The behaviour of dimension functions on unions of closed subsets I

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1 Introduction

All spaces we shall consider here are separable metrizable spaces.

It is well known that there exist (transfinite) dimension functions \( d \) such that \( d(X_1 \cup X_2) > \max\{dX_1, dX_2\} \) even if the subspaces \( X_1 \) and \( X_2 \) are closed in the union \( X_1 \cup X_2 \).

Let \( \mathcal{K} \) be a class of spaces, \( \beta, \alpha \) be ordinals such that \( \beta < \alpha \), and \( X \) be a space from \( \mathcal{K} \) with \( dX = \alpha \) which is the union of finitely many closed subsets with \( d \leq \beta \). Define \( m(X, d, \beta, \alpha) = \min\{k : X = \bigcup_{i=1}^{k} X_i, \text{ where } X_i \text{ is closed in } X \text{ and } dX_i \leq \beta\} \), \( m_\mathcal{K}(d, \beta, \alpha) = \min\{m(X, d, \beta, \alpha) : X \in \mathcal{K} \text{ and } m(X, d, \beta, \alpha) \text{ exists}\} \) and \( M_\mathcal{K}(d, \beta, \alpha) = \sup\{m(X, d, \beta, \alpha) : X \in \mathcal{K} \text{ and } m(X, d, \beta, \alpha) \text{ exists}\} \).

We will say that \( m_\mathcal{K}(d, \beta, \alpha) \) and \( M_\mathcal{K}(d, \beta, \alpha) \) do not exist if there is no space \( X \) from \( \mathcal{K} \) with \( dX = \alpha \) which is the union of finitely many closed subsets with \( d \leq \beta \). It is evident that either \( m_\mathcal{K}(d, \beta, \alpha) \) and \( M_\mathcal{K}(d, \beta, \alpha) \) satisfy \( 2 \leq m_\mathcal{K}(d, \beta, \alpha) \leq M_\mathcal{K}(d, \beta, \alpha) \leq \infty \) or they do not exist.

Two natural questions arise.

**Question 1.1** Determine the values of \( m_\mathcal{K}(d, \beta, \alpha) \) and \( M_\mathcal{K}(d, \beta, \alpha) \) for given \( \mathcal{K}, d, \beta, \alpha \).

**Question 1.2** Find a (transfinite) dimension function \( d \) having for given pair \( 2 \leq k \leq l \leq \infty \), \( m_\mathcal{K}(d, \beta, \alpha) = k \) and \( M_\mathcal{K}(d, \beta, \alpha) = l \).

Let \( \mathcal{C} \) be the class of metrizable compact spaces and \( \mathcal{P} \) be the class of separable completely metrizable spaces. By \( \text{trind(trInd)} \) we denote Hurewicz’s (Smirnov’s) transfinite extension of \( \text{ind (Ind)} \) and \( \text{Cmp} \) is the large inductive compactness degree introduced by de Groot. We shall recall their definitions in the next section. Let \( \alpha = \lambda(\alpha) + n(\alpha) \) be the natural
decomposition of the ordinal $\alpha \geq 0$ into the sum of a limit number $\lambda(\alpha)$ (observe that $\lambda(\text{an integer } \geq 0) = 0$) and a nonnegative integer $n(\alpha)$. Let $\beta < \alpha$ be ordinals, put $p(\beta, \alpha) = \left(\frac{n(\alpha)+1}{n(\beta)+1}\right)$ and $q(\beta, \alpha) = \text{the smallest integer } \geq p(\beta, \alpha)$. We have the following theorems. The outline of the proof will be presented in section 2.

Theorem 1.1 1. Let $0 \leq \beta < \alpha$ be finite ordinals. Then we have $m_{\beta, \alpha}(\text{Cmp }, \beta, \alpha) = q(\beta, \alpha)$ and $M_{\beta, \alpha}(\text{Cmp }, \beta, \alpha) = \infty$.  
2. Let $\beta < \alpha$ be finite ordinals. Then we have

$$m_{\beta, \alpha}(\text{trInd}, \beta, \alpha) = \begin{cases} q(\beta, \alpha), & \text{if } \lambda(\beta) = \lambda(\alpha), \\
does\ not\ exist, & \text{otherwise} \\
\infty, & \text{if } \lambda(\beta) = \lambda(\alpha), \\
does\ not\ exist, & \text{otherwise} \end{cases}$$

$$M_{\beta, \alpha}(\text{trInd}, \beta, \alpha) = \begin{cases} \infty, & \text{if } \lambda(\beta) = \lambda(\alpha), \\
does\ not\ exist, & \text{otherwise} \end{cases}$$

