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Some Remarks on Automata without Letichevsky Criteria

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Abstract: In this paper we show some properties of finite automata having no Letichevsky criteria

Keywords: Finite automata, Letichevsky criterion.

1. Introduction

We start with some standard concepts and notations. The elements of an alphabet $X$ are called letters ($X$ is supposed to be finite and nonempty). A word over an alphabet $X$ is a finite string consisting of letters of $X$. The string consisting of zero letters is called the empty word, written by $\lambda$. The length of a word $w$, in symbols $|w|$, means the number of letters in $w$ when each letter is counted as many times it occurs. By definition, $|\lambda| = 0$. At the same time, for any set $H$, $|H|$ denotes the cardinality of $H$. If $u$ and $v$ are words over an alphabet $X$, then their catenation $uv$ is also a word over $X$. Catenation is an associative operation and the empty word $\lambda$ is the identity with respect to catenation: $w\lambda = \lambda w = w$ for any word $w$. For a word $w$ and positive integer $n$, the notation $w^n$ means the word obtained by catenating $n$ copies of the word $w$. $w^0$ equals the empty word $\lambda$. $w^n$ is called the $m$-th power of $w$ for any non-negative integer $m$.

Let $X^+$ be the set of all words over $X$, moreover, let $X^+ = X^+ \setminus \{\lambda\}$. $X^+$ and $X^*$ are the free monoid and the free semigroup, respectively, generated by $X$ under catenation.

A (finite) directed graph (or, in short, a digraph) $D = (V, E)$ (of order $|V| > 0$) is a pair consisting of sets of vertices $V$ and edges $E \subseteq V \times V$. A walk in $D = (V, E)$ is a sequence of vertices $v_1, \ldots, v_n, n \geq 1$ such that $(v_i, v_{i+1}) \in E, i = 1, \ldots, n - 1$. A walk is closed if $v_1 = v_n$. By a (directed) path from a vertex $a$ to a vertex $b \neq a$ we shall mean a sequence $v_1 \ldots v_n, n > 1$ of pairwise distinct vertices such that $a = v_1, b = v_n$.

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and \((v_i, v_{i+1}) \in E\) for every \(i = 1, \ldots , n - 1\). The positive integer \(n - 1\) is called the length of the path. Thus a path is a walk with all \(n\) vertices distinct. A closed walk with all vertices distinct except \(v_1 = v_n\) is a cycle of length \(n - 1\).

By an automaton we mean a finite automaton without outputs. Given an automaton \(A = (X, \delta)\) with set of states \(X\), set of input letters \(X\), and transition \(\delta : A \times X \to A\), it is understood that \(\delta\) is extended to \(\delta^* : A \times X^* \to A\) with \(\delta^*(a, \lambda) = a\), \(\delta^*(a, xq) = \delta^*(\delta(a, x), q)\). In the sequel, we will consider the transition of an automaton in this extended form and thus we will denote it by the same Greek letter \(\delta\). Let \(A = (X, \delta)\) be an automaton. It is said that a state \(a \in A\) generates a state \(b \in A\) if \(\delta(a, p) = b\) holds for some \(p \in X^*\). For every state \(a \in A\) define the state subautomaton \(B = (B, X, \delta')\) generated by \(a\) such that \(B = \{b \mid b = \delta(a, p), p \in X^*\}\), moreover, \(\delta'(b, x) = \delta(b, x)\) for every pair \(b, x \in X\). \(A\) is called strongly connected if for every pair \(a, b \in A\) there exists \(p \in X^*\) such that \(\delta(a, p) = b\).

We say that \(A\) satisfies Letichevsky's criterion if there are a state \(a \in A\), input letters \(x, y \in X\), input words \(p, q \in X^*\) such that \(\delta(a, x) \neq \delta(a, y)\) and \(\delta(a, xp) = \delta(a, yq) = a\). It is said that \(A\) satisfies the semi-Letichevsky criterion if it does not satisfy Letichevsky's criterion but there are a state \(a \in A\), input letters \(x, y \in X\), an input word \(p \in X^*\) such that \(\delta(a, x) \neq \delta(a, y)\), \(\delta(a, xp) = a\) and for every \(q \in X^*\), \(\delta(a, yq) \neq a\). If \(A\) do not satisfy either Letichevsky's criterion or the semi-Letichevsky criterion then we say that \(A\) does not satisfy any Letichevsky criteria or is without any Letichevsky criteria.

