Extension of Bing Maps

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Abstract

In [7], M. Levin proved that the set of all Bing maps of a compact metric space to the unit interval constitutes a $G_{\delta}$-dense subset of the space of maps. In [6], J. Krasinkiewicz independently proved that the set of all Bing maps of a compact metric space to an $n$-dimensional manifold ($n \geq 1$) constitutes a $G_{\delta}$-dense subset of the space of maps. In [9], J. Song and E. D. Tymchatyn solved some problems of J. Krasinkiewicz [6]: They proved that the set of all Bing maps of a compact metric space to a nondegenerate connected polyhedron (or a 1-dimensional locally connected continuum) constitutes a $G_{\delta}$-dense subset of the space of maps. In this note, by using methods of Levin [7] and Krasinkiewicz [6], we prove the extension theorem of Bing maps which is slightly precise than the theorem of J. Song and E. D. Tymchatyn.

1 Introduction

In this note, all spaces are separable metrizable spaces and maps are continuous functions. We denote the unit interval $[0, 1]$ by $I$. An arc is a space which is homeomorphic to $I$. If $X$ is a compact metrizable space and $Y$ is a space, $C(X, Y)$ denotes the space of all continuous maps from $X$ to $Y$ endowed with sup metric. A compact metrizable space is called a compactum, and a continuum means a connected compactum. A map $f$ is called a $\epsilon$-map if all diameters of fibers of $f$ are smaller than $\epsilon$. A continuum is said to be indecomposable if it is not sum of two proper subcontinua. A compactum is called a Bing compactum (or said to be hereditarily indecomposable) if each of its subcontinua is indecomposable. A map is called a Bing map if each of its fibers is a Bing compactum. In [7], M. Levin proved the following theorem.
Theorem 1 (M. Levin [7]) For each compactum $X$, the set of all Bing maps in $C(X, I)$ is a $G_{δ}$-dense subset in $C(X, I)$.

On the other hand, J. Krasinkiewicz proved the next theorem independently.

Theorem 2 (J. Krasinkiewicz [6]) Let $X$ be a compactum and let $Y$ be an $n$-dimensional manifold ($n \geq 1$). Then the set of all Bing maps in $C(X, Y)$ is a $G_{δ}$-dense subset in $C(X, Y)$.

Note that Theorem 2 is a generalization of Theorem 1. In [6], J. Krasinkiewicz poses the following problem: If $Y$ in Theorem 2 is an other space (for example, dendrite, dendroid, polyhedron, locally connected continuum, the Menger universal curve, AR, ANR), does Theorem 2 hold? In [9], J. Song and E. D. Tymchatyn solved the problems of J. Krasinkiewicz: In particular, they proved the following:

Theorem 3 (J. Song and E. D. Tymchatyn [9]) The set of all Bing maps of a compact metric space to a nondegenerate connected polyhedron (or a 1-dimensional locally connected continuum) constitutes a $G_{δ}$-dense subset of the space of maps.

In this note, we prove the following theorem by using methods of Levin [7] and Krasinkiewicz [6], which is more precise than the above theorem of J. Song and E. D. Tymchatyn. The proofs are somewhat different from one of J. Song and E. D. Tymchatyn [9]. For case of graphs, we use an idea of M. Levin [7], and for general case of polyhedra, we will use an idea of J. Krasinkiewicz [6].

Theorem 4 (Extension Theorem of Bing Maps) Let $X$ be a compactum and let $A$ be a closed subset in $X$. Let $\mathcal{K}$ be a finite simplicial complex such that $|\mathcal{K}|$ is a nondegenerate connected polyhedron and let $\mathcal{L}$ be a subcomplex of $\mathcal{K}$. If $f : A \to |\mathcal{L}|$ is a Bing map and $\tilde{f} : X \to |\mathcal{K}|$ is a map with $\tilde{f}|A = f$ and $\tilde{f}^{-1}(|\mathcal{L}|) = A$, then for any $\epsilon > 0$ there exists a Bing map $g : X \to |\mathcal{K}|$ such that $g|A = f$ and $d(\tilde{f}, g) < \epsilon$.

As a corollary, we obtain the theorem of J. Song and E. D. Tymchatyn. Also, we investigate surjective Bing maps from continua to polyhedra.
2 Preliminaries

In this section, first we give some definitions which are used in this paper.

Notation 5 Let $X$ be a space and let $d$ be a metric on $X$. We denote
the identity map of $X$ by $id_X$. For a subset $A \subset X$ and $\delta > 0$, denote
\[ B(A, \delta) = \{x \in X \mid \text{there exists } a \in A \text{ such that } d(x, a) < \delta\} \]
\[
= \{x \in X \mid \text{if } U \text{ is a neighborhood of } x \text{ then } U \cap A \neq \emptyset\}, \]
\[ intA = \{x \in X \mid \text{there exists a neighborhood } V \text{ of } x \text{ such that } V \subset A\}. \]
If $\mathcal{A}$ is a family of subsets of $X$, denote $\text{mesh}\mathcal{A} = \sup\{diam(A) \mid A \in \mathcal{A}\}$. If $\sigma$ is a simplex, we denote the boundary of $\sigma$ by $\partial\sigma$. If $\mathcal{K}$ is a simplicial complex and $n \in \mathbb{N}$, we denote $\mathcal{K}^{(n)} = \{\sigma \in \mathcal{K} \mid \text{dim } \sigma \leq n\}$ and $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$.

For each arc $I$ and $x \neq y \in I$, $[x, y]_I$ means an arc in $I$ from $x$ to $y$ and $[x, y]_I$, $(x, y)_I$, $(x, y)_I$ mean $[x, y]_I \setminus \{y\}$, $[x, y]_I \setminus \{x\}$, $[x, y]_I \setminus \{x, y\}$ respectively.

Now we will give the definition of $D$-crooked. The definition of $D$-crooked
was originally introduced in [2], and the definition below was given in [7].

