \quad a \in \mathcal{A}, \alpha \in \Sigma.
\]

Conversely

**Proposition 4.3.** For a Hilbert $C^*$-right $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi : \mathcal{A} \to L(\mathcal{H}_\mathcal{A})$, define $\rho_\alpha$ for $\alpha \in \Sigma$ by setting

$$
\rho_\alpha(a) = \langle u_\alpha | \phi(a)u_\alpha \rangle, \quad a \in \mathcal{A}.
$$

Then $\rho_\alpha$ gives rise to an endomorphism of $\mathcal{A}$ such that $(\mathcal{A}, \rho, \Sigma)$ yields a $C^*$-symbolic dynamical system.

A Hilbert $C^*$-right $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ with a left action $\phi : \mathcal{A} \to L(\mathcal{H}_\mathcal{A})$ is called a Hilbert $C^*$-bimodule over $\mathcal{A}$ ([Pim], cf.[KW], [KPW], [MS]). Two Hilbert $C^*$-bimodules $(\phi, \mathcal{H}_\mathcal{A})$ and $(\phi', \mathcal{H}'_\mathcal{A})$ over $\mathcal{A}$ are said to be unitary equivalent if there exists a bimodule isomorphism $\Phi : \mathcal{H}_\mathcal{A} \to \mathcal{H}'_\mathcal{A}$ such that $\Phi$ is unitary with respect to their respect inner products.

**Definition.** A Hilbert $C^*$-right $\mathcal{A}$-module $\mathcal{H}_\mathcal{A}$ with an orthogonal essential finite basis $\{u_\alpha\}_{\alpha \in \Sigma}$ and a unital faithful diagonal left action $\phi : \mathcal{A} \to L(\mathcal{H}_\mathcal{A})$ is called a Hilbert $C^*$-symbolic bimodule over $\mathcal{A}$. It is written as $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma})$.

A Hilbert $C^*$-symbolic bimodule $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma})$ over $\mathcal{A}$ bijectively corresponds to a $C^*$-symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ by the above discussions. Two Hilbert $C^*$-symbolic bimodules $(\phi, \mathcal{H}_\mathcal{A}, \{u_\alpha\}_{\alpha \in \Sigma})$ and $(\phi', \mathcal{H}'_\mathcal{A}, \{u'_\alpha\}_{\alpha' \in \Sigma'})$ over $\mathcal{A}$ are said to be unitary equivalent if there exists a bimodule isomorphism $\Phi : \mathcal{H}_\mathcal{A} \to \mathcal{H}'_\mathcal{A}$ and a bijection $\pi : \Sigma \to \Sigma'$ such that $\Phi$ is unitary with respect to their respect inner products and satisfies $\Phi(u_\alpha) = u'_\pi(\alpha), \alpha \in \Sigma$. Let $\rho_\alpha, \alpha \in \Sigma$ and $\rho'_\alpha', \alpha' \in \Sigma'$ be their respect endomorphisms of $\mathcal{A}$. In this case, we have
\[ \rho_\alpha(a) = \rho_{\pi(\alpha)}'(a), a \in A \] because the equality (4.2) implies \( \phi'(a)\Phi(u_\alpha) = \Phi(u_\alpha)\rho_\alpha(a) \) and hence \( \phi'(a)u_{\pi(\alpha)}' = u_{\pi(\alpha)}\rho_\alpha(a) \). This means that \( \rho_\alpha(a) = \rho_{\pi(\alpha)}'(a), a \in A \).

Two \( C^\ast \)-symbolic dynamical systems \( (A, \rho, \Sigma) \) and \( (A, \rho', \Sigma') \) are said to be inner conjugate if there exists an element \( U_{\alpha, \beta} \in A \) for \( \alpha \in \Sigma, \beta \in \Sigma' \) such that

(i) \( \rho_\alpha(a)U_{\alpha, \beta} = U_{\alpha, \beta}\rho_\beta'(a) \),
(ii) \( \sum_{\epsilon \in \Sigma} U_{\alpha, \epsilon}U_{\gamma, \epsilon} = \delta_{\alpha, \gamma}\rho_\alpha(1) \), \( \sum_{\gamma \in \Sigma} U_{\gamma, \beta}U_{\gamma, \epsilon} = \delta_{\beta, \epsilon}\rho_\beta'(1) \) and
(iii) \( \rho_\alpha(1)U_{\alpha, \beta} = U_{\alpha, \beta} = U_{\alpha, \beta}\rho_\beta'(1) \)

for \( \alpha, \gamma \in \Sigma, \beta, \epsilon \in \Sigma' \) and \( a \in A \). The family \( \{U_{\alpha, \beta}\}_{\alpha \in \Sigma, \beta \in \Sigma'} \) is called an intertwiner between \( (A, \rho, \Sigma) \) and \( (A, \rho', \Sigma') \).

**Proposition 4.4.** Two \( C^\ast \)-symbolic dynamical systems \( (A, \rho, \Sigma) \) and \( (A, \rho', \Sigma') \) are inner conjugate if and only if their associated Hilbert \( C^\ast \)-bimodules \((\rho, \mathcal{H}_A^\rho)\) and \((\rho', \mathcal{H}_A^{\rho'})\) are unitary equivalent as a Hilbert \( C^\ast \)-bimodule.

We note that if \( (A, \rho, \Sigma) \) and \( (A, \rho', \Sigma') \) are inner conjugate with intertwiner \( \{U_{\alpha, \beta}\}_{\alpha \in \Sigma, \beta \in \Sigma'} \), then the equalities for \( \alpha \in \Sigma, \beta \in \Sigma' \) and \( a \in A \)

\[ \rho_\alpha(a) = \sum_{\epsilon \in \Sigma'} U_{\alpha, \epsilon}U_{\gamma, \epsilon}^*, \quad \rho_\beta'(a) = \sum_{\gamma \in \Sigma} U_{\gamma, \beta}^*U_{\gamma, \beta}, \]

hold. For \( (A, \rho, \Sigma) \), let \( D_\rho(a) \) for \( a \in A \) be the \( |\Sigma| \times |\Sigma| \)-diagonal matrix \( D_\rho(a) \) with diagonal entries \( [\rho_\alpha(a)]_{\alpha \in \Sigma} \). One knows \( (A, \rho, \Sigma) \) and \( (A, \rho', \Sigma') \) are inner conjugate if and only if there exists an \( |\Sigma| \times |\Sigma'| \)-matrix \( U \) over \( A \) such that

\[ D_\rho(a) = UD_\rho'(a)U^* \quad \text{for } a \in A, \quad \text{and} \]
\[ UU^* = D_\rho(1), \quad U^*U = D_{\rho'}(1). \]

Let \( A \) be an \( n \)-dimensional commutative \( C^\ast \)-algebra. By Proposition 3.2, a \( C^\ast \)-symbolic dynamical system \( (A, \rho, \Sigma) \) defines a left-resolving labeled graph \( G^\rho = (G^\rho, \lambda^\rho) \) over \( \Sigma \) with underlying finite directed graph \( G^\rho \). Let \( v_1, \ldots, v_n \) denote the vertex set of \( G^\rho \). We denote by \( A^\rho(i, j) \) the cardinal number of the edges \( E^\rho(i, j) \) whose source vertex is \( v_i \) and terminal vertex is \( v_j \). In this case, inner conjugacy is completely characterized as in the following way.