The Letichevsky criterion has a central role in the investigations of products of automata (see [1],[2],[3],[4]). Automata having semi-Letichevsky criterion and automata without any Letichevsky criteria are also important in the classical result of Z. Ésik and Gy. Horváth (see [2],[3]). In this paper we investigate automata without any Letichevsky criteria.

2. Results

First we observe

**Proposition 1** Given an automaton \(A = (X, \delta)\), a state \(a_0 \in A\), four input words \(u, v, p, q \in X^*\) with \(|up|, |vq| > 0\) under which \(\delta(a_0, u) \neq \delta(a_0, v)\), and \(\delta(a_0, up) = \delta(a_0, vq) = a_0\). Then \(A\) satisfies Letichevsky's criterion.

**Proof:** First we suppose \(|u|, |v| > 0\). Then there exist input words \(w, w', w_1, w_2 \in X^*\) and input letters \(x, y \in X\) such that \(u = wxw_1, v = w'yw_2\) and \(\delta(a_0, wx) \neq \delta(a_0, wy) = \delta(a_0, w'y)\). Therefore, we can reach Letichevsky's criterion substituting \(a_0, u, v, p, q\) for \(\delta(a_0, w), x, y, w_1pw, w_2qw\).

Now we assume, say, \(|v| = 0\). Then, by our assumptions, \(|q| > 0\) with \(\delta(a_0, q) = a_0\). On the other hand, \(\delta(a_0, u) \neq \delta(a_0, v) = a_0\) implies \(|u| > 0\). In addition, then we have \((a_0 = \delta(a_0, v)) = \delta(a_0, q) \neq \delta(a_0, u)\). Therefore, there are input words \(w, w', w_1, w_2 \in X^*\) and input letters \(x, y \in X\) such that \(u = wxw_1, q = w'yw_2\) and \(\delta(a_0, wx) \neq \delta(a_0, wy) = \delta(a_0, w'y)\).
\(\delta(a_0, wy) = \delta(a_0, w' y)\). We obtain again Letichevsky's criterion substituting \(a_0, u, v, p, q\) for \(\delta(a_0, w), x, y, w_1pw, w_2w\).

Now we study automata having no Letichevsky's criteria. The following statement is obvious.

**Proposition 2** \(\mathcal{A} = (A, X, \delta)\) is a automaton without any Letichevsky criteria if and only if for every state \(a_0 \in A\), input letters \(x, y \in X\) and an input word \(p \in X^*\) having \(\delta(a_0, xp) = a_0\), it holds that \(\delta(a_0, x) = \delta(a_0, y)\).

Obviously, if \(\mathcal{A} = (A, X, \delta)\) has the above properties then there exists a nonnegative integer \(n\) such that for every \(p \in X^*\) with \(|p| \geq n\), each \(\delta(a, p)\) generates an autonomous state-subautomaton of \(\mathcal{A}\). Denote by \(n_\mathcal{A}(\leq n)\) the minimal nonnegative integer having this property.

**Proposition 3** \(n_\mathcal{A} \leq \max(|A| - 2, 0)\).

*Proof:* Take out of consideration the trivial cases. Thus we may assume \(|A| > 2\). Consider \(a \in A, x_1, \ldots, x_{m+2} \in X\) having \(\delta(a, x_1 \cdots x_m x_{m+1}) \neq \delta(a, x_1 \cdots x_m x_{m+2})\). If \(a, \delta(a, x_1), \delta(a, x_1 x_2), \ldots, \delta(a, x_1 \cdots x_m), \delta(a, x_1 \cdots x_m x_{m+1}), \delta(a, x_1 \cdots x_m x_{m+2})\) are not distinct states then \(\mathcal{A}\) satisfies either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. Hence, \(m \leq |A| - 3\). Thus \(n_\mathcal{A} \leq |A| - 2\).