Definition 6 Let $\mathcal{D} = \{(F_0, F_1, V_0, V_1) \mid F_0, F_1 \text{ are disjoint closed subsets in } \mathbb{R}^n \text{ and } V_0, V_1 \text{ are disjoint open neighborhoods of } F_0, F_1 \text{ in } \mathbb{R}^n\}$ and $D = (F_0, F_1, V_0, V_1) \in \mathcal{D}$. A subspace $X \subset \mathbb{R}^n$ is $D$-crooked if there exists an
open neighborhood $U$ of $X$ in $\mathbb{R}^n$ such that for any map $f : I \to U$ with the
property $f(0) \in F_0$ and $f(1) \in F_1$, there exist $t_0, t_1$ with $0 < t_0 < t_1 < 1$
such that $f(t_0) \in V_1$ and $f(t_1) \in V_0$. A map is said to be $D$-crooked if each
of its fibers is $D$-crooked.

Clearly, subspaces of $D$-crooked spaces are also $D$-crooked. M. Levin
obtained the following propositions in [7].

Proposition 7 (M. Levin [7]) If $A \subset \mathbb{R}^n$ is $D$-crooked, then there exists a
neighborhood $U \subset \mathbb{R}^n$ of $A$ such that $U$ is $D$-crooked.

Proposition 8 (M. Levin [7]) A compactum $A \subset \mathbb{R}^n$ is a Bing compactum
if and only if $A$ is $D$-crooked for each $D \in \mathcal{D}$.

Proposition 9 (M. Levin [7]) There exist $D_1, D_2, \ldots \in \mathcal{D}$ such that for any
compactum $A \subset \mathbb{R}^n$, $A$ is a Bing compactum if and only if $A$ is $D_i$-crooked
for each $i = 1, 2, \ldots$.
The next theorem was proved by R. H. Bing. Many authors used the theorem to reach important conclusions (for example, the theorem is used in the proof of Theorem 1).

**Theorem 10** (R. H. Bing [2]) Let $X$ be a compactum and let $A$, $B$ be disjoint closed subsets in $X$. Then there exists a Bing compactum $L$ such that $L$ is a partition between $A$ and $B$.

Now, we recall the definition of inverse limits. Let $\{X_i, f_i\}_{i=1}^\infty$ be a double sequence of spaces $X_i$, called coordinate spaces, and maps $f_i : X_{i+1} \to X_i$, called bonding maps. Then inverse limit of $\{X_i, f_i\}_{i=1}^\infty$, denoted by $\lim\{X_i, f_i\}$, is the subspace of $\prod_{i=1}^\infty X_i$ defined by $\lim\{X_i, f_i\} = \{(x_i) \in \prod_{i=1}^\infty X_i | f_i(x_{i+1}) = x_i$ for each $i = 1, 2, \ldots \}$. For $Y = \lim\{X_i, f_i\}$ and $i = 1, 2, \ldots$, a map $p_i : Y \to X_i$ is called a $i$-th projection if $p_i$ satisfies $p_i((x_j)_{j=1}^\infty) = x_i$ for each $(x_j)_{j=1}^\infty \in Y$. It is well known that every $n$-dimensional continuum is an inverse limit of $n$-dimensional compact connected polyhedra with onto bonding maps.

### 3 Bing maps to Peano curves

A space is called a Peano space if the space is locally connected. A space $X$ is called a Peano curve if $X$ is a 1-dimensional Peano continuum. In this section, we prove the theorem of J. Song and E. D. Tymchatyn for graphs by using Levin's idea [7].

**Theorem 11** (J. Song and E. D. Tymchatyn [9]) Let $X$ be a compactum and let $Y$ be a Peano curve. Then the set of all Bing maps in $C(X,Y)$ is a $G_\delta$-dense subset in $C(X,Y)$.

Before we prove Theorem 11, we prove some lemmas. The next lemma follows from Theorem 10 which plays very important role in the proof of Lemma 13.

**Lemma 12** Let $X$ be a compactum and let $F_1, F_2, \ldots, F_k$ $(k \geq 2)$ be pairwise disjoint closed subsets in $X$. Then there exist pairwise disjoint open subsets $U_1, U_2, \ldots, U_k$ such that $F_i \subset U_i$ for $i = 1, 2, \ldots, k$ and $X \setminus \bigcup_{i=1}^k U_i$ is a Bing compactum.
Proof. We will prove Lemma 12 by the induction on $k$. For $k = 2$, Lemma 12 holds by Theorem 10. Suppose that Lemma 12 holds for $k = 2, 3, \ldots, n - 1$ ($n \geq 3$). Let $F_1, F_2, \ldots, F_n$ be pairwise disjoint closed subsets in $X$. By the inductive assumption there exist pairwise disjoint open subsets $U_1, U_2, \ldots, U_{n-2}, V_{n-1}$ such that $F_1 \subset U_1, F_2 \subset U_2, \ldots, F_{n-2} \subset U_{n-2}, F_{n-1} \cup F_n \subset V_{n-1}$ and $L_1 = X \setminus (\bigcup_{i=1}^{n-2} U_i \cup V_{n-1})$ is a Bing compactum. Since $F_{n-1}$ and $(X \setminus V_{n-1}) \cup F_n$ are disjoint, there exist disjoint open subsets $U_{n-1}, W_{n-1}$ such that $F_{n-1} \subset U_{n-1}$, $(X \setminus V_{n-1}) \cup F_n \subset W_{n-1}$ and $X \setminus (U_{n-1} \cup W_{n-1})$ is a Bing compactum. Let $U_n = W_{n-1} \setminus (X \setminus V_{n-1})$. We see that $F_i \subset U_i$ for $i = 1, 2, \ldots, n$ and $U_1, U_2, \ldots, U_n$ are pairwise disjoint. And since $X \setminus (U_1 \cup U_2 \cup \cdots \cup U_n) = L_1 \cup (X \setminus (U_{n-1} \cup W_{n-1}))$ and $L_1, X \setminus (U_{n-1} \cup W_{n-1})$ are pairwise disjoint Bing compacta, $X \setminus (U_1 \cup U_2 \cup \cdots \cup U_n)$ is a Bing compactum. So $U_1, U_2, \ldots, U_n$ have the required property. This completes the proof.

