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Kyoto University
FACTORIZATION AND HAAGERUP TYPE NORMS ON OPERATOR SPACES

群馬大学・教育学部 伊藤 隆 (Takashi Itoh)
Dept. of Math., Fac. of Edu., Gunma University

This is joint work with M. Nagisa (Chiba Univ.). The problem of the factorization through a Hilbert space for a bounded linear map was considered in Banach space theory and its study was started by Grothendieck [7]. Let $X$ and $Y$ be Banach spaces. It is called that $T : X \rightarrow Y$ factors through a Hilbert space if there exist a Hilbert space $H$ and bounded linear maps $a : X \rightarrow H$, $b : H \rightarrow Y$ such that $T = ba$.

\[
\begin{array}{ccc}
X & \stackrel{T}{\longrightarrow} & Y \\
a & \downarrow & b \\
\mathcal{H} & \nearrow & \mathcal{K}
\end{array}
\]

We note that given $T : X \rightarrow Y$, if $T : X \rightarrow Y^{**}$ factors through a Hilbert space $H$ then $T$ itself factors through a Hilbert space which is a closed subspace of $H$. So it is essential to consider the problem in case that $Y$ is a dual space.

Grothendieck introduced the norm $\| \|_H$ on the algebraic tensor product $X \otimes Y$ in [7] by

\[
\| u \|_H = \inf \{ \sup \{ \left( \sum_{i=1}^{n} |f(x_i)|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |g(y_i)|^2 \right)^{\frac{1}{2}} \} \}
\]

where the supremum is taken over all $f \in X^*$, $g \in Y^*$ with $\| f \|, \| g \| \leq 1$ and the infimum is taken over all representation $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. In this note, we denote by $X \otimes_\alpha Y$ the completion of $X \otimes Y$ by the norm $\| \|_\alpha$, and denote by $\| \|_{\alpha^*}$ the norm of the dual space $(X \otimes_\alpha Y)^*$. He showed that $T : X \rightarrow Y^*$ factors through a Hilbert space if and only if $T \in (X \otimes_H Y)^*$ by the natural identification $\langle T(x), y \rangle = T(x \otimes y)$ for $x \in X, y \in Y$, moreover \( \inf \{ \| b \| \| a \| \mid T = ba \} = \| T \|_{H^*} \).

In [15], Lindenstrauss and Pelczynski studied a bounded linear map $T : X \rightarrow Y$ with the condition:
given any \( n \) and \( n \times n \) matrices \( [a_{ij}] \in M(C) \) with \( \|a_{ij}\| \leq 1 \), then

\[
\sum_{i=1}^{n} \| \sum_{j=1}^{n} a_{ij} T(x_{j}) \| \leq C \sum_{j=1}^{n} \| x_{j} \|^{2} \quad \text{for any } x_{1}, \ldots, x_{n}.
\]

We consider \( T \otimes \alpha : X \otimes \ell_{n}^{2} \rightarrow Y \otimes \ell_{n}^{2} \) for \( T : X \rightarrow Y \) and define a norm \( \| \sum_{i=1}^{n} x_{i} \otimes e_{i} \|^{2} = \sum_{i=1}^{n} \| x_{i} \|^{2} \). Then the above condition is equivalent to \( \| T \otimes \alpha \| \leq C \| \alpha \| \) for all \( \alpha : \ell_{n}^{2} \rightarrow \ell_{n}^{2} \).

Their theorems are summarized for a bounded linear map \( T : X \rightarrow Y^{*} \) as follows:

The following are equivalent:

1. \( \| T \otimes \alpha \| \leq \| \alpha \| \) for all \( \alpha : \ell_{n}^{2} \rightarrow \ell_{n}^{2} \) and \( n \in \mathbb{N} \).
2. \( \| T \|_{H^{*}} \leq 1 \).
3. \( T \) factors through a Hilbert space \( K \) by bounded linear maps \( a : X \rightarrow K \) and \( b : K \rightarrow Y^{*} \) such that

\[
i.e., \quad T = ba \quad \text{with } \| a \| \| b \| \leq 1.
\]

In \( C^{*}\)-algebra theory and operator space theory, many important factorization theorems have been proved.

**Theorem 1. (Haagerup, [8])** Suppose that \( A \) and \( B \) are \( C^{*}\)-algebras, and \( T : A \rightarrow B^{*} \) is a bounded linear map. Then \( T \) factors through a Hilbert space such that \( T = ba \) with \( \| T \| \leq 2\| b \| \| a \| \).