We also note the next direct consequence of Proposition 2.

**Proposition 4** If \(\mathcal{A}\) is a strongly connected automaton without any Letichevsky criteria then \(\mathcal{A}\) is autonomous.

By this observation, we get immediately the following

**Proposition 5** Suppose that \(\mathcal{A} = (A, X, \delta)\) is a strongly connected automaton without any Letichevsky criteria. There exists a \(k > 0\) such that for every \(a, b \in A\), \(a = b\) if and only if there exists a pair \(p, q \in X^*\) with \(|p| \equiv |q|(\text{mod } k)\) and \(\delta(a, p) = \delta(b, q)\).

**Lemma 6** Given an automaton \(\mathcal{A} = (A, X, \delta)\) be without any Letichevsky criteria, \(a \in A\) is a state of a strongly connected state-subautomaton of \(\mathcal{A}\) if and only if there exists a nonempty word \(p \in X^*\) with \(\delta(a, p) = a\).

*Proof:* Let \(a \in A\) be a state of a strongly connected state-subautomaton of \(\mathcal{A}\). By definition, for every nonempty word \(q \in X^*\), there exists a word \(r \in X^*\) with \(\delta(a, qr) = a\). Conversely, suppose that \(\delta(a, p) = a\) for some \(a \in A\) and \(p \in X^*, p \neq \lambda\). Then for every prefix \(p'\) of \(p\) and input letters \(x, y \in X\), \(\delta(a, p'x) = \delta(a, p'y)\). Therefore, for every \(q \in X^*\), \(\delta(a, q) = \delta(a, r)\), where \(r\) is a prefix of \(p\) with \(|q| \equiv |r|(\text{mod } |p|)\). But then \(a\) generates a strongly connected state-subautomaton of \(\mathcal{A}\).

We shall use the following consequence of the above statement.
Proposition 7 Let $A = (A, X, \delta)$ be an automaton without any Letichevsky criteria. Moreover, suppose that $a \in A$ is not a state of any strongly connected state-subautomaton of $A$. If $\delta(b, p) = a$ for some $b \in A$ and nonempty $p \in X^*$ then $\delta(a, q) \neq b, q \in X^*$. Conversely, if $\delta(a, r) = c$ for some $c \in A$ and nonempty $r \in X^*$ then $\delta(c, q) \neq a, q \in X^*$.

Lemma 8 Let $A = (A, X, \delta)$ be a automaton without any Letichevsky's criteria. If there are $a \in A, q, q' \in X^*, |q| = |q'| \geq |A| - 1, \delta(a, q) \neq \delta(a, q')$ then for every pair of words $r, r' \in X^*, |r| = |r'|$ we have $\delta(a, q r) \neq \delta(a, q' r')$.

Proof: Suppose that our state does not hold, i.e., there are $a \in A, q, q' r, r' \in X^*, |q| = |q'| \geq |A| - 1, |r| = |r'|$ having $\delta(a, q) \neq \delta(a, q')$ and $\delta(a, q r) = \delta(a, q' r')$.

Then, of course, $|r| = |r'| > 0$. We distinguish the following three cases.

