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On Infimal Convolution of M-Convex Functions

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Abstract

The infimal convolution of M-convex functions is M-convex. This is a fundamental fact in discrete convex analysis that is often useful in its application to mathematical economics and game theory. M-convexity and its variant called $M^s$-convexity are closely related to gross substitutability, and the infimal convolution operation corresponds to an aggregation. This note provides a succinct description of the present knowledge about the infimal convolution of M-convex functions.

1 Definitions

Let $V$ be a nonempty finite set, and let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of integers and reals, respectively. We denote by $\mathbb{Z}^V$ the set of integral vectors indexed by $V$, and by $\mathbb{R}^V$ the set of real vectors indexed by $V$. For a vector $x = (x(v) | v \in V) \in \mathbb{Z}^V$, where $x(v)$ is the $v$th component of $x$, we define the positive support $\text{supp}^+(x)$ and the negative support $\text{supp}^-(x)$ by

$$\text{supp}^+(x) = \{v \in V | x(v) > 0\}, \quad \text{supp}^-(x) = \{v \in V | x(v) < 0\}.$$

We use notation $x(S) = \sum_{v \in S} x(v)$ for a subset $S$ of $V$. For each $S \subseteq V$, we denote by $\chi_S$ the characteristic vector of $S$ defined by: $\chi_S(v) = 1$ if $v \in S$ and $\chi_S(v) = 0$ otherwise, and write $\chi_v$ for $\chi_{\{v\}}$ for $v \in V$. For a vector $p = (p(v) | v \in V) \in \mathbb{R}^V$ and a function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$, we define functions $\langle p, x \rangle$ and $f[p](x)$ in $x \in \mathbb{Z}^V$ by

$$\langle p, x \rangle = \sum_{v \in V} p(v)x(v), \quad f[p](x) = f(x) + \langle p, x \rangle.$$

We also denote the set of minimizers of $f$ and the effective domain of $f$ by

$$\text{arg \ min} f = \{x \in \mathbb{Z}^V | f(x) \leq f(y) (\forall y \in \mathbb{Z}^V)\}, \quad \text{dom} f = \{x \in \mathbb{Z}^V | f(x) < +\infty\}.$$

We say that a function $f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom} f \neq \emptyset$ is $M$-convex if it satisfies the exchange axiom:

(M-EXC) For $x, y \in \text{dom} f$ and $u \in \text{supp}^+(x-y)$, there exists $v \in \text{supp}^-(x-y)$ such that
\[ f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v). \]  

The inequality (1) implicitly imposes the condition that \( x - \chi_u + \chi_v \in \text{dom} f \) and \( y + \chi_u - \chi_v \in \text{dom} f \) for the finiteness of the right-hand side. A function \( f \) is said to be \( M \)-concave if \( -f \) is \( M \)-convex.

As a consequence of (M-EXC), the effective domain of an \( M \)-convex function \( f \) lies on a hyperplane \( \{ x \in \mathbb{R}^V \mid x(V) = r \} \) for some integer \( r \), and accordingly, we may consider the projection of \( f \) along a coordinate axis. This means that, instead of the function \( f \) in \( n = |V| \) variables, we may consider a function \( f' \) in \( n - 1 \) variables defined by

\[ f'(x') = f(x_0, x') \quad \text{with} \quad x_0 = r - x'(V'), \]  

where \( V' = V \setminus \{v_0\} \) for an arbitrarily fixed element \( v_0 \in V \), and a vector \( x \in \mathbb{Z}^V \) is represented as \( x = (x_0, x') \) with \( x_0 = x(v_0) \in \mathbb{Z} \) and \( x' \in \mathbb{Z}^{V'} \). Note that the effective domain \( \text{dom} f' \) of \( f' \) is the projection of \( \text{dom} f \) along the chosen coordinate axis \( v_0 \). A function \( f' \) derived from an \( M \)-convex function by such projection is called an \( M^\# \)-convex function.