We recall the column (resp. row) Hilbert space \( H_{c}(\text{resp. } H_{r}) \) for a Hilbert space \( H \). If \( \xi = [\xi_{ij}] \in M_{n}(H) \), then we define a map \( C_{n}(\xi) \) by

\[
C_{n}(\xi) : \mathbb{C}^{n} \ni [\lambda_{1}, \ldots, \lambda_{n}] \mapsto [\sum_{j=1}^{n} \lambda_{j} \xi_{ij}]_{i} \in H^{n}
\]

and denote the column matrix norm by \( \| \xi \|_{c} = \| C_{n}(\xi) \| \). This operator space structure on \( H \) is called the column Hilbert space and denoted by \( H_{c} \).

To consider the row Hilbert space, let \( H \) be the conjugate Hilbert space for \( H \). We define a map \( R_{n}(\xi) \) by

\[
R_{n}(\xi) : H^{n} \ni [\bar{\eta_{1}}, \ldots, \bar{\eta_{n}}] \mapsto [\sum_{j=1}^{n} (\xi_{ij} | \eta_{j})]_{i} \in \mathbb{C}^{n}
\]

and the row matrix norm by \( \| \xi \|_{r} = \| R_{n}(\xi) \| \). This operator space structure on \( H \) is called the row Hilbert space and denoted by \( H_{r} \).
Let $A$ and $B$ be operator spaces. The Haagerup norm [4] on $A \otimes B$ is defined by

$$||u||_h = \inf \{ ||[x_1, \ldots, x_n]||[y_1, \ldots, y_n]^t || \mid u = \sum_{i=1}^{n} x_i \otimes y_i \},$$

where $[x_1, \ldots, x_n] \in M_{1,n}(A)$ and $[y_1, \ldots, y_n]^t \in M_{n,1}(B)$.

**Theorem 2.** (Effros-Ruan, [5]) Suppose that $A$ and $B$ are operator spaces, and $T : A \to B^*$ is a completely bounded map. Then $T$ factors through a row Hilbert space $\mathcal{H}_r$ if and only if $T \in (A \otimes B)^*$ with $||T||_{h^*} = \inf \{ ||b||_{cb} ||a||_{cb} \mid T = ba \}$.

**Theorem 3.** (Pisier-Shlyakhtenko, [21]) Suppose that $A$ and $B$ are $C^*$-algebras, and $T : A \to B^*$ is a completely bounded map. If one of the algebras $A, B$ is exact, then $T$ factors through $\mathcal{H}_c \oplus \mathcal{K}_r$ the direct sum of the column and row Hilbert spaces.

These factorizations form that

$$
\begin{array}{ccc}
A & \xrightarrow{T} & B^* \\
\downarrow a & & \downarrow b \\
\mathcal{K} & & \mathcal{K}
\end{array}
$$

On the other hand, in [12], it has been shown that the following factorization of a linear map $T$ from $\ell^1$ to $\ell^\infty$ in connection with a Schur multiplier:

$$
\begin{array}{ccc}
\ell^1 & \xrightarrow{T} & \ell^\infty \\
\downarrow a & & \uparrow a^t \\
\ell^2 & \longrightarrow & \ell^2^*
\end{array}
$$

where $a^t$ is the transposed map of $a$.

Motivated by this factorization, the aim of this note is to explain a square factorization theorem of a bounded linear map through a pair of Hilbert spaces $\mathcal{H}$ between an operator space and its dual space [13].

More precisely, let us suppose that $A$ and $B$ are operator spaces in $\mathbb{B}(\mathcal{H})$ and denote by $C^*(A)$ the $C^*$-algebra in $\mathbb{B}(\mathcal{H})$ generated by $A$. We define the **numerical radius Haagerup norm** of an element $u \in A \otimes B$ by

$$||u||_{wh} = \inf \{ \frac{1}{2} ||[x_1, \ldots, x_n, y_1^*, \ldots, y_n^*]^2 \mid u = \sum_{i=1}^{n} x_i \otimes y_i \}. $$
By the identity
\[
\inf_{\lambda > 0} \frac{\lambda \alpha + \lambda^{-1} \beta}{2} = \sqrt{\alpha \beta}
\]
for positive real numbers $\alpha, \beta \geq 0$, the Haagerup norm can be rewritten as
\[
\|u\|_h = \inf \left\{ \frac{1}{2} (\|x_1, \ldots, x_n\|^2 + \|y_1^*, \ldots, y_n^*\|^2) \mid u = \sum_{i=1}^n x_i \otimes y_i \right\}.
\]

Then it is easy to check that
\[
\frac{1}{2} \|u\|_h \leq \|u\|_{\text{wh}} \leq \|u\|_h
\]
and $\|u\|_{\text{wh}}$ is a norm.