Case 1. There are $q_1, r_1, q_2, r_2, q'_1, q'_2, r'_2$ with $q = q_1 r_1 = q_2 r_2, q' = q'_1 r'_1 = q'_2 r'_2, |q_1| < |q_2|, |q'_1| < |q'_2|$ such that $\delta(a, q_1) = \delta(a, q_2)$, $\delta(a, q'_1) = \delta(a, q'_2)$. But then, by Proposition 2, $\delta(a, q_1 w) = \delta(a, q_2 w)$ and $\delta(a, q'_1 w) = \delta(a, q'_2 w)$ for every $w, w' \in X^*, |w| = |w'|$. Thus, because of $\delta(a, q_1) = \delta(a, q_2)$ and $\delta(a, q'_1) = \delta(a, q'_2)$, we obtain that, for every $w, w' \in X^*$ there are $z, z' \in X^*$ with $\delta(a, q_1 w z) = \delta(a, q_1)$ and $\delta(a, q'_1 w z') = \delta(a, q'_1)$. Thus $q_1 r_1 = q, q'_1 r'_1 = q'$ imply that $\delta(a, q r z) = \delta(a, q_1)$ and $\delta(a, q' r' z') = \delta(a, q'_1)$ hold for some $z, z' \in X^*$. This means that $\delta(a, q r z r_1) = \delta(a, q)$ and $\delta(a, q' r' z' r'_1) = \delta(a, q')$. Put $b = \delta(a, q r)(= \delta(a, q' r'))$, $c = \delta(a, q), c' = \delta(a, q')$. Then $\delta(b, z r_1) = c \neq c' = \delta(b, z' r'_1)$ and $\delta(c, r) = \delta(c', r') = b$. But then $|r| = |r'| > 0$ implies $|z r_1 r|, |z' r'_1 r'| > 0$. Therefore, by Proposition 1, $A$ satisfies Letichevsky's criterion, a contradiction.

Case 2. There are $q_1, r_1, q_2, r_2$ with $q = q_1 r_1 = q_2 r_2, |q_1| < |q_2|$, such that $\delta(a, q_1) = \delta(a, q_2)$, but $\delta(a, q'_1) \neq \delta(a, q'_2)$ holds for every distinct prefixes $q'_1, q'_2$ of $q$. Then, because of $|q| = |q'| \geq |A| - 1$, we necessarily have $|q| = |q'| = |A| - 1$, moreover, we also have that for every $d \in A$ there exists a prefix $q'_1$ of $q'$ with $\delta(a, q'_1) = d$. (Indeed, we assumed $\delta(a, q'_1) \neq \delta(a, q'_2)$ for every distinct prefixes $q'_1, q'_2$ of $q'$, where $|q'| = |A| - 1$.)

And then for every $d \in A$ there exists an $r'_1 \in X^*$ having $\delta(d, r'_1) = \delta(a, q')$. On the other hand, we may assume $\delta(a, q r z r'_1) = \delta(a, q)$ as in the previous case.

Now we suppose again $\delta(a, q r) = \delta(a, q' r')$ as before. Substituting $d$ for $\delta(a, q r z r_1)$, there exists an $r'_1 \in X^*$ holding $\delta(a, q' r z r'_1) = \delta(a, q'_1)$. Put $b = \delta(a, q r), c = \delta(a, q), c' = \delta(a, q')$. But then $|r| = |r'| > 0$ implies $|z r_1 r|, |z r'_1 r'| > 0$. Therefore, by Proposition 1 we obtain again that $A$ satisfies Letichevsky's criterion contrary of our assumptions.

Case 3. Let $\delta(a, q_1) \neq \delta(a, q_2)$ and $\delta(a, q'_1) \neq \delta(a, q'_2)$ for every distinct prefixes $q_1, q_2$ of $q$ and $q'_1, q'_2$ of $q'$, respectively. Then for every $d \in A$ there are $r_1, r'_1 \in X^*$ having $\delta(d, r_1) = \delta(a, q)$ and $\delta(d, r'_1) = \delta(a, q')$. Therefore, assuming $\delta(a, q r) = \delta(a, q' r')$ for some $r, r' \in X^*$, and substituting $d$ for $\delta(a, q r) = \delta(a, q' r')$, we obtain $\delta(a, q r r_1) = \delta(a, q), \delta(a, q r r'_1) = \delta(a, q')$ (with $\delta(a, q r) = \delta(a, q' r')$). Put $c = \delta(a, q), c' = \delta(a, q')$.