The proof of the next lemma is inspired by the proof of Theorem 1. Let us recall that a compactum $X$ is called a graph if $X$ is a 1-dimensional polyhedron.

**Lemma 13** Let $X$ be a compactum and let $G$ be a connected graph. Then the set of all Bing maps in $C(X, G)$ is a $G_\delta$-dense subset in $C(X, G)$.

**Proof.** Let $X \subset I^n$ be a compactum, $f \in C(X, G)$ and $\epsilon > 0$. Set $D$ as in Definition 6 and $D_1, D_2, \ldots \in D$ as in Proposition 9. Put $D_i(X, G) = \{g \in C(X, G) \mid g$ is a $D_i$-crooked map$\}$ for each $i = 1, 2, \ldots$

By Proposition 8 and 9, $\{g \in C(X, G) \mid g$ is a Bing map$\} = \bigcap_{i=1}^{\infty} D_i(X, G)$. By Baire theorem it is sufficient to show that $D_i(X, G)$ is an open dense subset in $C(X, G)$.

Claim 1. $D_i(X, G)$ is an open subset in $C(X, G)$. This result has been proved in [7]. For completeness, we give the proof.

Proof of Claim 1. Let $g \in D_i(X, G)$. By Proposition 7 and since $g$ is a closed map, we can take an open cover $V$ of $G$ such that $f^{-1}(V)$ is $D_i$-crooked for each $V \in V$. Let $\delta$ be a Lesbegue number of the restriction of this cover to $g(X)$. Then $h \in D_i(X, G)$ for each $h \in C(X, G)$ with $d(h, g) < \delta/2$.

Claim 2. $D_i(X, G)$ is a dense subset of $C(X, G)$.

Proof of Claim 2. Let $f \in C(X, G)$ and $\epsilon > 0$. Take a simplicial complex $K$ of $G$ such that mesh $K < \epsilon$. At first we will show that $f$ can be
approximated by a map \( f' \in C(X, G) \) with the property that \( f^{-1}(p) \) is a Bing compactum for each \( p \in \mathcal{K}^{(0)} \). Let \( \{p_j\}_{j=1}^{m} = \mathcal{K}^{(0)} \setminus \{p \in \mathcal{K}^{(0)} | p \text{ is an endpoint of } G \} \). For \( j = 1, \) let \( I_{1j}, I_{2j}, \ldots, I_{kj} \in \mathcal{K}^{(1)} \) be all edges which contain \( p_1 \) as their endpoint, and let \( p_{1\ell} \) be the other endpoint of \( I_{1\ell} \) for \( \ell = 1, 2, \ldots, k \). Take \( r_{\ell} \in I_{1\ell} \setminus \{p_{1\ell} \} \) for \( \ell = 1, 2, \ldots, k \). Let \( A_1 = \bigcup_{\ell=1}^{k} I_{1\ell} \). Since \( f^{-1}([r_1,p_{11}], I_{11}), f^{-1}([r_2,p_{12}], I_{12}), \ldots, f^{-1}([r_k,p_{1k}], I_{1k}) \) are pairwise disjoint closed subsets in \( f^{-1}(A_1) \), by Lemma 12, there exist open subsets \( U_1, U_2, \ldots, U_k \in f^{-1}(A_1) \) such that \( f^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell}) \subset U_{\ell} \) for \( \ell = 1, 2, \ldots, k \), and \( L_1 = f^{-1}(A_1) \setminus \bigcup_{\ell=1}^{k} U_{\ell} \) is a Bing compactum. Now, we construct \( f_1 : L_1 \cup U_\ell \rightarrow I_{1\ell} \) such that \( f_1| f^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell}) = f| f^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell}) \) and \( f_1^{-1}(p_{1\ell}) = L_1 \) for \( \ell = 1, 2, \ldots, k \). We can take a map \( f_1 : L_1 \cup (U_\ell \setminus f^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell})) \rightarrow [p_{1\ell}, r_{1\ell}]_{t_1} \) such that \( f_1^{-1}(p_{1\ell}) = L_1 \) and \( f_1^{-1}(r_{1\ell}) = f^{-1}(r_{1\ell}) \). Then a map \( f_1 : L_1 \cup U_\ell \rightarrow I_{1\ell} \), defined by \( f_1(x) = f_1(x) \) if \( x \in L_1 \cup U_\ell \setminus f^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell}) \) and \( f_1(x) = f(x) \) if \( x \in f^{-1}([r_{\ell}, p_{1\ell}], I_{1\ell}) \) has the required property. Define \( f_1' : f^{-1}(A_1) \rightarrow A_1 \) by \( f_1'(x) = f_1(x) \) if \( x \in L_1 \cup U_\ell \) for \( \ell = 1, 2, \ldots, k \). Define \( f_1 : X \rightarrow G \) by \( f_1(x) = f(x) \) if \( x \in \text{cl}(X \setminus f^{-1}(A_1)) \) and \( f_1(x) = f_1'(x) \) if \( x \in f^{-1}(A_1) \). Then \( d(f, f_1') < \varepsilon \) and \( f_1^{-1}(p_1) \) is a Bing compactum.

If the step above has been done for \( j \leq n - 1 \) (\( 2 \leq n \leq m \)), then do the same step for \( j = n \). Then \( f \) can be approximated by a map \( f_1 : X \rightarrow G \) such that \( f^{-1}(p_1) \) is a Bing compactum for \( j = 1, 2, \ldots, n \). So we can take a map \( f' : X \rightarrow G \) such that \( d(f, f') < \varepsilon \) and \( f^{-1}(p_1) \) is a Bing compactum for each \( j = 1, 2, \ldots, m \). And we may assume that \( f'(x) \) is not an endpoint of \( G \) for each \( x \in X \). So \( f \) can be approximated by a map \( f' : X \rightarrow G \) such that \( f'^{-1}(p) \) is a Bing compactum for each \( p \in \mathcal{K}^{(0)} \). So we may assume that \( A = \bigcup_{p \in \mathcal{K}^{(0)}} f^{-1}(p) \) is a Bing compactum.