Theorem 1.2 1. For every finite $\alpha \geq 1$ there exists a space $X_{\alpha} \in \mathcal{P}$ such that
(a) $\text{Cmp}X_{\alpha} = \alpha$;
(b) $X_{\alpha} = \bigcup_{i=1}^{\infty} Y_{i}$, where each $Y_{i}$ is closed in $X_{\alpha}$ and $\text{Cmp}Y_{i} \leq 0$;
(c) $X_{\alpha} \neq \bigcup_{i=1}^{m} Z_{i}$, where each $Z_{i}$ is closedin $X_{\alpha}$ and $\text{Cmp}Z_{i} \leq \alpha - 1$ and $m$ is any integer $\geq 1$.

2. For every infinite $\alpha$ with $n(\alpha) \geq 1$ there exists a space $X_{\alpha} \in \mathcal{C}$ such that
(a) $\text{trInd}X_{\alpha} = \alpha$;
(b) $X_{\alpha} = \bigcup_{i=1}^{\infty} Y_{i}$, where each $Y_{i}$ is closed in $X_{\alpha}$ and finite-dimensional;
(c) $X_{\alpha} \neq \bigcup_{i=1}^{m} Z_{i}$, where each $Z_{i}$ is closed in $X_{\alpha}$ and $\text{trInd}Z_{i} \leq \alpha - 1$ and $m$ is any integer $\geq 1$.

2 Evaluations of $m_{\mathbb{K}}(d, \beta, \alpha)$ and $M_{\mathbb{K}}(d, \beta, \alpha)$

The notation $X \sim Y$ means that the spaces $X$ and $Y$ are homeomorphic. At first we consider the following construction.

**Step 1.** Let $X$ be a space without isolated points and $P$ a countable dense subset of $X$. Consider Alexandroff's dublicate $D = X \cup X^{1}$ of $X$, where each point of $X^{1}$ is clopen in $D$. Remove from $D$ those points of $X^{1}$ which do not correspond to any point from $P$. Denote the obtained space by $L(X, P)$. Observe that $L(X, P)$ is the disjoint union of $X$ with the countable dense subset $P^{1}$ of $L(X, P)$ consisting of points from $X^{1}$ corresponding to the points from $P$. The space $L(X, P)$ is separable and metrizable. It will be compact if $X$ is compact. Put $L_{1}(X, P) = L(X, P)$. Assume that $X$ is a completely metrizable space (recall that the increment $bX \setminus X$ in any compactification $bX$ of $X$ is an $F_{\sigma}$-set in $bX$). Observe that $L(bX, P)$ is a compactification of $L(X, P)$ and the increment
$L(bX, P) \setminus L(X, P) (\sim bX \setminus X)$ is an $F_\sigma$-set in $L(bX, P)$. Hence $L(X, P)$ is also completely metrizable.

**Step 2.** Let $X$ be a space with a countable subset $R$ consisting of isolated points. Let $Y$ be a space. Substitute each point of $R$ in $X$ by a copy of $Y$. The obtained set $W$ has the natural projection $pr: W \to X$. Define the topology on $W$ as the smallest topology such that the projection $pr$ is continuous and each copy of $Y$ has its original topology as a subspace of this new space. The obtained space is denoted by $L(X, R, Y)$. It is separable and metrizable and it will be compact (completely metrizable) if $X$ and $Y$ are the same. Moreover $L(X, R, Y)$ is the disjoint union of the closed subspace $X \setminus R$ of $X$ (which we will call basic for the space $L(X, R, Y)$) and countably many clopen copies of $Y$.