More formally, an \( M^\# \)-convex function is defined as follows. Let \( \emptyset \) denote a new element not in \( V \) and put \( \tilde{V} = \{0\} \cup V \). A function \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) is called \( M^\# \)-convex if the function \( \tilde{f} : \mathbb{Z}^{\tilde{V}} \to \mathbb{R} \cup \{+\infty\} \) defined by

\[ \tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 = -x(V) \\ +\infty & \text{otherwise} \end{cases} \quad (x_0 \in \mathbb{Z}, x \in \mathbb{Z}^{\tilde{V}}) \]  

is an \( M \)-convex function. It is known (see [4, Theorem 6.2]) that an \( M^\# \)-convex function \( f \) can be characterized by a similar exchange property:

\[(\text{M}^\#\text{-EXC})\] For \( x, y \in \text{dom} f \) and \( u \in \text{supp}^+(x - y) \),

\[ f(x) + f(y) \geq \min \left[ f(x - \chi_u) + f(y + \chi_u), \quad \min_{v \in \text{supp}^+(x-y)} \{ f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \} \right], \]  

where the minimum over an empty set is \( +\infty \) by convention. A function \( f \) is said to be \( M^\# \)-concave if \( -f \) is \( M^\# \)-convex.

Whereas \( M^\# \)-convex functions are conceptually equivalent to \( M \)-convex functions, the class of \( M^\# \)-convex functions is strictly larger than that of \( M \)-convex functions. This follows from the implication: \( (\text{M-EXC}) \Rightarrow (\text{M}^\#\text{-EXC}) \). The simplest example of an \( M^\# \)-convex function that is not \( M \)-convex is a one-dimensional (univariate) discrete convex function, depicted in Fig. 1.

**Proposition 1** ([4, Theorem 6.3]). An \( M \)-convex function is \( M^\# \)-convex. Conversely, an \( M^\# \)-convex function is \( M \)-convex if and only if the effective domain is contained in a hyperplane \( \{ x \in \mathbb{Z}^V \mid x(V) = r \} \) for some \( r \in \mathbb{Z} \).

1) "\( M^\# \)-convex" should be read "\( M \)-natural-convex."
$\vec{x}$

Figure 1: Univariate discrete convex function

$\mathbf{M}^\mathfrak{h}$-convex functions enjoy a number of nice properties that are expected of "discrete convex functions." Furthermore, $\mathbf{M}^\mathfrak{h}$-concave functions provide with a natural model of utility functions (see [4, §11.3] and [5]). In particular, it is known that $\mathbf{M}^\mathfrak{h}$-concavity is equivalent to gross substitutes property, and that $\mathbf{M}^\mathfrak{h}$-concavity implies submodularity, which is the discrete version of decreasing marginal returns.

It follows from (M-EXC) that the effective domain of an $\mathbf{M}$-convex function $f$ satisfies the exchange axiom:

(\textbf{B-EXC}) For $x, y \in B$ and $u \in \text{supp}^+(x - y)$, there exists $v \in \text{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$,

since $x - \chi_u + \chi_v \in \text{dom} f$ and $y + \chi_u - \chi_v \in \text{dom} f$ for $x, y \in \text{dom} f$ in (1). A nonempty set $B$ of integer points satisfying (B-EXC) is referred to as an $\mathbf{M}$-convex set.

2 Convolution Theorem

For a pair of functions $f_1, f_2 : \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\}$, the integer infimal convolution is a function $f_1 \square_\mathbb{Z} f_2 : \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{-\infty\}$ defined by

$$(f_1 \square_\mathbb{Z} f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x = x_1 + x_2, x_1, x_2 \in \mathbb{Z}^V\} \quad (x \in \mathbb{Z}^V).$$

(5)

Provided that $f_1 \square_\mathbb{Z} f_2$ is away from the value of $-\infty$, we have

$$\text{dom}(f_1 \square_\mathbb{Z} f_2) = \text{dom} f_1 + \text{dom} f_2,$$

(6)

where the right-hand side means the Minkowski sum of the effective domains.

The convolution theorem reads as follows.

Theorem 2 ([4, Theorem 6.13]). For $\mathbf{M}$-convex functions $f_1$ and $f_2$, the integer infimal convolution $f = f_1 \square_\mathbb{Z} f_2$ is $\mathbf{M}$-convex, provided $f > -\infty$.