Now, we will use an idea of the proof of [7, Theorem 1.8]. Let \( D_t = (F_0, F_1, V_0, V_1) \in \mathcal{D} \). Take closed neighborhoods \( E_0, E_1 \) of \( F_0, F_1 \) such that \( F_0 \subset E_0 \subset V_0 \) and \( F_1 \subset E_1 \subset V_1 \). Since \( D_t = (E_0, E_1, V_0, V_1) \in \mathcal{D} \) and \( A \) is a Bing compactum, by Proposition 8 \( A \) is \( D_t \)-crooked. By Proposition 7 there exists a neighborhood \( B \) of \( A \) such that \( B \) is \( D_t \)-crooked. We claim that \( H = B \cup \text{int}E_0 \cup \text{int}E_1 \) is \( D_t \)-crooked.

Let \( \varphi : I \rightarrow H \) be a map with \( \varphi(0) \in F_0 \) and \( \varphi(1) \in F_1 \). Let \( b_0 = \max\{b \in I | \varphi(b) \in E_0\} \) and \( b_1 = \min\{b \in I | b > b_0 \) and \( \varphi(b) \in E_1\} \). Since \( B \) is \( D_t \)-crooked, there exist \( t_0, t_1 \in I \) with \( b_0 < t_0 < t_1 < b_1 \) such that \( \varphi(t_0) \in V_1 \) and \( \varphi(t_1) \in V_0 \). So \( H \) is \( D_t \)-crooked.

Since \( (X \setminus H) \cap F_0 = \phi \) \( (X \setminus H) \cap F_1 \), \( X \setminus H \) is \( D_t \)-crooked and since \( (X \setminus H) \cap A = \phi \), \( A \cup (X \setminus H) \) is \( D_t \)-crooked. Let \( \mathcal{K}^{(1)} = \{I_1, I_2, \ldots, I_s\} \). For
each $I_j \in \mathcal{K}^{(1)}$, let $p_{j_1}, p_{j_2}$ be the endpoints of $I_j$, and $X_j = f^{-1}(I_j)$ and $S_j = (X \setminus H) \cap X_j$. Define $g_j : X_j \rightarrow I_j$ such that $g_j^{-1}(p_{j_1}) = f^{-1}(p_{j_1}) \cup S_j$ and $g_j^{-1}(p_{j_2}) = f^{-1}(p_{j_2})$ for $j = 1, 2, \ldots, s$. Define $g : X \rightarrow G$ by $g(x) = g_j(x)$ if $x \in X_j$. Then, $d(f, g) < \varepsilon$ and for each $y \in G$, $g^{-1}(y) \subset H$ or $g^{-1}(y) \subset (X \setminus H) \cup A$. In both cases, $g^{-1}(y)$ is $D_\varepsilon$-crooked. This completes the proof.

**Remark 14** In the proof of Claim 1 we only use the fact that $X$ is compact. So for each compactum $X$ and space $Y$, the set of all Bing maps in $C(X, Y)$ is a $G_\delta$-subset in $C(X, Y)$.

The following definition was given in [6].

**Definition 15** Let $Y$ be a space. We say that $Y$ is free if for every compactum $X$ the set of all Bing maps in $C(X, Y)$ is a dense subset in $C(X, Y)$.

A map $f : X \rightarrow Y$ is called an $n$-dimensional map if $\dim f^{-1}(y) \leq n$ for each $y \in Y$. Note that 0-dimensional maps are Bing maps. By the theorem of Hurewicz for mappings and dimension, we see that if $X$ is a compactum and $P$ is a polyhedron such that $\dim X > \dim P$, then there is no 0-dimensional map $f$ from $X$ to $P$.

We need the next lemma.

**Lemma 16** Let $Y$ be a space. If for each $\varepsilon > 0$ there exist a free compactum $Z$ and maps $p : Y \rightarrow Z$ and $q : Z \rightarrow Y$ such that $d(q \circ p, id_Y) < \varepsilon$ and $q$ is a 0-dimensional map, then $Y$ is a free space.

Proof. Let $X$ be a compactum and let $h : X \rightarrow Y$ be a map. By the assumption there exists a free compactum $Z$ and maps $p : Y \rightarrow Z$ and $q : Z \rightarrow Y$ such that $d(q \circ p, id_Y) < \varepsilon$ and $q$ is a 0-dimensional map. Since $q$ is uniformly continuous, there exists $\delta > 0$ such that if $a, b \in Z$ satisfy $d(a, b) < \delta$, then $d(q(a), p(b)) < \varepsilon$. Since $Z$ is free, there exists a Bing map $\varphi : X \rightarrow Z$ such that $d(p \circ h, \varphi) < \delta$. Let $\psi = q \circ \varphi$, then $\psi$ is a Bing map because $q$ is 0-dimensional and $\varphi$ is a Bing map. And $d(h, \psi) = d(h, q \circ \varphi) \leq d(h, q \circ p \circ h) + d(q \circ p \circ h, q \circ \varphi) < \varepsilon + \varepsilon = 2\varepsilon$. So $Y$ is a free space.

Now, we will give the proof of Theorem 11.