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2This holds automatically if $|q| = |q'| \geq |A|$. 
Then $\delta(d, r_1) = c, \delta(d, r_1') = c', \delta(c, r) = \delta(c', r') = d$ such that, by $|r| = |r'| > 0, |r_1r_1'| > 0$. By Proposition 1, this implies that $A$ satisfies Letichevsky’s criterion, a contradiction again. 

\[ \square \]

**Theorem 9** Let $A = (A, X, \delta)$ be a automaton without any Letichevsky’s criteria. For every state $a \in A$ we have one of the following two possibilities:

(i) there exist $q, q' \in X^*, |q| = |q'| \geq |A| - 1$ such that $\delta(a, qr) \neq \delta(a, q'r')$ for every $r, r' \in X^*, |r| = |r'|$,

(ii) $\delta(a, q) = \delta(a, q')$ for every $q, q' \in X^*, |q| = |q'| \geq |A| - 1$.

**Proof:** Suppose that (i) does not hold. Then for every $q, q' \in X^*, |q| = |q'| \geq |A| - 1$ there exist $r, r' \in X^*, |r| = |r'|$ having $\delta(a, qr) = \delta(a, q'r')$. Using Lemma 8, $\delta(a, qr) = \delta(a, q'r'), |r| = |r'|$ and $|q| = |q'| \geq |A| - 1$ implies $\delta(a, q) = \delta(a, q')$. Thus (ii) holds whenever (i) does not hold. 

The following statement is obvious.

**Lemma 10** Given a digraph $D = (V, E)$, let $v \in V, p_1, p_2, p_3, p_4 \in V^*$ such that $p_1p_2p_3vp_4v$ and $p_1p_2p_3vp_4v$ are walks and $vp_4v$ is a cycle. $|p_2| \equiv |p_2| (\text{mod } |p_4v|)$ if and only if there are positive integers $k, \ell$ having $|p_1p_2p_3v(p_4v)^k| = |p_1p_2p_3v(p_4v)^\ell|$. 

We finish the paper studying both types of states given in Theorem 9.

**Proposition 11** Let $A = (A, X, \delta)$ be an automaton without any Letichevsky’s criteria. Consider a state $a \in A$ and suppose that there are $q, q' \in X^*, |q| = |q'| \geq |A| - 1, \delta(a, q) \neq \delta(a, q')$. Then there are $q, q'$ having this property for which $q = uv$ and $q' = uv'$ for some $u, v, v' \in X^*$ such that for every prefixes $r$ of $v$ and $r'$ of $v'$ with $|r| = |r'| > 0$ we have $\delta(a, ur) \neq \delta(a, ur')$, and simultaneously, for every $w, z_1, z_2, w', z_1', z_2', |w|, |w'| > 0$ with $v = wz_1z_2, v' = w'z_1'z_2'$ we obtain $z_1 = z_1'$ whenever $\delta(a, uw) = \delta(a, uw')$, and $|z_1| = |z_1'|$.

**Proof:** Consider $a \in A$ and suppose that our conditions hold, i.e., there are $q, q' \in X^*$ having $|q| = |q'| \geq |A| - 1, \delta(a, q) \neq \delta(a, q')$. Then Proposition 3 implies that $\delta(a, q)$ and $\delta(a, q')$ generate autonomous state subautomata of $A$. We will distinguish the following cases (omitting some of the analogous cases):

**Case 1.** There are $u, u', v, v' \in X^*$ such that $q = uv, q' = u'v', \delta(a, u) = \delta(a, u')$ and for every nonempty prefixes $r$ of $v$ and $r'$ of $v'$, $\delta(a, ur) \neq \delta(a, u'r'), \delta(a, ur') \neq \delta(a, ur), \delta(a, ur) \neq \delta(a, u'r')$. Let, say, $|u| \geq |u'|$ and let $v''$ be a prefix of $v'$ with $|v''| = |v|$. Change $q'$ for $uv''$ and then we will have our requirements.

**Case 2.** There exist a prefix $u$ of $q$ having $\delta(a, u) = \delta(a, q')$. Let $t_2 \in X^*$ be a nonempty word with minimal length having $\delta(a, q't_1t_2) = \delta(a, q't_1)$ for some word $^3u = u' = \lambda$ is possible.
\[ t_1 \in X^\ast \] and assume that \( t_2 \) is minimal in the sense that for every nonempty \( p \in X^\ast, \delta(a, q't_1p) = \delta(a, q't_1) \) implies \( |t_2| \leq |p| \).\(^4\) Then, using that \( \delta(a, q') \) generates an autonomous state subautomaton of \( \mathcal{A} \), we have \( q = uv \), where \( v \) is a nonempty prefix of \( t_1t_2^k \) for some \( k \geq 0 \).