We also define a norm of an element $u \in C^*(A) \otimes C^*(A)$ by
\[
\|u\|_{\text{wh}} = \inf \left\{ \|x_1, \ldots, x_n\|^2 w(\alpha) \mid u = \sum x_i^* \alpha_{ij} \otimes x_j \right\},
\]
where $w(\alpha)$ is the numerical radius norm of $\alpha = [\alpha_{ij}]$ in $M_n(\mathbb{C})$.

$A \otimes_{\text{Wh}} A$ is defined as the closure of $A \otimes A$ in $C^*(A) \otimes_{\text{Wh}} C^*(A)$.

**Theorem 4.** Let $A$ be an operator space in $\mathcal{B}(\mathcal{H})$. Then $A \otimes_{\text{wh}} A = A \otimes_{\text{Wh}} A$.

Let $a : C^*(A) \rightarrow \mathcal{H}_c$ be a completely bounded map. We define a map $d : C^*(A) \rightarrow \mathcal{H}$ by $d(x) = \overline{a(x^*)}$. It is not hard to check that $d : C^*(A) \rightarrow \mathcal{H}_r$ is completely bounded and $\|a\|_{\text{cb}} = \|d\|_{\text{cb}}$ when we introduce the row Hilbert space structure to $\mathcal{H}$. In this paper, we define the adjoint map $a^*$ of $a$ by the transposed map of $d$, that is, $d^* : ((\mathcal{H}_r)^*)^* = (\mathcal{H}^*)^* = (\mathcal{H}^{**})_c = \mathcal{H}_c \rightarrow C^*(A)^*$ (c.f. [5]). More precisely, we define
\[
\langle a^*(\eta), x \rangle = \langle \eta, d(x) \rangle = \langle \eta | a(x^*) \rangle \quad \text{for} \quad \eta \in \mathcal{H}, x \in C^*(A).
\]

Now we can state a square factorization theorem.

**Theorem 5.** Suppose that $A$ is an operator space in $\mathcal{B}(\mathcal{H})$, and that $T : A \times A \rightarrow \mathbb{C}$ is bilinear. Then the following are equivalent:

1. $\|T\|_{\text{wh}^*} \leq 1$.
2. There exists a state $p_0$ on $C^*(A)$ such that
\[
|T(x, y)| \leq p_0(x x^*)^{\frac{1}{2}} p_0(y^* y)^{\frac{1}{2}} \quad \text{for} \quad x, y \in A.
\]
(3) There exist a \(*\)-representation \(\pi : C^*(A) \to \mathbb{B}(\mathcal{K})\), a unit vector \(\xi \in \mathcal{K}\) and a contraction \(b \in \mathbb{B}(\mathcal{K})\) such that
\[
T(x, y) = (\pi(x)b\pi(y)\xi | \xi) \quad \text{for } x, y \in A.
\]

(4) There exist an extension \(T' : C^*(A) \to C^*(A)^*\) of \(T\) and completely bounded maps \(a : C^*(A) \to \mathcal{K}_c\), \(b : \mathcal{K}_c \to \mathcal{K}_c\) such that
\[
C^*(A) \xrightarrow{T'} C^*(A)^*
\]
\[
\begin{array}{ccc}
\downarrow & & \uparrow a^* \\
\mathcal{K}_c & \rightarrow & \mathcal{K}_c \\
b & & \\
\end{array}
\]
i.e., \(T' = a^*ba\) with \(||a||_{cb}^2||b||_{cb} \leq 1\).

Remark 6. (i) If we replace the linear map \(\langle T(x), y \rangle = T(x, y)\) with \(\langle x, T(y) \rangle = T(x, y)\), then we have a factorization of \(T\) through a pair of the row Hilbert spaces \(\mathcal{H}_r\). More precisely, the following condition (4)' is equivalent to the above conditions.

(4)' There exist an extension \(T' : C^*(A) \to C^*(A)^*\) of \(T\) and completely bounded maps \(a : C^*(A) \to \mathcal{K}_r\), \(b : \mathcal{K}_r \to \mathcal{K}_r\) such that
\[
C^*(A) \xrightarrow{T'} C^*(A)^*
\]
\[
\begin{array}{ccc}
\downarrow & & \uparrow a^* \\
\mathcal{K}_r & \rightarrow & \mathcal{K}_r \\
b & & \\
\end{array}
\]
i.e., \(T' = a^*ba\) with \(||a||_{cb}^2||b||_{cb} \leq 1\).