Proof of Theorem 11. By Remark 14, it is sufficient to show that $Y$ is free. So we will show that $Y$ satisfies the condition of Lemma 16. Let $h \in C(X, Y)$
and $\epsilon > 0$. Since $Y$ is a 1-dimensional continuum, $Y$ can be written as $Y = \lim_{i} \{G_i, f_i\}_{i=1}^{\infty}$, where $G_i$ is a graph and $f_i : G_{i+1} \to G_i$ is surjective for $i = 1, 2, \ldots$ Since $Y$ is Peano continuum, there exists $\varepsilon_1 > 0$ such that if $x, y \in Y$ satisfy $d(x, y) < \varepsilon_1$, then there exists an arc $A$ in $Y$ such that $A$ contains $x$ and $y$ as its endpoints and $\text{diam}A < \varepsilon$. Let $\epsilon_2 = \min\{\varepsilon, \epsilon_1\}$. Take $i$ sufficient large so that the projection $p_i : Y \to G_i$ is an $\epsilon_2$-mapping. Since $p_i$ is a closed map to a compactum, there exists $\varepsilon_3 > 0$ such that if $B \subseteq G_i$ satisfies $\text{diam}B < \varepsilon_3$, then $\text{diam}p_i^{-1}(B) < \varepsilon_2$. Let $K$ be a subdivision of $G_i$ with mesh$K < \varepsilon_3$. Let $K^{(0)} = \{v_j\}_{j=1}^{m}$ and $K^{(i)} = \{I_{\ell}\}_{\ell=1}^{n}$. Take $a_j \in p^{-1}(v_j)$ for $j = 1, 2, \ldots$ Let $I_{\ell} \in K^{(i)}$ and let $v_{\ell_1}, v_{\ell_2}$ be endpoints of $I_{\ell}$. Since $\text{diam}I_{\ell} < \varepsilon_3$, it follows that $\text{diam}(p_{\ell}^{-1}(I_{\ell})) < \varepsilon_2$. Take $a_{\ell_1} \in p^{-1}(v_{\ell_1}) \cap \{a_j\}_{j=1}^{m}$ and $a_{\ell_2} \in p^{-1}(v_{\ell_2}) \cap \{a_j\}_{j=1}^{m}$. Since $d(a_{\ell_1}, a_{\ell_2}) < \varepsilon_2$, there exists an embedding $q_{t} : I_{t} \to Y$ such that $\text{diam}(q_{t}(I_{\ell})) < \varepsilon$, $q_{t}(p_{t}) = a_{t_{1}}$ and $q_{t}(p_{t}) = a_{t_{2}}$. Define $q_{i} : G_i \to Y$ by $q_{i}(x) = q_{t}(x)$ if $x \in I_{\ell}$ for $\ell = 1, 2, \ldots, n$. Then $d(id_{Y}, q_{t} \circ p_{t}) < 2\varepsilon$, and $|q_{t}^{-1}(y)| < \infty$ for each $y \in Y$. So $Y$ satisfies the condition of Lemma 16. This completes the proof.

**Remark 17** In the proof of Lemma 13, we used an idea of M. Levin (see the proof of [7, Theorem 1.8]). Also, we can prove Lemma 13 by using an idea of J. Krasinkiewicz [6, Lemma (5.2)] (compare the proof of Lemma 13 with the proofs of Lemmas 22, 23 and Theorem 24 in the next section).

## 4 Bing maps to polyhedra

In this section, by the method of J. Krasinkiewicz [6] we prove Theorem 24 and as an application of this theorem, we show the theorem of J. Song and E. D. Tymchatyn: the set of all Bing maps in $C(X, \mathcal{P})$ is a $G_{\delta}$-dense subset in $C(X, \mathcal{P})$, where $X$ is any compactum and $\mathcal{P}$ is any nondegenerate connected polyhedron. The next definition was given in [6].

**Definition 18** Let $X$ be a compactum and let $p \in C(X, \mathcal{I})$. We say that $X$ is fold$\text{ed relatively } p$ (folded rel $p$) if there exist closed subsets $F_0, F_{1/2}, F_1$ such that

1. $F_0 \cup F_{1/2} \cup F_1 = X$.
2. $F_0 \cap F_1 = \emptyset$.
3. $p^{-1}(0) \subset F_0$, $p^{-1}(1) \subset F_1$.
4. $F_0 \cap F_{1/2} \subset p^{-1}((1/2, 1])$, $F_{1/2} \cap F_1 \subset p^{-1}([0, 1/2)).$
A subset $X' \subset X$ is said to be folded rel $p$ if $X'$ is folded rel $p|X'$. A map $f$ from $X$ to a compactum $Y$ is said to be folded rel $p$ if $f^{-1}(y)$ is folded rel $p$ for each $y \in Y$.

**Lemma 19** (J. Krasinkiewicz [6]) Let $X$ be a compactum and let $Y$ be a space. Then for each $p \in C(X, I)$ we have:

1. If $X$ is folded rel $p$, then for each $q \in C(Y, X)$ $Y$ is folded rel $p \circ q$.

In particular every subset of $X$ is folded $p$.

2. If $F$ is a subset of $X$ folded rel $p$, then some neighborhood of $F$ in $X$ is folded rel $p$.

**Lemma 20** (J. Krasinkiewicz [6]) For each compactum $X$, there exists $\mathcal{P} = \{p_i\}_{i=1}^{\infty} \subset C(X, I)$ such that a closed subset $B \subset X$ is a Bing compactum if and only if $B$ is folded rel $p_i$ for each $i = 1, 2, \ldots$

**Lemma 21** (J. Krasinkiewicz [6]) Let $X$ be a compactum and let $Y$ be a space. Then for each $p \in C(X, I)$, the set $\{f \in C(X, Y) | f \text{ is folded rel } p\}$ is an open subset in $C(X, Y)$.

The next lemma is a key lemma in this paper. The proof is based on an idea of J. Krasinkiewicz [6, Lemma (5.2)].

**Lemma 22** Let $X$ be a compactum and let $A$ be a closed subset in $X$. Let $\varepsilon > 0$, $\sigma^n$ ($n \geq 1$) an $n$-dimensional simplex, and $p : X \to I$ a map. If $f : A \to \partial \sigma^n$ is a Bing map and $\tilde{f} : X \to \sigma^n$ is a map with $\tilde{f}|A = f$, then there exists a map $g : X \to \sigma^n$ such that $g|A = f$, $d(\tilde{f}, g) < \varepsilon$ and $g$ is folded rel $p$.