A proof of this theorem is given in Section 3, whereas the $\mathbf{M}^\mathfrak{h}$-version below is an immediate corollary.

Corollary 3 ([4, Theorem 6.15]). For $\mathbf{M}^\mathfrak{h}$-convex functions $f_1$ and $f_2$, the integer infimal convolution $f = f_1 \square_\mathbb{Z} f_2$ is $\mathbf{M}^\mathfrak{h}$-convex, provided $f > -\infty$. 
Proof. Let \( \tilde{f}_1 \) and \( \tilde{f}_2 \) be the M-convex functions associated with the M\(^2\)-convex functions \( f_1 \) and \( f_2 \) as in (3). For \( x_0 \in \mathbb{Z}, x \in \mathbb{Z}^V \) we have

\[
(\tilde{f}_1 \square_{\mathbb{Z}} \tilde{f}_2)(x_0, x) = \inf \{ \tilde{f}_1(y_0, y) + \tilde{f}_2(z_0, z) | x = y + z, x_0 = y_0 + z_0 \}
\]

\[
= \inf \{ f_1(y) + f_2(z) | x = y + z, x_0 = y_0 + z_0, y_0 = -y(V), z_0 = -z(V) \}
\]

\[
= \inf \{ f_1(y) + f_2(z) | x = y + z, x_0 = -x(V) \}
\]

\[
= \left\{ \begin{array}{ll}
( f_1 \square_{\mathbb{Z}} f_2)(x) & \text{if } x_0 = -x(V) \\
+\infty & \text{otherwise.}
\end{array} \right.
\]

This shows \( \tilde{f}_1 \square_{\mathbb{Z}} \tilde{f}_2 = ( f_1 \square_{\mathbb{Z}} f_2)^\sim \) in the notation of (3), whereas \( \tilde{f}_1 \square_{\mathbb{Z}} \tilde{f}_2 \) is M-convex by Theorem 2 applied to \( \tilde{f}_1 \) and \( \tilde{f}_2 \). Therefore, \( f_1 \square_{\mathbb{Z}} f_2 \) is M\(^2\)-convex. \( \square \)

Remark 1. The convolution theorem (Theorem 2) originates in [1, Theorem 6.10], and is described in [2, p. 80, Theorem 2.44 (5)], [3, p. 118, Theorem 4.8 (8)], and [4, p. 143, Theorem 6.13 (8)]. The M\(^2\)-version (Corollary 3) is also stated in [2, p. 83], [3, p. 119, Theorem 4.10], and [4, p. 144, Theorem 6.15 (1)]. An application of this fact to the aggregation of utility functions can be found in [3, p. 275, Proposition 9.13] and [4, p. 337, Theorem 11.12]. In particular, the convolution theorem implies that if the individual utility functions enjoy gross substitutes property, so does the aggregated utility function.

### 3 Proof

The proof of Theorem 2 given here relies on two fundamental facts stated in the lemmas below. The first shows that the class of M-convex sets is closed under Minkowski addition, and the second gives a characterization of an M-convex function in terms of M-convex sets.

**Lemma 4 ([4, Theorem 4.23]).** The Minkowski sum of two M-convex sets is M-convex.

**Lemma 5 ([4, Theorem 6.30]).** Let \( f : \mathbb{Z}^V \to \mathbb{R} \cup \{+\infty\} \) be a function with a bounded nonempty effective domain. Then, \( f \) is M-convex if and only if \( \arg\min f[-p] \) is an M-convex set for each \( p \in \mathbb{R}^V \).