Prove that in this case \( u \equiv |q'|(\text{mod } |t_2|) \) is impossible. Assume the contrary. Recall again that \( \delta(a, q') \) generates an autonomous state subautomaton of \( \mathcal{A} \). But then, applying Lemma 10, there are words \( r, r' \in X^\ast, |r| = |r'| \) having \( \delta(a, qr) = \delta(a, q'r') \).

By Lemma 8, then \( |q| = |q'| < |A| - 1 \) contrary of our assumptions. Thus we have the following cases.

Case 2.1. Suppose \( u \not\equiv |q'|(\text{mod } |t_2|) \) such that for every prefixes \( u_1 \) of \( u \) and \( u'_1 \) of \( q' \) with \( u_1u'_1 \neq \lambda, \delta(a, u_1) = \delta(a, u'_1) \) implies \( u_1 = u \) and \( u'_1 = q' \). Then we obtain our requirements again (having \( q = uv \), where \( v \) is a nonempty prefix of \( t_1t_2^k \) for some \( k \geq 0 \)).

Case 2.2. Assume \( u \not\equiv |q'|(\text{mod } |t_2|) \), and simultaneously, let for some prefixes \( u_1 \) of \( u \) and \( u'_1 \) of \( q' \), \( \delta(a, u_1) = \delta(a, u'_1) \) such that \( u = u_1v_1, q' = u'_1v_1' \), furthermore, \( \lambda \in \{u_1, v_1\} \) implies \( \lambda \not\in \{u'_1, v'_1\} \) and \( \lambda \in \{u'_1, v'_1\} \) implies \( \lambda \not\in \{u_1, v_1\} \). If \( v_1 = \lambda \) and \( v'_1 \neq \lambda \) then \( \delta(a, u'_1) = \delta(a, u'_1v_1') \neq \delta(a, u'_1v_1v) = \delta(a, uv) \) such that \( v \) is a nonempty suffix of \( q \). But then \( \mathcal{A} \) has either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. Similarly, it also lead to a contradiction is we assume \( v_1 \neq \lambda \) and \( v'_1 = \lambda \). Thus \( \lambda \not\in \{v_1, v'_1\} \) can be assumed and we may also assume \( \lambda \not\in \{u_1, u'_1\} \) analogously.

By \( u \not\equiv |q'|(\text{mod } |t_2|) \), either \( |u_1| \neq |u'_1|(\text{mod } |t_2|) \), or \( |v_1| \neq |v'_1|(\text{mod } |t_2|) \).

Case 2.2.1. Suppose \( |u_1| \neq |u'_1|(\text{mod } |t_2|) \) and let, say, \( |v_1| \geq |v'_1| \). Take a prefix \( v' \) of \( t_1t_2^k \) for a suitable \( k \geq 0 \) with \( |u'_1v_1v'| = |q| \) and let us consider \( u'_1v_1v' \) instead of \( q' \).

Case 2.2.2. Suppose \( |u_1| \equiv |u'_1|(\text{mod } |t_2|) \). Then \( |v_1| \neq |v'_1|(\text{mod } |t_2|) \). Let, say, \( |u_1| \geq |u'_1| \). Take a prefix \( v' \) of \( t_1t_2^k \) for a suitable \( k \geq 0 \) with \( |u_1v_1v'| = |q| \) and change \( u_1v_1v' \) for \( q' \).