**Proposition 4.5.** Let \( A \) be the \( n \)-dimensional commutative \( C^\ast \)-algebra. Then \( C^\ast \)-symbolic dynamical systems \( (A, \rho, \Sigma) \) and \( (A, \eta, \Sigma) \) are inner conjugate if and only if \( A^\rho(i, j) = A^\eta(i, j) \) for all \( i, j = 1, 2, \ldots, n \). That is, the directed graphs \( G^\rho \) and \( G^\eta \) are isomorphic.

5. **Crossed Products by Symbolic Dynamical Systems**

We will study \( C^\ast \)-algebras constructed from Hilbert \( C^\ast \)-symbolic bimodules. A general construction of \( C^\ast \)-algebras from Hilbert \( C^\ast \)-bimodules has been established by Pimsner [Pim] (cf. [Kal]). The \( C^\ast \)-algebras are called Cuntz-Pimsner algebras. Its ideal structure and simplicity conditions have been studied by Kajiwara-Pinznari-Watatani [KPW] and Muhly-Solel [MS], see also [KW], [PWY], [Sch]. For a \( C^\ast \)-symbolic dynamical system \( (A, \rho, \Sigma) \), we have a \( C^\ast \)-algebra from the Hilbert \( C^\ast \)-symbolic bimodule \((\rho, \mathcal{H}_A^\rho, \{u_\alpha\}_{\alpha \in \Sigma})\) by using Pimsner's general construction of \( C^\ast \)-algebras from Hilbert \( C^\ast \)-bimodules. We denote the \( C^\ast \)-algebra by \( A \rtimes_\rho \Lambda \), where \( \Lambda \) is the subshift \( \Lambda_{(A, \rho, \Sigma)} \) associated with \( (A, \rho, \Sigma) \). We call the algebra \( A \rtimes_\rho \Lambda \) the \( C^\ast \)-symbolic crossed product of \( A \) by the subshift \( \Lambda \).
**Proposition 5.1.** The $C^*$-symbolic crossed product $A \times_\rho \Lambda$ is the universal unital $C^*$-algebra $C^*(A, S_\alpha, \alpha \in \Sigma)$ generated by $x \in A$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following operator relations:

\[
\sum_{\beta \in \Sigma} S_\beta S_\beta^* = 1, \quad S_\alpha^* x S_\alpha = \rho_\alpha(x), \quad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x
\]

for all $x \in A$ and $\alpha \in \Sigma$. Furthermore for $\alpha_1, \ldots, \alpha_k \in \Sigma$, a word $(\alpha_1, \ldots, \alpha_k)$ is admissible for the subshift $\Lambda = \Lambda(A, \rho, \Sigma)$ if and only if $S_{\alpha_1} \cdots S_{\alpha_k} \neq 0$.

As in [Pim] (cf. [KPW]), the gauge action, denoted by $\hat{\rho}$, on the algebra $A \times_\rho \Lambda$ of the torus $T = \{z \in \mathbb{C} \mid |z| = 1\}$ is defined by

\[
\hat{\rho}_z(x) = x, \quad \hat{\rho}_z(S_\alpha) = zS_\alpha, \quad x \in A, \alpha \in \Sigma, z \in T.
\]

We have the following theorem.

**Theorem 5.2.** Let $(A, \rho, \Sigma)$ be a $C^*$-symbolic dynamical system and $\Lambda$ be the associated subshift $\Lambda(A, \rho, \Sigma)$. Assume that $A$ is commutative.

(i) If $A = \mathbb{C}$, the subshift $\Lambda$ is the full shift $\Sigma^\mathbb{Z}$, and the $C^*$-algebra $A \times_\rho \Lambda$ is the Cuntz algebra $O_\Sigma$ of order $|\Sigma|$.

(ii) If $A$ is finite dimensional, the subshift $\Lambda$ is a sofic shift $\Lambda_G$ presented by a left-resolving labeled graph $G$, and the $C^*$-algebra $A \times_\rho \Lambda$ is a Cuntz-Krieger algebra $O_G$ associated with the labeled graph. Conversely, for any sofic shift $\Lambda_G$, that is presented by a left-resolving labeled graph $G$, there exists a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that the associated subshift is the sofic shift $\Lambda_G$, the algebra $A$ is finite dimensional, and the $C^*$-algebra $A \times_\rho \Lambda$ is the Cuntz-Krieger algebra $O_G$ associated with the labeled graph.

(iii) If $A$ is an AF-algebra, there uniquely exists a $\lambda$-graph system $\Lambda$ up to equivalence such that the subshift $\Lambda$ is presented by $\Lambda$ and the $C^*$-algebra $A \times_\rho \Lambda$ is the $C^*$-algebra $O_\Lambda$ associated with the $\lambda$-graph system $\Lambda$. Conversely, for any subshift $\Lambda_\Sigma$, that is presented by a left-resolving $\lambda$-graph system $\Sigma$, there exists a $C^*$-symbolic dynamical system $(A, \rho, \Sigma)$ such that the associated subshift is the subshift $\Lambda_\Sigma$, the algebra $A$ is a commutative AF-algebra, and the $C^*$-algebra $A \times_\rho \Lambda$ is the $C^*$-algebra $O_\Sigma$ associated with the $\lambda$-graph system $\Lambda$.

We remark that Pimsner showed the following fact [Pim]: For every Hilbert $C^*$-bimodule $E$ over a $C^*$-algebra $A$, if $A$ is commutative and finite dimensional, and if $E$ is projective and finitely generated, the associated $C^*$-algebra is a Cuntz-Krieger algebra.