(ii) Let \(\ell^2_n\) be an \(n\)-dimensional Hilbert space with the canonical basis \(\{e_1, \ldots, e_n\}\). Given \(\alpha : \ell^2_n \to \ell^2_n\) with \(\alpha(e_j) = \sum_i \alpha_{ij}e_i\), we set the map \(\hat{\alpha} : \ell^2_n \to \ell^2_n^*\) by \(\hat{\alpha}(e_j) = \sum_i \alpha_{ij}\overline{e}_i\) where \(\{\overline{e}_i\}\) is the dual basis. For notational convenience, we shall also denote \(\hat{\alpha}\) by \(\alpha\). For \(\sum_{i=1}^n x_i \otimes e_i \in C^*(A) \otimes \ell^2_n\), we define a norm by \(||\sum_{i=1}^n x_i \otimes e_i|| = ||[x_1, \ldots, x_n]||\). Let \(T : C^*(A) \to C^*(A)^*\) be a bounded linear map. Consider \(T \otimes \alpha : C^*(A) \otimes \ell^2_n \to C^*(A)^* \otimes \ell^2_n^*\) with a numerical radius type norm \(w(\cdot)\) given by
\[
w(T \otimes \alpha) = \sup\{|\langle \sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i) \rangle | \| \sum x_i \otimes e_i \| \leq 1\}.
\]
Then we have
\[
\sup\{ \frac{w(T \otimes \alpha)}{w(\alpha)} \mid \alpha : \ell^2_n \rightarrow \ell^2_n, \ n \in \mathbb{N} \} = ||T||_{wh^*},
\]
since \( T(\sum x_i^* \alpha_{ij} \otimes x_j) = (\sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i)) \).

(iii) Let \( u = \sum x_i \otimes y_i \in C^* (A) \otimes C^*(A) \). It is straightforward from Theorem 2.3 that
\[
||u||_{wh} = \sup \{ \sum \varphi(x_i)b\varphi(y_i) \}
\]
where the supremum is taken over all *-preserving completely contractions \( \varphi \) and contractions \( b \).

We also define a variant of the numerical radius Haagerup norm of an element \( u \in A \otimes B \) by
\[
||u||_{wh'} = \inf \{ \frac{1}{2} ||[x_1, \ldots, x_n, y_1, \ldots, y_n]^t||^2 \mid u = \sum_{i=1}^{n} x_i \otimes y_i \},
\]
where \([x_1, \ldots, x_n, y_1, \ldots, y_n]^t \in M_{2n,1}(A + B)\), and denote by \( A \otimes_{wh'} B \) the completion of \( A \otimes B \) with the norm \( || \cdot ||_{wh'} \).

We remark that \( || \cdot ||_{wh} \) and \( || \cdot ||_{wh'} \) are not equivalent, since \( || \cdot ||_h \) in [10] is equivalent to \( || \cdot ||_{wh'} \) and \( || \cdot ||_h \) and \( || \cdot ||_{wh} \) are not equivalent [10], [14].

In the next theorem, we use the transposed map \( a^t : (K_c)^* \rightarrow C^*(A)^* \) of \( a : C^*(A)^* \rightarrow K_c \) instead of \( a^* : K_c \rightarrow C^*(A)^* \). We note that \( (K_c)^* = (\overline{K}), \) and the relation \( a \) and \( a^t \) is given by
\[
\langle a^t(\overline{\eta}), x \rangle = \langle \overline{\eta}, a(x) \rangle = \langle \overline{\eta} | a(x) \rangle_{\overline{K}} \quad \text{for } \overline{\eta} \in \overline{K}, \ x \in C^*(A).
\]

**Theorem 7.** Suppose that \( A \) is an operator space in \( \mathcal{B}(\mathcal{H}) \), and that \( T : A \times A \rightarrow \mathbb{C} \) is bilinear. Then the following are equivalent:

(1) \( ||T||_{wh^*} \leq 1 \).