In both of the above Case 2.2.1 and Case 2.2.2, we have words\(^5\) \( w, w_1, w_2, w'_1, w'_2 \in X^\ast, \lambda \not\in \{\{u_1, w'_1\}, w_1 \neq |w'_1|(\text{mod } |t_2|) \), \( w_2 \) is a prefix of \( w_2 \) (or, in the opposite case, \( w_2 \) is a prefix of \( w_2 \)), \( q = w_1w_2, q' = w'_1w'_2 \), such that \( \delta(a, w_1) = \delta(a, w'_1) \). Then let \( w, w_1, w_2, w'_1, w'_2 \in X^\ast \) be arbitrary having these properties for which \( \min(|w_1|, |w_2|) \) is minimal.

If for every nonempty proper prefixes \( z_1 \) of \( w_1 \) and \( z'_1 \) of \( w'_1 \) we have \( \delta(a, w) \not\in \{\delta(a, wz'_1), \delta(a, w_1z'_1)\} \) and \( \delta(a, w_2z_2) \neq \delta(a, w_z') \) then we are ready having our properties for \( q = w_1w_2, q' = w'_1w'_2 \).

Now we assume \( |w_1| \neq |w'_1|(\text{mod } |t_2|) \) such that for some prefixes \( z_1 \) of \( w_1 \) and \( z'_2 \) of \( w'_1 \), \( \delta(a, z_1) = \delta(a, z'_1) \) such that \( w_1 = z_1z_2, w'_1 = z'_1z'_2 \), furthermore, \( \lambda \in \{z_1, z_2\} \) implies \( \lambda \not\in \{z'_1, z'_2\} \) and \( \lambda \in \{z'_1, z'_2\} \) implies \( \lambda \not\in \{z_1, z_2\} \). We can prove \( \lambda \not\in \{z_1, z'_1, z_2, z'_2\} \) similarly as before. Then either \( |z_1| \neq |z_1|(\text{mod } |t_2|) \) or \( |z_2| \neq |z_2|(\text{mod } |t_2|) \). It remains to prove that these cases are impossible.

\(^4\)The finiteness of the state set of \( \mathcal{A} \) implies the existence of \( t_1 \) and \( t_2 \).

\(^5\)in Case 2a, of course, \( w = \lambda \).
If \( |z_1| \neq |z'_1| (\text{mod} \ |t_2|) \) and, say, \( |z_2| \geq |z'_2| \) then considering the prefix \( w'_2 \) of \( w_2 \) having \( |z'_1 w'_2| = |z_1 w_2| \), we can take \( w, z_1, z_2 w_2, z'_1, z_2 w'_2 \) as \( w, w_1, w_2, w'_2 \) contrary of the minimality of \( \min(|w_1|, |w_2|) \).

If \( |z_1| = |z'_1| (\text{mod} \ |t_2|) \) with \( |z_2| \neq |z'_2| (\text{mod} \ |t_2|) \) and, say, \( |z_1| \geq |z'_1| \) then considering the prefix \( w''_2 \) of \( w'_2 \) having \( |z'_2 w''_2| = |z_2 w'_2| \), we can take \( w z_1, z_2, z'_2, w_2, w''_2 \) as \( w, w_1, w'_1, w_2, w''_2 \) contradicting the minimality of \( \min(|w_1|, |w_2|) \).

The proof is complete. \( \square \)

**Proposition 12** Let \( A = (A, X, \delta) \) be an automaton without any Letichevsky's criteria. Consider \( a, a_0 \in A, p \in X^* \) with \( \delta(a_0, p) = a \) and suppose that \( \delta(a, r) = \delta(a, r') \) holds for every \( r, r' \in X^*, |pr| = |pr'| \geq |A| - 1 \). Assume that \( \delta(a, q) \neq \delta(a, q') \) holds for some \( q, q' \in X^*, |pq| = |pq'| (< |A| - 1 \) and let \( q, q' \) be words of maximal length having this property. Then there are \( q, q' \) with this property having

(i) \( q = uu \) and \( q' = uv' \) for some \( u, v, v' \in X^* \) such that for every prefixes \( r \) of \( v \) and \( r' \) of \( v' \) with \( |r| = |r'| > 0 \) we have \( \delta(a, ur) \neq \delta(a, ur') \), and simultaneously, for every \( w, z_1, z_2, u, v' \), \( z'_1, z'_2 \) with \( v = wz_1 z_2, v' = wz'_1 z'_2 \) we obtain \( z_1 = z'_1 \) whenever \( \delta(a, uu) = \delta(a, uv') \), and \( |z_1| = |z'_1| \);

(ii) for every distinct prefixes \( p_1, p_2 \) of \( pq, \delta(a_0, p_1) \neq \delta(a_0, p_2) \).