**Proof.** Let $\varepsilon > 0$. Let $f : A \to \partial \sigma^n$ be a Bing map and let $\tilde{f} : X \to \sigma^n$ be a map with $\tilde{f}|A = f$. Let $\varphi : \sigma^n \times I \to \sigma^n$ and $\psi : \sigma^n \times I \to I$ be projections. We may assume that $\tilde{f}$ satisfies $\tilde{f}^{-1}(\partial \sigma^n) = A$. Since $\tilde{f}$ is a closed map, by (2) of Lemma 19 there exists $V$ which is a family of open subsets in $\sigma^n$ such that $\partial \sigma^n \subset \bigcup V$ and $\tilde{f}^{-1}(V)$ is folded rel $p$ for each $V \in V$. Let $r = d(\partial \sigma^n, \sigma^n \setminus \bigcup V)$ and let $Z = \{y \in \sigma^n | d(y, \partial \sigma^n) \geq r/2\}$. There exists $\delta \geq 0$ such that if $B \subset \sigma^n$ satisfies $\text{diam} B < \delta$ and $B \cap \{y \in \sigma^n | d(y, \partial \sigma^n) = r/2\} \neq \emptyset$ then $B$ is contained in some member of $V$. Let $\mathcal{U} = \{U_i\}_{i=1}^{k}$ be a finite family of open $n$-discs in $\sigma^n$ such that $Z \subset \bigcup \mathcal{U}$ and mesh $\mathcal{U} < \min\{\delta/2, r/2, \varepsilon\}$. We can assume that no proper subfamily of $\mathcal{U}$ covers $Z$. Take $a_i \in U_i \setminus \bigcup_{j \neq i} U_j$ for $i = 1, 2, \ldots$. There exist compact sets
$Z_1, Z_2, \ldots, Z_k$ such that $\bigcup_{i=1}^k Z_i = Z$ and $Z_i \subset U_i$ for $i = 1, 2, \ldots, k$. For each $i = 1, 2, \ldots, k$ there exists a PL $n$-disc $D_i$ in $U_i$ such that $Z_i \subset \operatorname{int}D_i$. For each $i = 1, 2, \ldots, k$, there exists an open $n$-disc $G_i$ such that $D_i \cup \{a_i\} \subset G_i \subset \overline{\operatorname{cl}G_i} \subset U_i$. For each $i = 1, 2, \ldots, k$, there exists a neighborhood $W_i$ of $a_i$ such that $W_i \subset G_i \setminus \bigcup_{j \neq i} G_j$. Let $O_1, O_2, \ldots, O_k$ be pairwise disjoint open intervals in $(0, 1/2)$. For each $i = 1, 2, \ldots, k$, take $r_i, s_i, t_i \in O_i$ such that $r_i < s_i < t_i$. For each $i = 1, 2, \ldots, k$, there exists PL $(n+1)$-disc $E_i$ such that $(a_i, 3/4) \in \operatorname{int}E_i \subset E_i \subset G_i \times O_i \cup W_i \times (t_i, 1)$, $D_i \times [r_i, t_i] \cap E_i \subset \partial D_i \times [r_i, t_i]$ and $Q_i = D_i \times [r_i, t_i] \cup E_i$ is closed PL $(n+1)$-disc. Since $\{G_i \times O_i \cup W_i \times (t_i, 1)\}_{i=1}^k$ is pairwise disjoint and $Q_i \subset G_i \times O_i \cup W_i \times (t_i, 1)$ for $i = 1, 2, \ldots, k$, $Q_1, Q_2, \ldots, Q_k$ are pairwise disjoint. So there exists an isotopy $H_t : \sigma^n \times I \to \sigma^n \times I$ such that $H_t|((\sigma^n \times I \setminus \bigcup_{i=1}^k \operatorname{int}Q_i) = t \sigma^n \times I \setminus \bigcup_{i=1}^k \operatorname{int}Q_i$, $H_t|Q_i$ is a homeomorphism of $Q_i$ to itself and $H_t(Z_i \times \{s_i\}) \subset \psi^{-1}((1/2, 1))$ for $i = 1, 2, \ldots, k$. Let $g = \varphi \circ H_1^{-1} \circ (\tilde{f} \times p) : X \to \sigma^n$. Since $H_t|\partial\sigma^n \times I = \partial\sigma^n \times I$, $g|A = f$. Since mesh$U < \epsilon$, $d(\tilde{f}, g) < \epsilon$. Let $y \in \sigma^n$. Now we consider next three cases.

**Case 1.** If $y \in \sigma^n \setminus \bigcup \mathcal{U}$, then there exists $V \in \mathcal{V}$ such that $y \in V$. Since $g^{-1}(y) = \tilde{f}^{-1}(y) \subset \tilde{f}^{-1}(V)$, $g^{-1}(y)$ is folded rel $p$.

**Case 2.** Suppose that $y \in \bigcup \mathcal{U} \cap (\sigma^n \setminus Z)$. Let $U_1, U_2, \ldots, U_k$ be the all members of $\mathcal{U}$ which contain $y$. Let $U' = \bigcup_{i=1}^k U_i$. Since $U' \cap \{y \in \sigma^n|d(y, \partial\sigma^n) = r/2\} \neq \emptyset$ and diam$U' < \delta$, there exists $V' \in \mathcal{V}$ such that $U' \subset V'$. Then $g^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_I \circ \varphi^{-1}(y) \subset (\tilde{f} \times p)^{-1} \circ \varphi^{-1}(U') = \tilde{f}^{-1}(U') \subset \tilde{f}^{-1}(V')$. So $g^{-1}(y)$ is folded rel $p$.

**Case 3.** Case 3. If $y \in Z$, there exists $i = 1, 2, \ldots, k$ such that $y \in Z_i$. Since $H_I(\{y\} \times I) = H_I(\{y\} \times [0, s_i]) \cup H_I(\{y\} \times [s_i, t_i]) \cup H_I(\{y\} \times [t_i, 1])$, $H_I(\{y\} \times I)$ is folded rel $\psi$. Since $g^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_I \circ \varphi^{-1}(y) = (\tilde{f} \times p)^{-1} \circ H_I(\{y\} \times I)$ and by (1) of Lemma 19 $g^{-1}(y)$ is folded rel $\psi \circ (\tilde{f} \times p) = p$.

So $g$ is folded rel $p$. This completes the proof.