Let \( f_1 \) and \( f_2 \) be M-convex functions, and put \( f = f_1 \square_{\mathbb{Z}} f_2 \). First we treat the case where \( \text{dom} f_1 \) and \( \text{dom} f_2 \) are bounded. The expression (6) shows that \( \text{dom} f \) is bounded. For each \( p \in \mathbb{R}^V \) we have

\[
f[-p] = ( f_1[-p] ) \square_{\mathbb{Z}} ( f_2[-p] ),
\]

from which follows

\[
\arg\min f[-p] = \arg\min f_1[-p] + \arg\min f_2[-p]
\]
by (5). In this expression, both \( \arg \min f_1[-p] \) and \( \arg \min f_2[-p] \) are \( \mathbb{M} \)-convex sets by Lemma 5 (only if part), and therefore, their Minkowski sum (the right-hand side) is \( \mathbb{M} \)-convex by Lemma 4. This means that \( \arg \min f[-p] \) is \( \mathbb{M} \)-convex for each \( p \in \mathbb{R}^V \), which implies the \( \mathbb{M} \)-convexity of \( f \) by Lemma 5 (if part).

The general case without the boundedness assumption on effective domains can be treated via limiting procedure as follows. For \( i = 1, 2 \) and \( k = 1, 2, \ldots \), define \( f^{(k)}_i : \mathbb{Z}^V \rightarrow \mathbb{R} \cup \{+\infty\} \) by

\[
\begin{align*}
f_i^{(k)}(x) &= \begin{cases} f_i(x) & \text{if } \|x\|_\infty \leq k \\
+\infty & \text{otherwise} \end{cases} \quad (x \in \mathbb{Z}^V),
\end{align*}
\]

which is an \( \mathbb{M} \)-convex function with a bounded effective domain, provided that \( k \) is large enough for \( \text{dom} f_i^{(k)} \neq \emptyset \). For each \( k \), the infimal convolution \( f^{(k)} = f_1^{(k)} \square_z f_2^{(k)} \) is \( \mathbb{M} \)-convex by the above argument, and moreover, \( \lim_{k \to \infty} f^{(k)}(x) = f(x) \) for each \( x \). It remains to demonstrate the property (M-EXC) for \( f \). Take \( x, y \in \text{dom} f \) and \( u \in \text{supp}^-(x-y) \). There exists \( k_0 = k_0(x, y) \), depending on \( x \) and \( y \), such that \( x, y \in \text{dom} f^{(k)} \) for every \( k \geq k_0 \). Since \( f^{(k)} \) is \( \mathbb{M} \)-convex, there exists \( v_k \in \text{supp}^-(x-y) \) such that

\[
f^{(k)}(x) + f^{(k)}(y) \geq f^{(k)}(x - \chi_u + \chi_{v_k}) + f^{(k)}(y + \chi_u - \chi_{v_k}).
\]

Since \( \text{supp}^-(x-y) \) is a finite set, at least one element of \( \text{supp}^-(x-y) \) appears infinitely many times in the sequence \( v_1, v_2, \ldots \). More precisely, there exists \( v \in \text{supp}^-(x-y) \) and an increasing subsequence \( k_1 < k_2 < \cdots \) such that \( v_{k_j} = v \) for \( j = 1, 2, \ldots \). By letting \( k \to \infty \) along this subsequence in the above inequality we obtain

\[
f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).
\]

Thus \( f \) satisfies (M-EXC). This completes the proof of Theorem 2.

**Remark 2.** Here is an example to demonstrate the necessity of the limiting argument in the above proof. For \( \mathbb{M} \)-convex functions \( f_1, f_2 : \mathbb{Z}^2 \rightarrow \mathbb{R} \) defined by

\[
f_1(x) = \begin{cases} \exp(-x(1)) & \text{if } x(1) + x(2) = 0, \\
+\infty & \text{otherwise}, \end{cases} \quad f_2(x) = \begin{cases} \exp(x(1)) & \text{if } x(1) + x(2) = 0, \\
+\infty & \text{otherwise}, \end{cases}
\]

we have

\[
f(x) = (f_1 \square_z f_2)(x) = \inf \{\exp(-t) + \exp(x(1) - t) \mid t \in \mathbb{Z}\} = 0
\]

for all \( x \in \mathbb{Z}^2 \) with \( x(1) + x(2) = 0 \). The infimum is not attained by any finite \( t \), and consequently, \( f^{(k)}(x) \) is not equal to \( f(x) \) for any finite \( k \). This is why we need the limiting argument in the proof. \( \blacksquare \)