We will give some examples

(i) Let $\alpha_1, \ldots, \alpha_m \in \text{Aut}(B)$ be automorphisms of a unital $C^*$-algebra $B$. Let $G = (G, \lambda)$ be a left-resolving labeled graph with symbols $\Sigma = \{\alpha_1, \ldots, \alpha_m\}$. Let $V = \{v_1, \ldots, v_n\}$ be the vertex set. Let $[A^\theta(i, \alpha_k)]_{i,j=1,\ldots,n}$ be the $n \times n$-matrix for $\alpha_k \in \Sigma$ with entries in $\{0,1\}$ defined by (3.1). We put $A = B \oplus \cdots \oplus B$ the
direct sum of the \( n \)-copies of \( \mathcal{B} \). For \( \alpha_k \in \Sigma \), define \( \rho_{\alpha_k}^G \in \text{End}(A) \) by setting
\[
\rho_{\alpha_k}^G(b_1, \ldots, b_n) = \left( \sum_{i=1}^n A^G(i, \alpha_k, 1)\alpha_k(b_i), \ldots, \sum_{i=1}^n A^G(i, \alpha_k, n)\alpha_k(b_i) \right), \quad (b_1, \ldots, b_n) \in A.
\]

Since we assume that every vertex of \( G \) has an incoming edge, one has \( \sum_{k=1}^n \rho_{\alpha_k}^G(1) \geq 1 \). Since we also assume that every vertex of \( G \) has an outgoing edge, the family \( \{\rho_{\alpha_k}^G\}_{k=1}^n \) is faithful. Hence we have a \( C^* \)-symbolic dynamical system \((A, \rho^G, \Sigma)\). The associated subshift \( \Lambda_{(A, \rho^G, \Sigma)} \) is the sofic shift \( \Lambda_G \) presented by the labeled graph \( G \). If the underlying directed graph \( G \) is irreducible with condition (I) in the sense of [CK] and each automorphism \( \alpha_k \) has no nontrivial invariant ideal of \( \mathcal{B} \), the associated crossed product \( A \times_{\rho} \Lambda_G \) is simple and purely infinite.

The following example is a special case of this example.

(ii) Let \( A = C(\mathbb{T}) \) and \( \Sigma = \{1, 2, \ldots, n\} \), \( n > 1 \). Take irrational numbers \( \theta_1, \ldots, \theta_n \in \mathbb{R} \setminus \mathbb{Q} \). Define \( \rho_i(f)(z) = f(e^{2\pi\sqrt{-1}\theta_i}z) \) for \( f \in C(\mathbb{T}), z \in \mathbb{T} \). We have a \( C^* \)-symbolic dynamical system \((C(\mathbb{T}), \rho, \Sigma)\). Since the endomorphisms \( \rho_i, i = 1, \ldots, n \) are automorphisms and hence the associated subshift is the full shift \( \Sigma^\mathbb{Z} \). We denote by \( \mathcal{O}_{\theta_1,\ldots,\theta_n} \) the \( C^* \)-symbolic crossed product \( C(\mathbb{T}) \rtimes \theta_1,\ldots,\theta_n \Sigma^\mathbb{Z} \). As the algebra \( \mathcal{O}_{\theta_1,\ldots,\theta_n} \) is the universal unital \( C^* \)-algebra generated by \( n \) isometries and one unitary \( U \) satisfying the following relations:
\[
\sum_{j=1}^n S_j S_j^* = 1, \quad S_i^* S_i = 1, \quad US_i = e^{2\pi\sqrt{-1}\theta_i}S_i U, \quad i = 1, \ldots, n.
\]

Hence \( \mathcal{O}_{\theta_1,\ldots,\theta_n} \) is realized as the ordinary crossed product \( \mathcal{O}_n \rtimes \alpha_{\theta_1,\ldots,\theta_n} \mathbb{Z} \) of the Cuntz algebra \( \mathcal{O}_n \) by the automorphism \( \alpha_{\theta_1,\ldots,\theta_n} \) defined by \( \alpha_{\theta_1,\ldots,\theta_n}(S_i) = e^{2\pi\sqrt{-1}\theta_i}S_i \).

It is simple and purely infinite whose K-groups are
\[
K_0(\mathcal{O}_{\theta_1,\ldots,\theta_n}) = K_1(\mathcal{O}_{\theta_1,\ldots,\theta_n}) \cong \mathbb{Z}/(n-1)\mathbb{Z}.
\]

(iii) Let \( A = [A(i, j)]_{i,j=1,\ldots,n} \) be an \( n \times n \) matrix with entries in \( \{0, 1\} \). We denote by \( \Lambda_A^+ \) the compact Hausdorff space
\[
\Lambda_A^+ = \{(x_i)_{i \in \mathbb{N}} \in \{1, \ldots, n\}^\mathbb{N} | A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{N}\}
\]
of the right one-sided topological Markov shift associated with the matrix \( A \). Let \( S_i, i = 1, \ldots, n \) be the generating partial isometries of the Cuntz-Krieger algebra \( \mathcal{O}_A \) such that \( \sum_{j=1}^n S_j S_j^* = 1, S_i^* S_i = \sum_{j=1}^n A(i, j) S_j S_j^* \). The algebra \( \mathcal{A}_A = C(\Lambda_A^+) \) of all continuous functions on \( \Lambda_A^+ \) is identified with the subalgebra of \( \mathcal{O}_A \) generated by the projections \( S_\mu S_\mu^* \) for \( \mu = \mu_1, \ldots, \mu_k \), where \( S_\mu = S_{\mu_1} \cdots S_{\mu_k} \) for \( \mu_1, \ldots, \mu_k \in \{1, \ldots, n\} \). Let \( \Sigma = \{(1, 2, \ldots, n), (1, 2, \ldots, n)\} \) be \( 2n \)-brackets. We define \( 2n \)-endomorphisms of \( \mathcal{A}_A \) by setting
\[
\rho_i^A(a) = S_i^* a S_i, \quad \rho_j^A(a) = S_j a S_j^*, \quad i = 1, \ldots, n, \ a \in \mathcal{A}_A.
\]
We have a $C^*$-symbolic dynamical system $(\mathcal{A}_\Sigma, \rho^\Lambda, \Sigma)$. If in particular all entries $A(i,j), i, j = 1, \ldots, n$ of $A$ are 1, then $\Lambda_A^+$ is the right one-sided full shift $\{1, \ldots, n\}^\mathbb{N}$ and the associated subshift is the Dyck shift $D_n$ of the $2n$-brackets. Let $\mathcal{L}^{Ch(D_n)}$ be the corresponding $\lambda$-graph system for $(\mathcal{A}_\Sigma, \rho^\Lambda, \Sigma)$. It is called the Cantor horizon $\lambda$-graph system of the Dyck shift $D_n$ that has been studied in [KM]. The $C^*$-symbolic crossed product $C(\{1, \ldots, n\}^\mathbb{N}) \times_{\rho^\Lambda} D_n$ is a simple purely infinite $C^*$-algebra $\mathcal{O}_{\mathcal{L}^{Ch(D_n)}}$ that is the $C^*$-algebra associated with $\mathcal{L}^{Ch(D_n)}$. Its K-groups have been computed so that

$$K_0(C(\{1, \ldots, n\}^\mathbb{N}) \times_{\rho^\Lambda} D_n) = \mathbb{Z}/n\mathbb{Z} \oplus C(\mathbb{C}, \mathbb{Z}),$$

$$K_1(C(\{1, \ldots, n\}^\mathbb{N}) \times_{\rho^\Lambda} D_n) = 0$$

where $C(\mathbb{C}, \mathbb{Z})$ denotes the abelian group of all $\mathbb{Z}$-valued continuous functions on the Cantor set $\mathbb{C}([K'M])$.