**Step 3.** Let $X$ be a space without isolated points and $P$ be a countable dense subset of $X$. Define $L_n(X, P) = L(L_1(X, P), P^1, L_{n-1}(X, P)), n \geq 2$. Observe that for any open subset $O$ of $L_n(X, P)$ meeting the basic subset $X$ of $L_n(X, P)$ there is a copy of $L_{n-1}(X, P)$ contained in $O$. Put $L_*(X, P) = \{\ast\} \cup \bigoplus_{n=1}^{\infty} L_n(X, P)$ (Here by $\{\ast\} \cup \bigoplus_{i=1}^{\infty} X_i$ we mean the one-point extension of the free union $\bigoplus_{i=1}^{\infty} X_i$ such that a neighborhood base at the point $\ast$ consists of the sets $\{\ast\} \cup \bigoplus_{i=1}^{\infty} X_i, k = 1, 2, \ldots$). Observe that $L_*(X, P)$ is separable and metrizable, and it contains a copy of $L_q(X, P)$ for each $q$. $L_*(X, P)$ will be compact (completely metrizable) if $X$ is the same.

All our dimension functions $d$ are assumed to be monotone with respect to closed subsets and $d(\text{a point}) \leq 0$.

**Lemma 2.1** Let $d$ be a dimension function and $X$ be a space without isolated points which cannot be written as the union of $k \geq 1$ closed subsets with $d \leq \alpha$, where $\alpha$ is an ordinal. Let also $P$ be a countable dense subset of $X$. Then

(a) for every $q$ we have $L_q(X, P) \neq \bigcup_{i=1}^{k} X_i$, where each $X_i$ is closed in $L_q(X, P)$ and $dX_i \leq \alpha$;

(b) $L_*(X, P) \neq \bigcup_{i=1}^{m} X_i$, where each $X_i$ is closed in $L_*(X, P)$ and $dX_i \leq \alpha$, and $m$ is any integer $\geq 1$.

All our classes $\mathcal{K}$ of topological spaces are assumed to be monotone with respect to closed subsets and closed under operations $L(,)$ and $L(,,)$.

**Lemma 2.2** Let $\mathcal{K}$ be a class of topological spaces, $\alpha$ be an ordinal $\geq 0$ and $d$ be a dimension function such that $dL(L(S, P), P^1, T) \leq \alpha$ for any $S, T$ from $\mathcal{K}$ with $dS \leq \alpha$, $dT \leq \alpha$ and any $P$. Let $X \in \mathcal{K}$ such that $X = \bigcup_{i=1}^{k} X_i$, where each $X_i$ is closed in $X$, without isolated points and $dX_i \leq \alpha$. Let also $P_i$ be a countable dense subset of $X_i$ for each $i$. Then for each $q$ the space $L_q(X, \bigcup_{i=1}^{k} P_i)$ exists and is the union of $k^q$ closed subsets with $d \leq \alpha$. 
We will say that a dimension function \( d \) satisfies the sum theorem of type \( A \) if for any \( X \) being the union of two closed subspaces \( X_1 \) and \( X_2 \) with \( dX_i \leq \alpha_i \), where each \( \alpha_i \) is finite and \( \geq 0 \), we have \( dX \leq \alpha_1 + \alpha_2 + 1 \). A space \( X \) is completely decomposable in the sense of the dimension function \( d \) if \( dX = \alpha \), where \( \alpha \) is an integer \( \geq 1 \), and \( X = \bigcup_{i=1}^{\alpha+1} X_i \), where each \( X_i \) is closed in \( X \) and \( dX_i = 0 \). Observe that if this space \( X \) belongs to a class \( \mathcal{K} \) of topological spaces then \( m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq \alpha + 1 \) for each \( \beta \) with \( 0 \leq \beta < \alpha \).

We will say that a transfinite dimension function \( d \) satisfies the sum theorem of type \( A_{tr} \) if for any \( X \) being the union of two closed subspaces \( X_1 \) and \( X_2 \) with \( dX_i \leq \alpha_i \) and \( \alpha_2 \geq \alpha_1 \) we have \( dX \leq \alpha_2 \), if \( \lambda(\alpha_1) < \lambda(\alpha_2) \), and \( dX \leq \alpha_2 + n(\alpha_1) + 1 \), if \( \lambda(\alpha_1) = \lambda(\alpha_2) \). A space \( X \) is completely decomposable in the sense of the transfinite dimension function \( d \) if \( dX = \alpha \), where \( \alpha \) is an infinite ordinal with \( n(\alpha) \geq 1 \), and \( X = \bigcup_{i=1}^{\alpha+1} X_i \), where each \( X_i \) is closed in \( X \) and \( dX_i = \lambda(\alpha) \). Observe that if this space \( X \) belongs to a class \( \mathcal{K} \) of topological spaces then \( m_{\mathcal{K}}(d, \beta, \alpha) \leq m(X, d, \beta, \alpha) \leq n(\alpha) + 1 \) for each \( \beta \) with \( \lambda(\alpha) \leq \beta < \alpha \).