**Remark 3.** The infimal convolution operation of \( \mathbb{M} \)-convex functions can be formulated as a special case of the transformation of an \( \mathbb{M} \)-convex function by a network, and the convolution theorem (Theorem 2) can be understood as a special case of a theorem on network transformation.
The general framework of the network transformation is as follows. Let \( G = (V, A; S, T) \) be a directed graph with vertex set \( V \), arc set \( A \), entrance set \( S \) and exit set \( T \), where \( S \) and \( T \) are disjoint subsets of \( V \). We consider an integer-valued flow \( \xi = (\xi(a) \mid a \in A) \in \mathbb{Z}^A \). For each \( a \in A \), the cost of the flow \( \xi(a) \) through arc \( a \) is represented by a function \( f_a : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\} \). Given a function \( f : \mathbb{Z}^S \to \mathbb{R} \cup \{+\infty\} \) associated with the entrance set \( S \), we define another function \( \hat{f} : \mathbb{Z}^T \to \mathbb{R} \cup \{-\infty\} \) on the exit set \( T \) by

\[
\hat{f}(y) = \inf_{\xi,a} \left\{ f(x) + \sum_{a \in A} f_a(\xi(a)) \mid \partial \xi = (x, -y, 0), \xi \in \mathbb{Z}^A, (x, -y, 0) \in \mathbb{Z}^S \times \mathbb{Z}^T \times \mathbb{Z}^{V \setminus (S \cup T)} \right\} \quad (y \in \mathbb{Z}^T),
\]

where \( \partial \xi \in \mathbb{Z}^V \) denotes a vector defined by

\[
\partial \xi(v) = \sum\{\xi(a) \mid \text{arc } a \text{ leaves vertex } v\} - \sum\{\xi(a) \mid \text{arc } a \text{ enters vertex } v\} \quad (v \in V).
\]

We may think of \( \hat{f}(y) \) as the minimum cost of an integer-valued flow to meet a demand specification \( y \) at the exit, where the cost consists of two parts, the cost \( f(x) \) of supply or production of \( x \) at the entrance and the cost \( \sum_{a \in A} f_a(\xi(a)) \) of transportation through arcs; the sum of these is to be minimized over varying supply \( x \) and flow \( \xi \) subject to the flow conservation constraint \( \partial \xi = (x, -y, 0) \). We regard \( \hat{f} \) as a transformation of \( f \) by the network.

It is known ([4, Theorem 9.27]) that if \( f_a \) is a univariate discrete convex function for each \( a \in A \) and \( f \) is an \( M \)-convex function, then \( \hat{f} \) is an \( M \)-convex function, provided that \( \hat{f} > -\infty \) and \( \hat{f} \neq +\infty \).

For the infimal convolution of functions \( f_1 \) and \( f_2 \), let \( V_1 \) and \( V_2 \) be copies of \( V \) and consider a bipartite graph \( G = (S \cup T, A; S, T) \) (see Fig. 2) with \( S = V_1 \cup V_2, T = V \) and \( A = \{(v_1, v) \mid v \in V\} \cup \{(v_2, v) \mid v \in V\} \), where \( v_i \in V_i \) is the copy of \( v \in V \) for \( i = 1, 2 \). We regard \( f_i \) as being defined on \( V_i \) for \( i = 1, 2 \) and assume that the arc cost functions \( f_a \ (a \in A) \) are identically zero. The function \( \hat{f} \) induced on \( T \) coincides with the infimal convolution \( f_1 \square \mathbb{Z} f_2 \). In this case it is always true that \( \hat{f} \neq +\infty \). Thus the convolution theorem (Theorem 2) follows from [4, Theorem 9.27], as is explained in [4, Note 9.30].

The connection to network transformation also suggests that the infimal convolution \( f_1 \square \mathbb{Z} f_2 \) can be evaluated by solving an \( M \)-convex submodular flow problem; see [4, Section 9.2] for the definition of the problem and [4, Section 10.4] for algorithms.

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References

Figure 2: Bipartite graph for infimal convolution