**Lemma 23** Let $X$ be a compactum and let $A$ be a closed subset in $X$. Let $\varepsilon > 0$, $\sigma^n$ $(n \geq 1)$ an $n$-dimensional simplex. If $f : A \to \partial\sigma^n$ is a Bing map and $\tilde{f} : X \to \sigma^n$ is a map with $\tilde{f}|A = f$, then there exists a Bing map $g : X \to \sigma^n$ such that $d(\tilde{f}, g) < \varepsilon$ and $g|A = f$. 

Proof. Set \( \mathcal{P} \) as in Lemma 20. Let \( C(X, f|A) = \{ g \in C(X, \sigma^n) | g|A = f \} \). For each \( p_i \in \mathcal{P} \), let \( C(X, f|A, p_i) = \{ g \in C(X, f|A) | g \text{ is folded rel } p_i \} \).

Let \( B(X, f|A) = \{ g \in C(X, f|A) | g \text{ is a Bing map} \} \). Since \( B(X, f|A) = \bigcap_{i=1}^{\infty} C(X, f|A, p_i) \), by Lemma 21, 22 and Baire theorem \( B(X, f|A) \) is dense in \( C(X, f|A) \). This completes the proof.

The following theorem is a more precise result than the theorem of J. Song and E. D. Tymchatyn.

**Theorem 24** (Extension Theorem of Bing Maps) Let \( X \) be a compactum and let \( A \) be a closed subset in \( X \). Let \( \mathcal{K} \) be a finite simplicial complex such that \( |\mathcal{K}| \) is a nondegenerate connected polyhedron and let \( \mathcal{L} \) be a subcomplex of \( \mathcal{K} \). If \( f : A \rightarrow |\mathcal{L}| \) is a Bing map and \( \tilde{f} : X \rightarrow |\mathcal{K}| \) is a map with \( \tilde{f}|A = f \) and \( \tilde{f}^{-1}(|\mathcal{L}|) = A \), then for any \( \epsilon > 0 \) there exists a Bing map \( g : X \rightarrow |\mathcal{K}| \) such that \( g|A = f \) and \( d(\tilde{f}, g) < \epsilon \).

Proof. First, we prove the following claim:

The set \( C_{v0}(X, |\mathcal{K}|) = \{ f \in C(X, |\mathcal{K}|) | f^{-1}(v) \) is a Bing compactum for each vertex \( v \in \mathcal{K}^0 \} \) is a \( G_\delta \)-dense subset of \( C(X, |\mathcal{K}|) \).

Let \( v = v_0 \in \mathcal{K}^0 \) and let \( p : X \rightarrow \mathbb{I} \) be a map. We shall prove that \( C_v(X, |\mathcal{K}|, p) = \{ f \in C(X, |\mathcal{K}|) | f^{-1}(v) \) is folded rel \( p \} \) is an open and dense subset of \( C(X, |\mathcal{K}|) \). We can easily see that \( C_v(X, |\mathcal{K}|, p) \) is an open set of \( C(X, |\mathcal{K}|) \). We prove that \( C_v(X, |\mathcal{K}|, p) \) is dense in \( C(X, |\mathcal{K}|) \).

Let \( \epsilon > 0 \) and \( 0 < \alpha < \beta < 1 \). Consider the star \( St(v, \mathcal{K}) = \bigcup \{ \sigma \in \mathcal{K} | v \in \sigma \} \) of \( \mathcal{K} \) with \( v \). For each simplex \( \sigma = [v_0, v_1, ..., v_m] \in \mathcal{K} \) \( (v_0 = v, m \geq 1) \), put

\[
\sigma_\alpha = \{ \Sigma_{i=0}^{m} t_i v_i | t_i \geq 0 (i = 0, 1, 2, .., m), \Sigma_{i=0}^{m} t_i = 1, t_0 \geq \alpha \}
\]

\[
\sigma_\beta = \{ \Sigma_{i=0}^{m} t_i v_i | t_i \geq 0 (i = 0, 1, 2, .., m), \Sigma_{i=0}^{m} t_i = 1, t_0 \geq \beta \}.
\]

Let

\[
M = \bigcup_{v \in \epsilon_{\mathcal{K}}} \sigma_\alpha, \quad N = \bigcup_{v \in \epsilon_{\mathcal{K}}} \sigma_\beta.
\]

Choose positive numbers \( s_0, s_1, \) and \( s_2 \) with \( 0 < s_0 < s_1 < 1/2 < s_2 < 1 \). Consider the following set

\[
Z = (M \times [s_0, s_1]) \cup (cl(M - N) \times [s_1, s_2]) \subset St(v, \mathcal{K}) \times [0, 1].
\]

For each \( (m-\text{dimensional}) \) simplex \( \sigma \) containing \( v \) \( (m \geq 1) \), put \( \sigma_Z = Z \cap (\sigma \times \mathbb{I}) \). Also, consider the following map \( \phi : Z \rightarrow T = ([0, 1-\alpha] \times [s_0, s_1]) \cup \).
$([\beta - \alpha, 1 - \alpha] \times [s_1, s_2]) \subset \mathbb{I} \times \mathbb{I}$ defined by $\phi(z) = (1 - t_0, t) = (\Sigma^m_{i=1} t_i, t)$ for $z = (x, t), \ t \in \mathbb{I}$ and $x = \Sigma^m_{i=0} t_i v_i \in [v_0, v_1, \ldots, v_m]$. By identifying each $z \in Z$ with $\phi(z) \in T$, we obtain the adjunction space $W = (St(v, \mathcal{K}) \times \mathbb{I}) \cup \Phi T$. Let $q : (St(v, \mathcal{K}) \times \mathbb{I}) \to W$ be the natural projection.