To every space \( X \) one assigns the large inductive compactness degree \( \text{Cmp} \) as follows.

(i) \( \text{Cmp} X = -1 \) iff \( X \) is compact;
(ii) \( \text{Cmp} X = 0 \) iff there is a base \( B \) for the open sets of \( X \) such that the boundary \( \text{Bd} U \) is compact for each \( U \) in \( B \);
(iii) \( \text{Cmp} X \leq \alpha \), where \( \alpha \) is an integer \( \geq 1 \), if for each pair of disjoint closed subsets \( A \) and \( B \) of \( X \) there exists a partition \( C \) between \( A \) and \( B \) in \( X \) such that \( \text{Cmp} C \leq \alpha - 1 \);
(iv) \( \text{Cmp} X = \alpha \) if \( \text{Cmp} X \leq \alpha \) and \( \text{Cmp} X > \alpha - 1 \);
(v) \( \text{Cmp} X = \infty \) if \( \text{Cmp} X > \alpha \) for every positive integer \( \alpha \).

Recall also the definitions of the transfinite inductive dimensions \( \text{trInd} \) and \( \text{trInd} \).

(i) \( \text{trInd} X = -1 \) iff \( X = \emptyset \);
(ii) \( \text{trInd} X \leq \alpha \), where \( \alpha \) is an ordinal \( \geq 0 \), if for each pair of disjoint closed subsets \( A \) and \( B \) of \( X \) there exists a partition \( C \) between \( A \) and \( B \) in \( X \) such that \( \text{trInd} C \leq \alpha \);
(iii) \( \text{trInd} X = \alpha \) if \( \text{trInd} X \leq \alpha \) and \( \text{trInd} X \leq \beta \) holds for no \( \beta < \alpha \);
(iv) \( \text{trInd} X = \infty \) if \( \text{trInd} X \leq \alpha \) holds for no ordinal \( \alpha \).

The definition of \( \text{trInd} \) is obtained by replacing the set \( A \) in (ii) with a point of \( X \).

**Remark 2.1** (i) Note that \( \text{Cmp} \) satisfies the sum theorem of type \( A \) ([ChH, Theorem 2.2]) and for each integer \( \alpha \geq 1 \) there exists a separable completely metrizable space \( C_{\alpha} \) with \( \text{Cmp} C_{\alpha} = \alpha \) which is completely decomposable in the sense of \( \text{Cmp} \) ([ChH, Theorem 3.1]).

For the convenience of the reader, we recall that \( C_{\alpha} = \{0\} \times ([0,1]^{\alpha} \setminus [0,1)^{\alpha}) \cup \bigcup_{i=1}^{\infty} \{x_i\} \times [0,1]^{\alpha} \subset I^{\alpha+1} \), where \( \{x_i\}_{i=1}^{\infty} \) is a sequence of real numbers such that \( 0 < x_{i+1} < x_i \leq 1 \) for all \( i \) and \( \lim_{i \to \infty} x_i = 0 \). Note that the closed subsets in the decomposition of \( C_{\alpha} \) can be assumed without isolated points.

(ii) Note also that \( \text{trInd} \) satisfies the sum theorem of type \( A_{tr} \) ([E, Theorem 7.2.7]) and for each infinite ordinal \( \alpha \) with \( n(\alpha) \geq 1 \) there exists a metrizable compact space \( S^{\alpha} \) (Smirnov's
compactum) with $\text{trInd} S^\alpha = \alpha$ which is completely decomposable in the sense of $\text{trInd}$ ([Ch, Lemma 3.5]). Recall that Smirnov’s compacta $S^0, S^1, ..., S^\alpha, ..., \alpha < \omega_1$, are defined by transfinite induction: $S^0$ is the one-point space, $S^\alpha = S^3 \times [0, 1]$ for $\alpha = \beta + 1$, and if $\alpha$ is a limit ordinal, then $S^\alpha = \{ \alpha \} \cup \bigcup_{\beta < \alpha} S^\beta$ is the one-point compactification of the free union of all the previously defined $S^\beta$’s, where $\alpha$ is the compactifying point. Note that the closed subsets in the decomposition of $S^\alpha$ can be assumed without isolated points.