For a general matrix $A$ with entries in $\{0,1\}$, let $\mathcal{L}^{Ch(D_A)}$ be the corresponding $\lambda$-graph system to $(\mathcal{A}_\Sigma, \rho^\Lambda, \Sigma)$. It is easy to see that the associated subshift is a subshift of Dyck shift $D_n$ that has some forbidden words coming from the forbidden words of the topological Markov shift $\Lambda_A$. The subshift is a version of topological Markov shift of the Dyck shifts, and appear in [HIK], [KM2]. We call it the topological Markov Dyck shift associated with the matrix $A$ and write it as $D_A$. We then see that the $C^*$-symbolic crossed product $C(\Lambda_A^+) \times_{\rho^\Lambda} D_A$ is a simple purely infinite $C^*$-algebra $\mathcal{O}_{\mathcal{L}^{Ch(D_A)}}$ if the matrix $A$ is irreducible.

6. **Strong shift equivalence of $C^*$-symbolic dynamical systems and Hilbert $C^*$-bimodules**

As in the preceding section, we may regard a $\lambda$-graph system as a $C^*$-symbolic dynamical system. The matrix interpretation of a $\lambda$-graph system is called a symbolic matrix system. In [Ma], we have formulated strong shift equivalence of symbolic matrix systems, as a generalization of nonnegative square matrices ([Wil]) and symbolic square matrices ([N]). Strong shift equivalence of symbolic matrix systems is a basic equivalence relation related to topological conjugacy of subshifts. It has been proved that two subshifts $\Lambda$ and $\Lambda'$ are topologically conjugate if and only if their canonical symbolic matrix systems $(\mathcal{M}_{\Lambda}^+, I_{\Lambda}^+)$ and $(\mathcal{M}_{\Lambda'}^+, I_{\Lambda'}^+)$ are strong shift equivalent ([Ma]).

In this section, we will formulate strong shift equivalences and shift equivalences of $C^*$-symbolic dynamical systems and of Hilbert $C^*$-symbolic bimodules as generalizations of those of $\lambda$-graph systems.

**Definition.** Two $C^*$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be **strong shift equivalent in 1-step** if there exist finite sets $C$ and $D$, two families of homomorphisms $\eta_c : \mathcal{A} \to \mathcal{A}'$, $c \in C$ and $\zeta_d : \mathcal{A}' \to \mathcal{A}$, $d \in D$ and two into bijections $\kappa : \Sigma \to CD$ and $\kappa' : \Sigma' \to DC$ such that

$$\rho_a = \zeta_d \circ \eta_c \text{ if } \kappa(\alpha) = c_{\alpha} d_{\alpha}, \text{ and } \rho_{a'} = \eta_{c_{a'}} \circ \zeta_d, \text{ if } \kappa'(\alpha') = d_{a'} c_{a'}$$

and

$$\zeta_d \circ \eta_c = 0 \text{ if } cd \notin \kappa(\Sigma), \text{ and } \eta_c \circ \zeta_d = 0 \text{ if } dc \notin \kappa'(\Sigma').$$
We write this situation as $(\mathcal{A}, \rho, \Sigma) \approx (\mathcal{A}', \rho', \Sigma')$.

We set $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathcal{A}'$ and $\tilde{\Sigma} = C \cup D$ disjoint union of $C$ and $D$. Define $\tilde{\rho}_\tilde{\alpha} \in \text{End}(\tilde{\mathcal{A}})$ for $\tilde{\alpha} \in \tilde{\Sigma}$ by setting

$$\tilde{\rho}_\tilde{\alpha}(x, y) = \begin{cases} (0, \eta_c(x)) & \text{if } \tilde{\alpha} = c \in C, \\ (\zeta_d(y), 0) & \text{if } \tilde{\alpha} = d \in D \end{cases}$$

for $(x, y) \in \mathcal{A} \oplus \mathcal{A}'$. Then we have

**Lemma 6.1.** $(\tilde{\mathcal{A}}, \tilde{\rho}, \tilde{\Sigma})$ is a $C^*$-symbolic dynamical system.

We call $(\tilde{\mathcal{A}}, \tilde{\rho}, \tilde{\Sigma})$ the bipartite $C^*$-symbolic dynamical system related to $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$. If there exists an $N$-chain of strong shift equivalences in 1-step between $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$, they are said to be strong shift equivalent in $N$-step and written as $(\mathcal{A}, \rho, \Sigma) \approx_N (\mathcal{A}', \rho', \Sigma')$. They are simply said to be strong shift equivalent.

Recall that two $C^*$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be isomorphic if there exists an isomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ of $C^*$-algebras and a bijection $\pi : \Sigma \to \Sigma'$ such that $\rho_\alpha = \phi^{-1} \circ \rho'_{\pi(\alpha)} \circ \phi$ for all $\alpha \in \Sigma$.

**Lemma 6.2.**

(i) If $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are isomorphic, they are strong shift equivalent in 1-step.

(ii) Suppose that both sets $\Sigma$ and $\Sigma'$ are one points $\{\alpha\}$ and $\{\alpha'\}$ respectively and both $\rho_\alpha$ and $\rho'_\alpha$ are automorphisms. Then $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are isomorphic if and only if they are strong shift equivalent in 1-step.