**Proof:** Consider \( a \in A \) and suppose that our conditions hold.

First we suppose that, whenever \( uu' \neq \lambda, \delta(a, u) = \delta(a, u') \) implies \( u = q \) and \( u' = q' \) for every prefixes \( u \) of \( q \) and \( u' \) of \( q' \). It is clear that then we are ready.

Assume the opposite case and let \( q = uv, q' = u'v' \) with \( \lambda \notin \{uu', uv'\} \) such that \( \delta(a, u) = \delta(a, u') \).

Let \( \min(|u|, |u'|) \) be maximal with the above property and prove that in this case \( u = u' \) can be assumed. Indeed, if it true if \( |u| = |u'| \) because we can consider, say, \( uu' \) instead of \( uv' \).

Finally, prove that, say, \( |u| > |u'| \) is impossible. Indeed, otherwise we could change \( q' \) for \( uu' \), where \( uu' \) is a prefix of \( u' \) with \( |u'| = |u'| \). This contradicts the maximality of \( \min(|u|, |u'|) \).

Now we prove (ii) omitting some analogous cases. If there are no distinct prefixes \( p_1, p_2 \in X^* \) of \( pq \) with \( \delta(a_0, p_1) = \delta(a_0, p_2) \) for \( pq \) and \( pq \). Therefore, in this case, we are ready. Otherwise, we may suppose \( \delta(a_0, p'_1) = \delta(a_0, p'_2) \) for some distinct prefixes \( p'_1, p'_2 \in X^* \) of \( pq' \). Let, say, \( p'_1 = p'_2 r' \) for some nonempty \( r' \in X \). By Lemma 2 and \( \delta(a_0, pq) \neq \delta(a_0, pq') \), this implies that \( \delta(a_0, p'_2) \) generates an autonomous state-subautomaton \( B \) of \( A \). Moreover, \( \delta(a_0, p'_1) = \delta(a_0, p'_2 r') = \delta(a_0, p'_2) \), \( r' \neq \lambda \) implies that this autonomous state-subautomaton is strongly connected. On the other hand, by the maximality of \( |q| (= |q'|) \), \( \delta(a_0, pq x) = \delta(a_0, pq' x') \) holds for every \( x, x' \in X \). Thus, \( \delta(a_0, pq x) \) is also a state of the state-subautomaton \( B \) of \( A \). Recall that by the maximality of \( q \) and \( q' \), we have \( \delta(a_0, pq x) = \delta(a_0, pq' x') \), \( x, x' \in X \). Then \( \delta(a_0, pq) \neq \delta(a_0, pq') \) and \( \delta(a_0, pq x) = \delta(a_0, pq' x') \) imply that \( \delta(a_0, pq) \) is not a state of \( B \). Therefore, for every prefix \( p_1 \) of \( pq \), \( \delta(a_0, p_1) \) is not a state of \( B \).
Suppose that, contrary of our assumptions, $\delta(a_0, p_1) = \delta(a_0, p_2)$ holds for distinct prefixes $p_1$ and $p_2$ of $pq$ and put, say, $p_1 = p_2r_1$ (where $r_1 \neq \lambda$ is assumed). In other words, $\delta(a_0, p_2r_1) = \delta(a_0, p_2)$ holds such that $\delta(a_0, p_2)$ is not a state of $B$. But $\delta(a_0, pqx) = \delta(a_0, pq'x'), x, x' \in X$ implies that there exists an $r_2 \in X^*$ such that $\delta(a_0, p_2r_2)$ is a state of $B$. Clearly, then $A$ satisfies either Letichevsky's criterion or the semi-Letichevsky criterion, a contradiction. This completes the proof. \hfill \Box

References