(iii) Observe that $\text{trind}$ satisfies another sum theorem. Namely, for any $X$ being the union of two closed subspaces $X_1$ and $X_2$ with $\text{trind} X_i \leq \alpha_i$ and $\alpha_2 \geq \alpha_1$ we have $\text{trind} X \leq \alpha_2$, if $\lambda(\alpha_1) < \lambda(\alpha_2)$, and $\text{trind} X \leq \alpha_2 + 1$, if $\lambda(\alpha_1) = \lambda(\alpha_2)$ [Ch, Theorem 3.9].

**Proposition 2.1** (i) Let $K$ be a class of topological spaces, $d$ be a dimension function satisfying the sum theorem of type $A$, $\alpha$ be an integer $\geq 1$ and $X$ be a space from $K$ with $dX = \alpha$ which is completely decomposable in the sense of $d$. Then for any integer $0 \leq \beta < \alpha$ we have $m_K(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha)$.

(ii) Let $K$ be a class of topological spaces, $d$ be a transfinite dimension function satisfying the sum theorem of type $A_{tr}$, $\alpha$ be an infinite ordinal with $n(\alpha) \geq 1$ and $X$ be a space from $K$ with $dX = \alpha$ which is completely decomposable in the sense of $d$. Then for any infinite ordinal $\beta < \alpha$ we have $m_K(d, \beta, \alpha) = m(X, d, \beta, \alpha) = q(\beta, \alpha)$ if $\lambda(\beta) = \lambda(\alpha)$ and $m_K(d, \beta, \alpha)$ does not exist otherwise.

The deficiency def is defined in the following way: For a space $X$,

$$\text{def } X = \min \{ \dim (Y \setminus X) : Y \text{ is a metrizable compactification of } X \}.$$  

Recall that $\text{Cmp } X \leq \text{def } X$ and $\text{def } X = 0$ iff $\text{Cmp } X = 0$.

**Lemma 2.3** (i) def $L(L(X, P), P^1, Y) = \max \{ \text{def } X, \text{def } Y \}$ for any $X, P, Y$. In particular, we have def $L(L(X, P), P^1, Y) \leq 0$ if $\text{Cmp } X \leq 0$ and $\text{Cmp } Y \leq 0$.

(ii) $\text{trInd} L(L(X, P), P^1, Y) = \max \{ \text{trInd } X, \text{trInd } Y \}$ for any compacta $X, Y$ and any $P$.

**Proof.** (i) Let $bX$ and $bY$ be metrizable compactifications of $X$ and $Y$ respectively such that $\dim (bX \setminus X) = \text{def } X$ and $\dim (bY \setminus Y) = \text{def } Y$. Observe that the space $L(L(bX, P), P^1, bY)$ is a compactification of $L(L(X, P), P^1, Y)$ and the increment $Z = L(L(bX, P), P^1, bY) \setminus L(L(X, P), P^1, Y)$ is the union of countably many closed subsets, one of which is homeomorphic to $bX \setminus X$ and the others are homeomorphic to $bY \setminus Y$. So by the countable sum theorem for $\dim$ we get that $\dim Z = \max \{ \dim (bX \setminus X), \dim (bY \setminus Y) \} = \max \{ \text{def } X, \text{def } Y \}$. Hence $\text{def } L(L(X, P), P^1, Y) \leq \max \{ \text{def } X, \text{def } Y \}$, thereby $\text{def } L(L(X, P), P^1, Y) = \max \{ \text{def } X, \text{def } Y \}$.

(ii) At first let us prove the statement when $Y$ is a singleton. Observe that in this case $L(L(X, P), P^1, Y) = L(X, P)$. Consider two disjoint closed subsets $A$ and $B$ of $L(X, P)$.
Recall that $L(X, P)$ contains a copy of $X$. Choose a partition $C$ between $A \cap X$ and $B \cap X$ in $X$. Extend the partition to a partition $C_1$ between $A$ and $B$ in $L(X, P)$. Consider another partition $C_2$ between $A$ and $B$ in $L(X, P)$ which is "thin" (i.e. $\text{Int}_{L(X, P)} C_2 = \emptyset$) and is in $C_1$. Observe that $C_2 \subset C$. Hence $\text{trInd} L(X, P) = \text{trInd} X$.