We next formulate shift equivalence of $C^*$-symbolic dynamical systems

**Definition.** $C^*$-symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma)$ and $(\mathcal{A}', \rho', \Sigma')$ are said to be shift equivalent of lag $N$ if there exist two finite sets $C$ and $D$, two families $\eta_c : \mathcal{A} \to \mathcal{A}'$, $c \in C$ and $\zeta_d : \mathcal{A} \to \mathcal{A}'$, $d \in D$ of homomorphisms and four specifications $\kappa_C : \Sigma C \to \Sigma' C'$, $\kappa_D : \Sigma' D \to \Sigma D$, $\kappa_{\Sigma'} : \Sigma' N \to \Sigma D$ and $\kappa_{\Sigma} : \Sigma N \to \Sigma' D$ such that

$$\eta_c \circ \rho_\alpha = \rho'_\alpha \circ \eta_c$$

$$\zeta_d \circ \rho'_\alpha = \rho_\alpha \circ \zeta_d$$

and

$$\rho_{\alpha_N} \circ \cdots \circ \rho_{\alpha_2} \circ \rho_{\alpha_1} = \zeta_d \circ \eta_c$$

and $\kappa_C(\alpha_1 \alpha_2 \cdots \alpha_N) = cd$,

$$\rho'_{\alpha_N} \circ \cdots \circ \rho'_{\alpha_2} \circ \rho'_{\alpha_1} = \eta_c \circ \zeta_d$$

and $\kappa_{\Sigma'}(\alpha_1 \alpha_2 \cdots \alpha_N) = d'c'$.

We write this situation as $(\mathcal{A}, \rho, \Sigma) \sim_N (\mathcal{A}', \rho', \Sigma')$.

The following proposition is proved by similar ideas to the case of matrices ([Wi], cf.[LM]).
Proposition 6.3. Let \((A', \rho', \Sigma')\) and \((A'', \rho'', \Sigma'')\) be \(C^*\)-symbolic dynamical systems.

(i) \(\bigl(A, \rho, \Sigma\bigr) \approx_N (A', \rho', \Sigma')\) implies \(\bigl(A, \rho, \Sigma\bigr) \sim_N (A', \rho', \Sigma')\).

(ii) \(\bigl(A, \rho, \Sigma\bigr) \sim_N (A', \rho', \Sigma')\) implies \(\bigl(A, \rho, \Sigma\bigr) \sim_N (A', \rho', \Sigma')\) for all \(N' \geq N\).

(iii) \(\bigl(A, \rho, \Sigma\bigr) \sim (A', \rho', \Sigma')\) and \((A', \rho', \Sigma')\) imply \((A'\overline{\rho}'\overline{\Sigma}'\overline{\Sigma}'\overline{\Sigma})\) imply \((A, \rho, \Sigma) \sim (A'', \rho'', \Sigma'')\).

Thus shift equivalence of \(C^*\)-symbolic dynamical systems is an equivalence relation.

We will next formulate strong shift equivalence and shift equivalence of Hilbert \(C^*\)-bimodules. Let \(A\) and \(A'\) be \(C^*\)-algebras. We define a Hilbert \(C^*\)-symbolic right \(A'\)-module \((\varphi, \mathcal{A} A', \{w_\alpha\}_{\alpha \in \Sigma})\) over \(\Sigma\) with left \(A\)-action by a Hilbert \(C^*\)-right \(A'\)-module with orthogonal essential finite basis \(\{w_\alpha\}_{\alpha \in \Sigma}\) and a unital faithful diagonal left action \(\varphi\) of \(A\) on \(\mathcal{A} A'\). Let \((\varphi, \mathcal{A} A', \{w_\alpha\}_{\alpha \in \Sigma})\) be a Hilbert \(C^*\)-symbolic right \(A'\)-module over \(\Sigma\) with left \(A\)-action and \((\psi, \mathcal{A} A', \{w'_\alpha\}_{\alpha \in \Sigma'})\) a Hilbert \(C^*\)-symbolic right \(A''\)-module over \(\Sigma'\) with left \(A'\)-action. Define the relative tensor product

\[
(\varphi, \mathcal{A} A', \{w_\alpha\}_{\alpha \in \Sigma}) \otimes_{A'} (\psi, \mathcal{A} A', \{w'_\alpha\}_{\alpha \in \Sigma'}) := (\varphi \otimes 1, \mathcal{A} A' \otimes_{A'} \mathcal{A} A'', \{w_\alpha \otimes_{A'} w'_\alpha\}_{(\alpha, \alpha') \in \Sigma \otimes_{A'} \Sigma'})
\]

where \(\mathcal{A} A' \otimes_{A'} \mathcal{A} A''\) is the tensor product Hilbert \(C^*\)-right \(A''\)-module relative to \(A'\), and \(\varphi \otimes 1\) is the natural left \(A\)-action on it. The finite set \(\Sigma \otimes_{A'} \Sigma'\) is defined as follows: As both the left action \(\varphi\) and \(\psi\) are diagonal with respect to the bases \(\{w_\alpha\}_{\alpha \in \Sigma}\) and \(\{w'_\alpha\}_{\alpha \in \Sigma'}\), respectively, there exist \(\eta_\alpha(a) \in A'\) for \(a \in A\) and \(\zeta_{\alpha'}(b) \in A''\) for \(b \in A'\) such that

\[
\varphi(a)w_\alpha = w_\alpha \eta_\alpha(a), \quad \psi(b)w'_\alpha = w'_\alpha \zeta_{\alpha'}(b).
\]

The finite set \(\Sigma \otimes_{A'} \Sigma'\) is defined by

\[
\Sigma \otimes_{A'} \Sigma' = \{(\alpha, \alpha') \in \Sigma \times \Sigma' \mid \zeta_{\alpha'}(\eta_\alpha(1_A)) \neq 0\}.
\]

It is easy to check that

\[
(\varphi \otimes 1, \mathcal{A} A' \otimes_{A'} \mathcal{A} A'', \{w_\alpha \otimes_{A'} w'_\alpha\}_{(\alpha, \alpha') \in \Sigma \otimes_{A'} \Sigma'})
\]

is a Hilbert \(C^*\)-symbolic right \(A''\)-module over \(\Sigma \otimes_{A'} \Sigma'\) with left \(A\)-action.

Definition. Let \((\phi, \mathcal{A} A)\) be a Hilbert \(C^*\)-bimodule over \(A\) and \((\phi', \mathcal{A} A')\) a Hilbert \(C^*\)-right \(A'\)-module over \(A'\). They are said to be strong shift equivalent in 1-step and written as \((\phi, \mathcal{A} A) \approx (\phi', \mathcal{A} A')\) if there exist a Hilbert \(C^*\)-right \(A'\)-module \((\varphi, \mathcal{A} A')\) with left \(A\)-action and a Hilbert \(C^*\)-right \(A\)-module \((\psi, \mathcal{A} A)\) with left \(A'\)-action such that

\[
(\psi \otimes 1, \mathcal{A} A' \otimes_{A'} \mathcal{A} A', \psi \otimes 1, \mathcal{A} A \otimes A' A' \mathcal{A} A', \psi \otimes 1, \mathcal{A} A \mathcal{A} A' A' \mathcal{A} A', \psi \otimes 1, \mathcal{A} A \mathcal{A} A' A' \mathcal{A} A') = (\phi', \mathcal{A} A)\) as a Hilbert \(C^*\)-bimodule over \(A\),