Let $D$ be a closed disk (=2-cell) and consider an embedding $u : \{v\} \times [s_0, s_1] \to \partial(D)$, where $\partial(D)$ is the manifold boundary of $D$. Also, consider the adjunction space $E = (St(v, \mathcal{K}) \times \mathbb{I}) \cup_u D$. Note that for each simplex $\sigma$ containing $v$, $cl(q(\sigma \times \mathbb{I}) - q(\sigma_Z))$ is naturally homeomorphic to $\sigma \times \mathbb{I}$ and $q(Z)$ is homeomorphic to $D$. Hence $W$ is homeomorphic to $E$. More precisely, for each simplex $\sigma = [v, v_1, \ldots, v_m] \in \mathcal{K}$ containing $v$ ($m \geq 1$) there is an embedding $h_{\sigma} : q(\sigma \times \mathbb{I}) \to E$ such that $h_{\sigma}(cl(q(\sigma \times \mathbb{I}) - q(\sigma_Z))) = \sigma \times \mathbb{I}$ and $h_{\sigma}(q(\sigma_Z)) = D$ and $h_{\sigma}(q(H_{\sigma})) = \{v\} \times \mathbb{I}$, where

$$H_{\sigma} = \{v\} \times ([0, s_0] \cup [s_1, s_2]) \cup (\sigma_{\alpha} \times \{s_0\}) \cup (H_{\beta} \times [s_0, s_2]) \cup (H_{\beta} \times [s_1, s_2]) \cup (cl(\sigma_{\alpha} - \sigma_{\beta}) \times \{s_2\}) \cup (\sigma_{\beta} \times \{s_1\}),$$

and

$$H_{\alpha} = \{\Sigma^m_{i=0} t_i v_i \mid t_i \geq 0 \ (i = 0, 1, 2, \ldots, m), \ \Sigma^m_{i=0} t_i = 1, \ t_0 = \alpha\},$$

$$H_{\beta} = \{\Sigma^m_{i=0} t_i v_i \mid t_i \geq 0 \ (i = 0, 1, 2, \ldots, m), \ \Sigma^m_{i=0} t_i = 1, \ t_0 = \beta\}.$$

Also, choose a map $u' : E = (St(v, \mathcal{K}) \times \mathbb{I}) \cup_u D \to St(v, \mathcal{K}) \times \mathbb{I}$ such that $u'|((St(v, \mathcal{K}) \times \mathbb{I}) \cup \Phi T) = id$ and $u'^{-1}(\{v\} \times \mathbb{I}) = \{v\} \times \mathbb{I}$. By using these maps, we can obtain a map $h : |\mathcal{K}| \times \mathbb{I} \to |\mathcal{K}| \times \mathbb{I}$ such that for each simplex $[v, v_1, \ldots, v_m] \in \mathcal{K}$ containing $v$, $h([v_1, \ldots, v_m]) = id$ and $h^{-1}(\{v\} \times \mathbb{I}) = \cup_{\sigma \in \mathcal{K}} H_{\sigma}$. Consider the map $g = \varphi \circ h \circ (f \times p) : X \to |\mathcal{K}|$, where $\varphi : |\mathcal{K}| \times \mathbb{I} \to |\mathcal{K}|$ is the natural projection. Since $H_{\sigma}$ is crooked with respect to $p$, we see that $g^{-1}(v)$ is folded rel $p$ (see the following figures below). Since we can choose a positive number $\alpha$ with $1 - \alpha < \epsilon$, we see that $d(f, g) < \epsilon$. Hence we see that $C_\epsilon(X, |\mathcal{K}|) = \{f \in C(X, |\mathcal{K}|) \mid f^{-1}(v) \text{ is a Bing compactum} \}$ is a $G_\delta$-dense subset of $C(X, |\mathcal{K}|)$. Then

$$C_{\mathcal{K}\mathcal{V}}(X, |\mathcal{K}|) = \cap_{\nu \in \mathcal{K}\mathcal{V}} C_\epsilon(X, |\mathcal{K}|)$$

is a $G_\delta$-dense subset of $C(X, |\mathcal{K}|)$. Hence the claim is true.
Let $\dim|\mathcal{K}| = n$. For each $j = 0, 1, \ldots, n$, let $A_j = |L| \cup |\mathcal{K}(j)|$. By the claim, we may assume that $\bar{f}f^{-1}(A_0) : \bar{f}f^{-1}(A_0) \to A_0$ is a Bing map. Put $\bar{g}_0 = \bar{f}$. Note that for each simplex $\sigma \in \mathcal{K}$, the boundary $\partial \sigma$ is a $Z$-set of $\sigma$. By Lemma 23, we have a Bing map $g_1 : \bar{g}_0^{-1}(A_1) \to A_1$ such that $g_1|\bar{g}_0^{-1}(A_0) = \bar{g}_0|\bar{g}_0^{-1}(A_0)$. By the homotopy extension theorem, we may assume that there is a map $\tilde{g}_1 : X \to |K|$ such that $\tilde{g}_1$ is an extension of $g_1$, and $\tilde{g}_1^{-1}(A_1) = \bar{g}_0^{-1}(A_1)$. If we continue this process, we have a Bing map $g = g_n : X \to |K|$ such that $g|A = f$ and $d(\tilde{f}, g) < \epsilon$. This completes the proof.

The next result is the theorem of J. Song and E. D. Tymchatyn.

**Corollary 25** (J. Song and E. D. Tymchatyn [9]) Let $X$ be a compactum and let $P$ be an $n$-dimensional connected polyhedron $(n \geq 1)$. Then the set of all Bing maps in $C(X, P)$ is a $G_{\delta}$-dense subset in $C(X, P)$.

**Proof.** If we put $A = |L| = \phi$ in Theorem 24, we obtain this theorem.

**Corollary 26** (J. Song and E. D. Tymchatyn [9]) Let $M$ be a Menger manifold with $\dim M \geq 1$. Then the set of all Bing maps in $C(X, M)$ is a $G_{\delta}$-dense subset in $C(X, M)$ (see [1] for properties of Menger manifolds).

**Proof.** We only prove that $M$ is free. Let $\epsilon > 0$. There exists a non-degenerate connected polyhedron $P \subset M$ and map $p : M \to P$ such that
$d(x, p(x)) < \epsilon$ for each $x \in M$ (see [1]). Let $q : P \rightarrow M$ be a natural embedding. Then $q$ is 0-dimensional and $d(q \circ p, id_M) < \epsilon$. By Lemma 16 and Corollary 25, $M$ is free. This completes the proof.

References


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