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On splitting theorems for CAT(0) spaces

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The purpose of this note is to introduce main results of my recent paper [7] about splitting theorems for CAT(0) spaces.

We say that a metric space $X$ is a geodesic space if for each $x, y \in X$, there exists an isometry $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such $\xi$ is called a geodesic). Also a metric space $X$ is said to be proper if every closed metric ball is compact.

Let $X$ be a geodesic space and let $T$ be a geodesic triangle in $X$. A comparison triangle for $T$ is a geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbb{R}^2$ with same edge lengths as $T$. Choose two points $x$ and $y$ in $T$. Let $\overline{x}$ and $\overline{y}$ denote the corresponding points in $\overline{T}$. Then the inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$$

is called the $\text{CAT}(0)$-inequality, where $d_{\mathbb{R}^2}$ is the natural metric on $\mathbb{R}^2$. A geodesic space $X$ is called a $\text{CAT}(0)$ space if the $\text{CAT}(0)$-inequality holds for all geodesic triangles $T$ and for all choices of two points $x$ and $y$ in $T$.

A proper CAT(0) space $X$ can be compactified by adding its ideal boundary $\partial X$, and $X \cup \partial X$ is a metrizable compactification of $X$ ([2], [4]).

A geometric action on a CAT(0) space is an action by isometries which is proper ([2, p.131]) and cocompact. We note that every CAT(0) space on which some group acts geometrically is a proper space ([2,
Details of CAT(0) spaces and their boundaries are found in [2] and [4].

In [7], we first proved the following splitting theorem which is an extension of Proposition II.6.3 in [2].

**Theorem 1.** Suppose that a group \( \Gamma = \Gamma_1 \times \Gamma_2 \) acts geometrically on a CAT(0) space \( X \). If \( \Gamma_1 \) acts cocompactly on the convex hull \( C(\Gamma_1 x_0) \) of some \( \Gamma_1 \)-orbit, then there exists a closed, convex, \( \Gamma \)-invariant, quasi-dense subspace \( X' \subset X \) such that \( X' \) splits as a product \( X_1 \times X_2 \) and there exist geometric actions of \( \Gamma_1 \) and \( \Gamma_2 \) on \( X_1 \) and \( X_2 \), respectively. Here each subspace of the form \( X_1 \times \{x_2\} \) is the closed convex hull of some \( \Gamma_1 \)-orbit.

Using this theorem, we also proved the following splitting theorem which is an extension of Theorem II.6.21 in [2].

**Theorem 2.** Suppose that a group \( \Gamma = \Gamma_1 \times \Gamma_2 \) acts geometrically on a CAT(0) space \( X \). If the center of \( \Gamma \) is finite, then there exists a closed, convex, \( \Gamma \)-invariant, quasi-dense subspace \( X' \subset X \) such that \( X' \) splits as a product \( X_1 \times X_2 \) and the action of \( \Gamma = \Gamma_1 \times \Gamma_2 \) on \( X' = X_1 \times X_2 \) is the product action.

We also showed the following splitting theorem in more general case.

**Theorem 3.** Suppose that a group \( \Gamma = \Gamma_1 \times \Gamma_2 \) acts geometrically on a CAT(0) space \( X \). Then there exist closed convex subspaces \( X_1, X_2, X_1', X_2' \) in \( X \) such that

1. \( X_1 \times X_2' \) and \( X_1' \times X_2 \) are quasi-dense subspaces of \( X \),
2. \( X_1' \) and \( X_2' \) are quasi-dense subspaces of \( X_1 \) and \( X_2 \), respectively,
3. \( \Gamma_1 \) and \( \Gamma_2 \) act geometrically on \( X_1 \) and \( X_2 \) respectively, and
4. some subgroups of finite index in \( \Gamma_1 \) and \( \Gamma_2 \) act geometrically on \( X_1' \) and \( X_2' \) respectively.

A CAT(0) space \( X \) is said to have the geodesic extension property if every geodesic can be extended to a geodesic line \( \mathbb{R} \to X \). Concerning
CAT(0) spaces with the geodesic extension property, we obtained the following theorem as an application of the above splitting theorems.

**Theorem 4.** Suppose that a group $\Gamma = \Gamma_1 \times \Gamma_2$ acts geometrically on a CAT(0) space $X$ with the geodesic extension property. Then $X$ splits as a product $X_1 \times X_2$ and there exist geometric actions of $\Gamma_1$ and $\Gamma_2$ on $X_1$ and $X_2$, respectively. Moreover if $\Gamma$ has finite center, then $\Gamma$ preserves the splitting, i.e., the action of $\Gamma = \Gamma_1 \times \Gamma_2$ on $X = X_1 \times X_2$ is the product action.

Let $Y$ be a compact geodesic space of non-positive curvature. Then the universal covering $X$ of $Y$ is a CAT(0) space by the Cartan-Hadamard theorem (cf. [2, p.193, p.237]), and we can think of $Y$ as the quotient $\Gamma \backslash X$ of $X$, where $\Gamma$ is the fundamental group of $Y$ acting freely and properly by isometries on $X$. As an application of Theorem 2, we showed the following splitting theorem which is an extension of Corollary II.6.22 in [2].

**Theorem 5.** Let $Y$ be a compact geodesic space of non-positive curvature. Suppose that the fundamental group of $Y$ splits as a product $\Gamma = \Gamma_1 \times \Gamma_2$ and that $\Gamma$ has trivial center. Then there exists a deformation retract $Y'$ of $Y$ which splits as a product $Y_1 \times Y_2$ such that the fundamental group of $Y_i$ is $\Gamma_i$ for each $i = 1, 2$.

A group $\Gamma$ is called a **CAT(0) group**, if $\Gamma$ acts geometrically on some CAT(0) space. Theorem 3 implies the following.

**Theorem 6.** $\Gamma_1$ and $\Gamma_2$ are CAT(0) groups if and only if $\Gamma_1 \times \Gamma_2$ is a CAT(0) group.

In [3], Croke and Kleiner proved that there exists a CAT(0) group $\Gamma$ and CAT(0) spaces $X$ and $Y$ such that $\Gamma$ acts geometrically on $X$ and $Y$ and the boundaries of $X$ and $Y$ are not homeomorphic. A CAT(0) group $\Gamma$ is said to be **rigid**, if $\Gamma$ determines the boundary up to homeomorphism of a CAT(0) space on which $\Gamma$ acts geometrically. Then we denote $\partial \Gamma$ as the boundary of the rigid CAT(0) group $\Gamma$. 
A conclusion in [1] implies that if $\Gamma$ is a rigid CAT(0) group, then $\Gamma \times \mathbb{Z}^n$ is also a rigid CAT(0) group for each $n \in \mathbb{N}$. In [9], Ruane proved that if $\Gamma_1 \times \Gamma_2$ is a CAT(0) group and if $\Gamma_1$ and $\Gamma_2$ are hyperbolic groups (in the sense of Gromov) then $\Gamma_1 \times \Gamma_2$ is rigid. Concerning rigidity of products of rigid CAT(0) groups, we can obtain the following theorem from Theorem 3 which is an extension of these results.

**Theorem 7.** If $\Gamma_1$ and $\Gamma_2$ are rigid CAT(0) groups, then so is $\Gamma_1 \times \Gamma_2$, and the boundary $\partial(\Gamma_1 \times \Gamma_2)$ is homeomorphic to the join $\partial \Gamma_1 * \partial \Gamma_2$ of the boundaries of $\Gamma_1$ and $\Gamma_2$.

**REFERENCES**


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