\[
(\psi \otimes 1, \mathcal{A} A' \otimes_{A'} \mathcal{A} A', \psi \otimes 1, \mathcal{A} A \otimes A' A' \mathcal{A} A', \psi \otimes 1, \mathcal{A} A \mathcal{A} A' A' \mathcal{A} A', \psi \otimes 1, \mathcal{A} A \mathcal{A} A' A' \mathcal{A} A') = (\phi', \mathcal{A} A)\) as a Hilbert \(C^*\)-bimodule over \(A'\).
The above all equalities of Hilbert $C^*$-bimodules mean unitary equivalences as Hilbert $C^*$-bimodules. In this situation, we say that $(\varphi, \mathcal{H}_A)$ and $(\psi, \mathcal{H}_A)$ satisfy the strong shift equivalence relation between $(\phi, \mathcal{H}_A)$ and $(\phi', \mathcal{H}_A)$. Consider the direct sum

$$(\varphi, \mathcal{H}_A) \oplus (\psi, \mathcal{H}_A) := (\varphi \oplus \psi, \mathcal{H}_A' \oplus \mathcal{H}_A)$$

that is a Hilbert $C^*$-right $\mathcal{A}' \oplus \mathcal{A}$-module with left $\mathcal{A} \oplus \mathcal{A}'$-action. It is denoted by $(\xi, \mathcal{H}_X)$ and satisfies

$$\mathcal{A} \mathcal{H}_A' = \xi(\mathcal{A}) \mathcal{H}_X = \mathcal{H}_X \mathcal{A}', \quad \mathcal{A} \mathcal{H}_A = \xi(\mathcal{A}') \mathcal{H}_X = \mathcal{H}_X \mathcal{A}.$$ 

As $\mathcal{H}_X$ is regarded as a Hilbert $C^*$-right $\mathcal{A} \oplus \mathcal{A}'$-module, $(\xi, \mathcal{H}_X)$ is considered to be a Hilbert $C^*$-bimodule over $\mathcal{A} \oplus \mathcal{A}'$, that is called a bipartite Hilbert $C^*$-bimodule related to $(\phi, \mathcal{H}_A)$ and $(\phi', \mathcal{H}_A')$. We note that the condition (6.1) is equivalent to the condition:

$$(\xi \otimes 1, \mathcal{H}_X \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_X) = (\phi, \mathcal{H}_A) \oplus (\phi', \mathcal{H}_A')$$

as a Hilbert $C^*$-bimodule over $\mathcal{A} \oplus \mathcal{A}'$.

If there exists an $N$-chain of strong shift equivalences in 1-step between $(\phi, \mathcal{H}_A)$ and $(\phi', \mathcal{H}_A')$, they are said to be strong shift equivalent in $N$-step and we write it as $$(\phi, \mathcal{H}_A) \approx (\phi', \mathcal{H}_A').$$

They are simply said to be strong shift equivalent.

In particular, Hilbert $C^*$-symbolic bimodules $(\phi, \mathcal{H}_A, \{u_{\alpha}\}_{\alpha \in \Sigma})$ and $(\phi', \mathcal{H}_A', \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})$ are said to be strong shift equivalent in 1-step if there exist a Hilbert $C^*$-symbolic right $\mathcal{A}'$-module $(\varphi, \mathcal{H}_A'), \{w_{\alpha}\}_{\alpha \in \Sigma}$ with left $\mathcal{A}$-action and a Hilbert $C^*$-right $\mathcal{A}$-module $(\psi, \mathcal{H}_A, \{w'_{\alpha'}\}_{\alpha' \in \Sigma'})$ with left $\mathcal{A}'$-action such that the qualities (6.1) are taken to be unitary equivalent as Hilbert $C^*$-symbolic bimodules.

**Definition.** Let $(\phi, \mathcal{H}_A)$ be a Hilbert $C^*$-bimodule over $\mathcal{A}$ and $(\phi', \mathcal{H}_A')$ a Hilbert $C^*$-bimodule over $\mathcal{A}'$. They are said to be shift equivalent of lag $N$ if there exist a Hilbert $C^*$-right $\mathcal{A}'$-module $(\varphi, \mathcal{H}_A')$ with left $\mathcal{A}$-action and a Hilbert $C^*$-right $\mathcal{A}$-module $(\psi, \mathcal{H}_A)$ with left $\mathcal{A}'$-action such that

$$(\phi, \mathcal{H}_A \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_A) = (\varphi \otimes 1, \mathcal{H}_A' \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_A'),$$

and

$$(\phi', \mathcal{H}_A' \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_A') = (\psi \otimes 1, \mathcal{H}_A \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_A'),$$

and

$$(\varphi \otimes 1, \mathcal{H}_A' \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_A') = (\phi, \mathcal{H}_A \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_A'), \quad (\psi \otimes 1, \mathcal{H}_A \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_A) = (\phi', \mathcal{H}_A' \otimes_{\mathcal{A} \oplus \mathcal{A}'} \mathcal{H}_A')$$

We write this situation as $(\phi, \mathcal{H}_A) \approx (\phi', \mathcal{H}_A')$.

We similarly define a shift equivalence between Hilbert $C^*$-symbolic bimodules by equipping with finite bases.

The above formulations of a strong shift equivalence and a shift equivalence of Hilbert $C^*$-bimodules are generalizations of those of nonnegative square matrices defined by Willams (cf.[N],[Ma6]). The following proposition is parallel to Proposition 6.3. ([Wi], cf.[LM]).
Proposition 6.4. Let \((\phi, \mathcal{H}_A), (\phi', \mathcal{H}_{A'})\) and \((\phi'', \mathcal{H}_{A''})\) be Hilbert \(C^*\)-bimodules.

(i) \((\phi, \mathcal{H}_A) \approx_N (\phi', \mathcal{H}_{A'})\) implies \((\phi, \mathcal{H}_A) \cong_N (\phi', \mathcal{H}_{A'})\) for all \(N\).

(ii) \((\phi, \mathcal{H}_A) \cong_N (\phi', \mathcal{H}_{A'})\) implies \((\phi, \mathcal{H}_A) \approx_N (\phi', \mathcal{H}_{A'})\) for all \(N\).

(iii) \((\phi, \mathcal{H}_A) \cong_N (\phi', \mathcal{H}_{A'})\) and \((\phi'', \mathcal{H}_{A''})\) imply \((\phi, \mathcal{H}_A) \cong_N (\phi'', \mathcal{H}_{A''})\).

The similar statements hold for Hilbert \(C^*\)-symbolic bimodules.

Therefore shift equivalence of Hilbert \(C^*\)-bimodules and similarly shift equivalence of Hilbert \(C^*\)-symbolic bimodules are equivalence relations.