Now let us consider the general case. Assume that $A$ and $B$ are disjoint closed subsets in $L(L(X, P), P^1, Y)$. Recall that there is the natural continuous projection $pr : L(L(X, P), P^1, Y) \to L(X, P)$. Consider the closed subsets $pr A$ and $pr B$ of $L(X, P)$. If they are disjoint, choose a partition $C_2$ between $pr A$ and $pr B$ in $L(X, P)$ like in the previous part. Observe that $pr^{-1} C_2$ is a partition between $A$ and $B$ in $L(L(X, P), P^1, Y)$ such that $pr^{-1} C_2$ is homeomorphic to a closed subset of $C$. Assume now that $pr A \cap pr B \neq \emptyset$. Note that $Q^1 = pr A \cap pr B$ is finite and $L(L(X, P), P^1, Y)$ is the free union of $L(L(X, (P \setminus Q)), P^1 \setminus Q^1, Y)$, where $Q$ is the finite subset of $P$ corresponding to $Q^1$ and finitely many copies of $Y$. Choose a partition between $A$ and $B$ in $X$ and a partition between $A$ and $B$ in each of the copies of $Y$ corresponding to points of $Q$. It follows from the foregoing discussion that the free union of these partitions constitutes a partition in $L(L(X, P), P^1, Y)$ between $A$ and $B$. We conclude that $\text{trInd} L(L(X, P), P^1, Y) = \max\{\text{trInd} X, \text{trInd} Y\}$. □

**Proof of Theorem 1.1.**

(i) Because of Remark 2.1 and Proposition 2.1, we need only establish that $M_{\mathcal{P}}(\text{Cmp}, \beta, \alpha) = \infty$. Consider the space $C_\alpha = \bigcup_{i=1}^{\alpha+1} X_i$, where each $X_i$ is closed in $X$, without isolated points and $\text{Cmp} X_i = 0$, from Remark 2.1. Let $P_\alpha$ be a countable dense subset of $X_i$. Put $P = \bigcup_{i=1}^{\alpha+1} P_i$. Recall that def $C_\alpha = \alpha$ ([ChH, Theorem 3.1]). So by Lemma 2.3 for any integer $q$ we have def $L_q(C_\alpha, P) = \alpha$ and hence $\text{Cmp} L_q(C_\alpha, P) = \alpha$. Observe that by Lemmas 2.2 and 2.3, we get that the completely metrizable space $L_q(C_\alpha, P)$ is the union of $(\alpha+1)^q$ many closed subspaces with $\text{Cmp} \leq 0$. Hence $m(L_q(C_\alpha, P), \text{Cmp}, \beta, \alpha) \leq (\alpha+1)^q$. Since $\text{Cmp}$ satisfies the sum theorem of type $A$, $C_\alpha$ cannot be represented as $\alpha$-many closed subsets with $\text{Cmp} \leq 0$. By Lemma 2.1, we have $m(L_q(C_\alpha, P), \text{Cmp}, \beta, \alpha) \geq q \alpha \geq q$. Since $\lim_{q \to \infty} q = \infty$ we get $M_{\mathcal{P}}(\text{Cmp}, \beta, \alpha) = \infty$.

(ii) By similar arguments as in the proof of (i) one can prove $M_{\mathcal{C}}(\text{trInd}, \beta, \alpha) = \infty$, if $\lambda(\beta) = \lambda(\alpha)$; and does not exist otherwise. □

**Proof of Theorem 1.2.**

(i) Put $X_\alpha = \{\ast\} \cup \bigoplus_{i=1}^{\infty} L_i(C_\alpha, P)$. Observe that $X_\alpha$ is completely metrizable and is the union of countably many closed subspaces with $\text{Cmp} \leq 0$. Since def $X_\alpha = \alpha$, we have $\text{Cmp} X_\alpha = \alpha$. Now observe that $\lim_{i \to \infty} m(L_i(C_\alpha, P), \text{Cmp}, \alpha - 1, \alpha) = \infty$. Hence $X_\alpha$ cannot be written as the finite union of closed subsets with $\text{Cmp} \leq \alpha - 1$.