(2) There exists a state \( p_0 \) on \( C^*(A) \) such that
\[
|T(x, y)| \leq p_0(x^*x)^{1/2}p_0(y^*y)^{1/2} \quad \text{for } x, y \in A.
\]

(3) There exist a *-representation \( \pi : C^*(A) \rightarrow \mathcal{B}(\mathcal{K}) \), a unit vector \( \xi \in \mathcal{K} \) and a contraction \( b : \mathcal{K} \rightarrow \overline{\mathcal{K}} \) such that
\[
T(x, y) = (b\pi(y)\xi \mid \pi(x)\overline{\xi})_{\overline{\mathcal{K}}} \quad \text{for } x, y \in A.
\]
There exist a completely bounded map $a : A \to \mathcal{K}_c$ and a bounded map $b : \mathcal{K}_c \to (\mathcal{K}_c)^*$ such that

$$A \xrightarrow{T} A^*$$

$$\mathcal{K}_c \xrightarrow{b} (\mathcal{K}_c)^*$$

i.e., $T = a^t b a$ with $\|a\| \|b\| \leq 1$.

Now we can describe the above theorems in terms of Banach space theory.

Let $X$ be a Banach space. Recall that the minimal quantization $\text{Min}(X)$ of $X$. Let $\Omega_X$ be the unit ball of $X^*$, that is, $\Omega_X = \{ f \in X^* \mid \|f\| \leq 1 \}$. For $[x_{ij}] \in M_n(X)$, $\|[x_{ij}]\|_{\text{min}}$ is defined by

$$\|[x_{ij}]\|_{\text{min}} = \sup\{ \|[f(x_{ij})]\| \mid f \in \Omega_X \}.$$  

Then $\text{Min}(X)$ can be regarded as a subspace in the $C^*$-algebra $C(\Omega_X)$ of all continuous functions on the compact Hausdorff space $\Omega_X$. Here we define a norm of an element $u \in X \otimes X$ by

$$\|u\|_{wH} = \inf\{ \sup\{ \left( \sum_{i=1}^{n} |f(x_i)|^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} |f(y_i)|^2 \right)^{\frac{1}{2}} \} \mid f \in \Omega_X \},$$

where the supremum is taken over all $f \in X^*$ with $\|f\| \leq 1$ and the infimum is taken over all representation $u = \sum_{i=1}^{n} x_i \otimes y_i$.

Let $T : X \to X^*$ be a bounded linear map. We consider the map $T \otimes \alpha : X \otimes \ell_n^2 \to X^* \otimes \ell_n^{2*}$ and define a norm for $\sum x_i \otimes e_i \in X \otimes \ell_n^2$ by

$$\|\sum x_i \otimes e_i\| = \sup\{ \left( \sum |f(x_i)|^2 \right)^{\frac{1}{2}} \mid f \in \Omega_X \}.$$  

We note that, given $x \in X$, $x^*$ is regarded as $(x^*, f) = \overline{f(x)}$ for $f \in X^*$ in the definition of $w(T \otimes \alpha)$, that is,

$$w(T \otimes \alpha) = \sup\{ |\langle \sum x_i^* \otimes e_i, T \otimes \alpha(\sum x_i \otimes e_i) \rangle| \mid \sum x_i \otimes e_i \leq 1 \}.$$  

Finally we can state the following result which can be seen as a numerical radius norm version of Grothendieck, Lindenstrauss-Pelczynski's.
Corollary 8. Suppose that $X$ is a Banach space, and that $T : X \to X^*$ is a bounded linear map. Then the following are equivalent:

1. $w(T \otimes \alpha) \leq w(\alpha)$ for all $\alpha : \ell^2_n \to \ell^2_n$ and $n \in \mathbb{N}$.
2. $\|T\|_{wH} \leq 1$.
3. $T$ factors through a Hilbert space $K$ and its dual space $K^*$ by bounded linear maps $a : X \to K$ and $b : K \to K^*$ as follows:

$$
X \xrightarrow{T} X^*
$$
$$
\downarrow a \uparrow a^*\nK \xrightarrow{b} K^*
$$

i.e., $T = a^*ba$ with $\|a\|^2\|b\| \leq 1$.

4. $T$ has an extension $T' : C(\Omega_X) \to C(\Omega_X)^*$ which factors through a pair of Hilbert spaces $K$ by bounded linear maps $a : C(\Omega_X) \to K$ and $b : K \to K^*$ as follows:

$$
C(\Omega_X) \xrightarrow{T'} C(\Omega_X)^*
$$
$$
\downarrow a \uparrow a^*\nK \xrightarrow{b} K
$$

i.e., $T' = a^*ba$ with $\|a\|^2\|b\| \leq 1$.

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