Proposition 6.5. If \(C^*\)-symbolic dynamical systems \((A, \rho, \Sigma)\) and \((A', \rho', \Sigma')\) are strong shift equivalent in 1-step, their associated Hilbert \(C^*\)-symbolic bimodules \((\phi, \mathcal{H}_A, \{u_{\alpha}\}_{\alpha \in \Sigma})\) and \((\phi', \mathcal{H}_{A'}, \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})\) are strong shift equivalent in 1-step.

Its converse implication holds.

Proposition 6.6. If Hilbert \(C^*\)-symbolic bimodules \((\phi, \mathcal{H}_A, \{u_{\alpha}\}_{\alpha \in \Sigma})\) and \((\phi', \mathcal{H}'_{A'}, \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})\) are strong shift equivalent in 1-step, their associated \(C^*\)-symbolic dynamical systems \((A, \rho, \Sigma)\) and \((A', \rho', \Sigma')\) are strong shift equivalent in 1-step.

We may similarly see that two \(C^*\)-symbolic dynamical systems \((A, \rho, \Sigma)\) and \((A', \rho', \Sigma')\) are shift equivalent of lag \(N\) if and only if their associated Hilbert \(C^*\)-symbolic bimodules \((\phi, \mathcal{H}_A, \{u_{\alpha}\}_{\alpha \in \Sigma})\) and \((\phi', \mathcal{H}'_{A'}, \{u'_{\alpha'}\}_{\alpha' \in \Sigma'})\) are shift equivalent of lag \(N\).

7. Strong shift equivalence of gauge actions

In this section we introduce the notion of strong shift equivalence of \(C^*\)-symbolic crossed products with gauge actions.

Definition. Two \(C^*\)-symbolic crossed products \((A \times_{\rho} \Lambda, \hat{\rho}, T)\) and \((A' \times_{\rho'} \Lambda', \hat{\rho}', T)\) with gauge actions are said to be strong shift equivalent in 1-step if there exists a \(C^*\)-symbolic dynamical system \((A_0, \rho_0, \Sigma_0)\) and full projections \(p, p' \in A_0 \times_{\rho_0} \Lambda_0\) satisfying \(p + p' = 1\) and \(\rho_0(z)(p) = p, \rho_0(z)(p') = p'\) for \(z \in T\) where \(\Lambda_0\) is the subshift associated with \((A_0, \rho_0, \Sigma_0)\), and

\[
(p(A_0 \times_{\rho_0} \Lambda_0) p, \rho_0, T) = (A \times_{\rho} \Lambda, \hat{\rho}^2, T),
\]

\[
(p'(A_0 \times_{\rho_0} \Lambda_0) p', \rho_0, T) = (A' \times_{\rho'} \Lambda', \hat{\rho}'^2, T).
\]

We write this situation as \((A \times_{\rho} \Lambda, \hat{\rho}, T) \approx_1 (A' \times_{\rho'} \Lambda', \hat{\rho}', T)\). If there exists an \(N\)-chain of strong shift equivalences in 1-step, they are said to be strong shift equivalent in \(N\)-step and written as \((A \times_{\rho} \Lambda, \hat{\rho}, T) \equiv_N (A' \times_{\rho'} \Lambda', \hat{\rho}', T)\). It is simply said to be strong shift equivalent.

Theorem 7.1. Let \((A, \rho, \Sigma)\) and \((A', \rho', \Sigma')\) be two \(C^*\)-symbolic dynamical systems whose associated subshifts are denoted by \(\Lambda\) and \(\Lambda'\) respectively. If \((A, \rho, \Sigma)\) and \((A', \rho', \Sigma')\) are strong shift equivalent, the \(C^*\)-symbolic crossed products \((A \times_{\rho} \Lambda, \hat{\rho}, T)\) and \((A' \times_{\rho'} \Lambda', \hat{\rho}', T)\) with gauge actions are strong shift equivalent.

This theorem and its proof are generalizations of [Ma4: Theorem 3.15].
Suppose that \((A, \rho, \Sigma)\) and \((A', \rho', \Sigma')\) are strong shift equivalent in 1-step. There exist finite sets \(C\) and \(D\), two families of homomorphisms \(\eta_c : A \to A', c \in C\) and \(\zeta_d : A' \to A, d \in D\) and two into bijections \(\kappa : \Sigma \to CD\) and \(\kappa' : \Sigma' \to DC\) that give rise to the strong shift equivalence between \((A, \rho, \Sigma)\) and \((A', \rho', \Sigma')\). Let \((\widetilde{A}, \tilde{\rho}, \widetilde{\Sigma})\) be the bipartite \(C^*\)-symbolic dynamical system related to \((A, \rho, \Sigma)\) and \((A', \rho', \Sigma')\).

As \(\widetilde{A} = A \oplus A'\), we identify \(A\) and \(A'\) with the subalgebras of \(\widetilde{A}\) by regarding \(a \in A\) as \((a, 0) \in \widetilde{A}\) and \(a' \in A'\) as \((0, a') \in \widetilde{A}\) respectively. The symbolic crossed product

\[
\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda} = C^*(S_{\tilde{\alpha}}, x \mid \tilde{\alpha} \in \tilde{\Sigma}, x \in \widetilde{A})
\]

of \((\widetilde{A}, \tilde{\rho}, \tilde{\Sigma})\) is the universal \(C^*\)-algebra generated by partial isometries \(S_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{\Sigma} = C \cup D\) and elements \(x \in \widetilde{A}\) that satisfy the relations (5.1). Let \(C^*(S_{CD}, A)\) and \(C^*(S_{DC}, A')\) be the \(C^*\)-subalgebras of \(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda}\) defined by setting

\[
C^*(S_{CD}, A) = C^*(S_{c_a d_a}, (a, 0) \mid c_a d_a = \kappa(\alpha), \alpha \in \Sigma, a \in A) \quad \text{and}
\]

\[
C^*(S_{DC}, A') = C^*(S_{d_a' c_{a'}}, (0, a') \mid d_a' c_{a'} = \kappa'(\alpha'), \alpha' \in \Sigma', a' \in A')
\]

respectively, where \(S_{c_a d_a} = S_{c_a} S_{d_a}\) and \(S_{d_a' c_{a'}} = S_{d_a'} S_{c_{a'}}\). Put the projections

\[
P_C = \sum_{c \in C} S_c S_c^*, \quad P_D = \sum_{d \in D} S_d S_d^* \quad \text{in} \ \widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda}.
\]

Hence \(P_C + P_D = 1\).

We see that the following propositions hold.