(ii) Put $X_\alpha = \{\ast\} \cup \bigoplus_{i=1}^{\infty} L_i(S^\alpha, P)$. Observe that $X_\alpha$ is compact and is the union of countably many finite-dimensional closed subspaces (recall that $S^\alpha$ and therefore $L_i(S^\alpha, P)$ have the same property). Since for each $i$, $\text{trInd} L_i(S^\alpha, P) = \alpha$, we have $\text{trInd} X_\alpha = \alpha$. Now
observe that \( \lim_{i \to \infty} m(L_i(S^\alpha, P), \tr Ind, \alpha - 1, \alpha) = \infty \). Hence \( X_\alpha \) cannot be written as the finite union of closed subsets with \( \tr Ind \leq \alpha - 1 \). \( \square \)

**Remark 2.2** Let \( Q \) be the set of rational numbers of the closed interval \([0, 1]\). Recall that for the spaces \( X = Q \times [0, 1]^n \) and \( Y = ([0, 1] \setminus Q) \times I^n \) we have \( \text{Cmp } X = \text{def } X = \text{Cmp } Y = \text{def } Y = n \) ([AN, p. 18 and 56]). It is easy to observe that \( X \) satisfies points (a)-(c) of Theorem 1.2 (i). However, \( X \) is not completely metrizable. Note that \( Y \) is completely metrizable and satisfies points (a) and (c) of Theorem 1.2 (i) but not (b). Observe that Smirnov's compactum \( S^\alpha \) with \( n(\alpha) \geq 1 \) satisfies points (a) and (b) of Theorem 1.2 (ii) but not (c). Note also that any Cantor manifold \( Z \) with \( \tr Ind Z = \alpha \), where \( \alpha \) is infinite ordinal with \( n(\alpha) \geq 1 \), (see for such spaces for example in [O]) satisfies points (a) and (c) of Theorem 1.2 (ii) but not (b).

Let \( d \) be a (transfinite) dimension function. A space \( X \) with \( dX \neq \infty \) is said to have property \((\ast)_d \) if for every open nonempty subset \( O \) of the space \( X \) there exists a closed in \( X \) subset \( F \subset O \) with \( dF = dX \).

Observe that the spaces \( X, Y \) from Remark 2.2 have property \((\ast)_{\text{Cmp}} \) and \( Z \) has property \((\ast)_{\tr Ind} \).

**Proposition 2.2** Let \( X \) be a completely metrizable space with \( dX \neq \infty \). Then \( X \neq \bigcup_{i=1}^{\infty} X_i \), where each \( X_i \) is closed in \( X \) and \( dX_i < dX \) iff there exists a closed subspace \( Y \) of \( X \) such that

(i) \( dY = dX \) and
(ii) \( Y \) has the property \((\ast)_d \).

**Remark 2.3** This remark concerns non-metrizable compact spaces. Using the construction of Lokucievskij's example ([E, p. 140]), Chatyrko, Kozlov and Pasynkov [ChKP, Remark 3.15 (b)] presented for each \( n = 3, 4, \ldots \) a compact Hausdorff space \( X_n \) such that \( \ind X_n = 2 \) and \( m(X_n, \text{ind}, 1, 2) = n \). Hence it is clear that \( m_N(\text{ind}, 1, 2) = 2 \) and \( M_N(\text{ind}, 1, 2) = \infty \), where \( N \) is the class of compact Hausdorff spaces. In [K] Kotkin constructed a compact Hausdorff space \( X \) with \( \ind X = 3 \) which is the union of three one-dimensional in the sense of \( \ind \) closed subspaces. Hence, \( m_N(\text{ind}, 1, 3) = 3 \) and \( m_N(\text{ind}, 2, 3) = 2 \). Filippov in [F] presented for every \( n \) a compact Hausdorff space \( F_n \) with \( \ind F_n = n \), which is the union of finitely many one-dimensional in the sense of \( \ind \) closed subspaces, thereby \( m_N(\text{ind}, k, n) < \infty \) for each \( 1 \leq k < n \). By the sum theorem from Remark 2.1 (iii) for \( \ind \) which is valid in fact for all regular spaces, one can get that \( m_N(\text{ind}, 1, n) \geq 2^{n-2} + 1 \) for each \( n \).
参考文献