**Proposition 7.2.**

\[
C^*(S_{CD}, A) = P_C(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda})P_C, \quad C^*(S_{DC}, A') = P_D(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda})P_D.
\]

**Proposition 7.3.** The \(C^*\)-symbolic crossed products \(A \times_\rho \Lambda\) and \(A' \times_{\rho'} \Lambda'\) are canonically isomorphic to the algebras \(C^*(S_{CD}, A)\) and \(C^*(S_{DC}, A')\) respectively.

The following lemma shows that the subalgebras \(P_C(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda})P_C\) and \(P_D(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda})P_D\) are complementary full corners in \(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda}\).

**Lemma 7.4.** The projections \(P_C, P_D\) are full in the algebra \(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda}\).

**Proof of sketch of Theorem 7.1.** By Proposition 7.2 and Proposition 7.3, we may identify the algebras \(A \times_\rho \Lambda\) with \(P_C(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda})P_C\), and \(A' \times_{\rho'} \Lambda'\) with \(P_D(\widetilde{A} \times_{\tilde{\rho}} \tilde{\Lambda})P_D\). By these identifications, one has

\[
\tilde{\rho}_2^2(s_a) = \tilde{\rho}_z(S_c S_d), \quad \tilde{\rho}_2^2(s'_{a'}) = \tilde{\rho}_z(S_d S_c)
\]

for \(\kappa(\alpha) = cd \in CD\), \(\kappa'(\alpha') = dc \in DC\). Thus \((A \times_\rho \Lambda, \tilde{\rho}, \tilde{T})\) and \((A' \times_{\rho'} \Lambda', \tilde{\rho}', \tilde{T})\) are strong shift equivalent in 1-step. \(\square\)

**Remark.** It is possible to generalize the above discussions such as strong shift equivalent Hilbert \(C^*\)-bimodules give rise to strong shift equivalent \(C^*\)-algebras of the Hilbert \(C^*\)-bimodules. We will discuss this generalization in a forthcoming paper [Ma6].

We present the following theorem.
Theorem 7.5. Let $(A, \rho, \Sigma)$ and $(A', \rho', \Sigma')$ be two $C^*$-symbolic dynamical systems whose associated subshifts are denoted by $\Lambda$ and $\Lambda'$ respectively. If $(A, \rho, \Sigma)$ and $(A', \rho', \Sigma')$ are strong shift equivalent, then we have

(i) the subshifts $\Lambda$ and $\Lambda'$ are topologically conjugate,

(ii) the $C^*$-symbolic crossed products $(A \rtimes_\rho \Lambda, \hat{\rho}, T)$ and $(A' \rtimes_{\rho'} \Lambda', \hat{\rho}', T)$ with gauge actions are strong shift equivalent, and

(iii) the stabilized gauge actions $(A \rtimes_\rho \Lambda \otimes K, \hat{\rho} \otimes \id, T)$ and $(A' \rtimes_{\rho'} \Lambda' \otimes K, \hat{\rho}' \otimes \id, T)$ are cocycle conjugate, where $K$ denotes the $C^*$-algebra of all compact operators on a separable infinite dimensional Hilbert space.

In the rest of this section, we will concern K-theory for the $C^*$-algebra $A \rtimes_\rho \Lambda$ constructed from a $C^*$-dynamical system $(A, \rho, \Sigma)$. The endomorphisms $\rho_\alpha : A \to A$ for $\alpha \in \Sigma$ yield endomorphisms $\rho_\alpha : K_*(A) \to K_*(A)$ for $\alpha \in \Sigma$ on the K-theory groups of $A$. Define an endomorphism

$$\rho_* : K_* (A) \to K_* (A), \quad *=0,1$$

by setting $\rho_*(g) = \sum_{\alpha \in \Sigma} \rho_{\alpha *}(g), g \in K_* (A)$. By [Pim] (cf. [KPW]), one has the following six term exact sequence of K-theory:

$$K_0(A) \xrightarrow{id-\rho_*} K_0(A) \xrightarrow{i_*} K_0(A \rtimes_\rho \Lambda)$$

$$\uparrow \quad \quad \quad \quad \downarrow$$

$$K_1(A \rtimes_\rho \Lambda) \quad \quad \quad \quad K_1(A) \xleftarrow{i_*} K_1(A) \xleftarrow{id-\rho_*} K_1(A).$$

Hence if in particular $K_1(A) = 0$, one has

$$K_0(A \rtimes_\rho \Lambda) = K_0(A)/(id - \rho_*)K_0(A),$$

$$K_1(A \rtimes_\rho \Lambda) = \text{Ker}(id - \rho_*) \text{ in } K_0(A).$$

This formula is a generalization of K-theory formulae proved in [C2] and [Ma3]. As in [Ma3:Lemma 5.2], one sees that the fixed point algebra $F_{(A, \rho, \Sigma)}$ of $A \rtimes_\rho \Lambda$ under the gauge action $\hat{\rho}$ is stably isomorphic to $(A \times_\rho \Lambda) \times_\hat{\rho} \mathbb{T}$. We define the K-groups $K_*(A, \rho, \Sigma)$ and the dimension groups $D_*(A, \rho, \Sigma)$ for $(A, \rho, \Sigma)$ by setting

$$K_*(A, \rho, \Sigma) = K_*(A \rtimes_\rho \Lambda)$$

$$D_*(A, \rho, \Sigma) = (K_*(F_{(A, \rho, \Sigma)}), \hat{\rho}_*) \quad *=0,1$$

where $\hat{\rho}_*$ is the automorphism on the abelian group $K_*(F_{(A, \rho, \Sigma)})$ induced by the dual action $\hat{\rho}$ of the gauge action $\hat{\rho}$. We also define the Bowen-Franks groups $BF^*(A, \rho, \Sigma)$ for $(A, \rho, \Sigma)$ by setting

$$BF^*(A, \rho, \Sigma) = \text{Ext}_*(A \rtimes_\rho \Lambda), \quad *=0,1$$

Then Theorem 7.5 (iii) implies
Proposition 7.6. The abelian groups $K_*(A, \rho, \Sigma), BF^*(A, \rho, \Sigma)$ and the abelian group with automorphisms $D_*(A, \rho, \Sigma)$ for $(A, \rho, \Sigma)$ are invariant under strong shift equivalence of $C^*$-symbolic dynamical systems.

The above results are generalization of [Ma4] see also [C2], [CK], [Ma2].

In [Ma8], dynamical property of a "subshift"

$$S_{(A, \rho, \Sigma)} = \{(\rho \alpha_i)_{i \in \mathbb{Z}} | (\rho \alpha_i \circ \cdots \circ \rho \alpha_{i+k})(1) \neq 0, i \in \mathbb{Z}, k \in \mathbb{Z}^+\}$$

will be studied.

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