Al-Hawārī's commentary on Ibn al-Bannā''s *Talkhīṣ*: Contents and influences

Mahdi Abdeljaouad and Jeffrey A. Oaks

Abstract:
In 1305 al-Hawārī completed his commentary on Ibn al-Bannā''s famous arithmetic book *Talkhīṣṣ aʿmāl al-ḥisāb*. This is the only commentary, apart from Ibn al-Bannā''s own, to have been written during the author's lifetime. What distinguishes al-Hawārī's book from the numerous later commentaries is its focus on numerical examples of the rules of calculation. We present here what we know about the author, his book, its salient features, and its influences.

Keywords: al-Hawārī, Ibn al-Bannā', Ibn Ghāzī, Şeker Zāde, arithmetic, double false position, algebra.

The famous mathematician and jurist Ibn al-Bannāʾ (1256-1321) wrote his *Talkhīṣ aʿmāl al-ḥisāb* (*Condensed [Book] on the Operations of Arithmetic*) as a concise, easy to memorize introduction to the rules for calculating with Indian numerals on whole numbers, fractions, and roots, and for finding unknown numbers by double false position and algebra. The work was completed sometime before 1301 and became one of the most popular textbooks in Western Islam. It has been widely accessible to scholars since it was edited and translated into French first by Aristide Marre in 1865, then again by Mohammad Souissi in 1969.[1]

Popular and brief books naturally inspired commentaries. In fact, the first commentary of the *Talkhīṣ* was written by Ibn al-Bannāʾ himself. In 701H/1301 he completed his *Rafʿ al-ḥijāb ʿan wujūh aʿmāl al-ḥisāb* (*Removal of the Veil from the Face of the Workings of Arithmetic*) in which he expanded the text of the *Talkhīṣ* by adding further explanations of the techniques, some numerical examples, and elements of number theory and combinatorics.[2] Among the many authors who composed commentaries of the *Talkhīṣ* after the death of the author are al-Ghurbī (2nd half 14th c.), al-Mawāhidī (ca. 1382), Ibn Zakariyyā (d. 1403-4), Ibn Qunfudh (d. 1407-8), al-ʿUqbānī (d. 1408), Ibn Haydūr (d. 1413), Ibn Majdī (d. 1447), al-Ḥabbāk (d. 1463), al-Qalaṣādī (d. 1486), and Muḥammad al-Ghazzī (16th c.). In the other direction, Ibn al-Hāʾim (d. ca. 1412) wrote an even more condensed version of the *Talkhīṣ*, and Ibn Marzūq (d. 1438), al-Wansharīsī (d. 1548-9), Ibn Ghāzī (d. 1513), and Ibn al-Qāḍī (d. 1616) put the contents of the *Talkhīṣ* to verse.[3]

One person is known to have written a commentary of the *Talkhīṣ* while the author was still living. Al-Hawārī (fl. early 14th c.), one of Ibn al-Bannāʾ's students, completed his *al-Lubāb fī sharḥ Talkhīṣ aʿmāl al-ḥisāb* (*The Essential Commentary on Condensed [Book] on the Operations of Arithmetic*) in 1305. We are currently preparing an edition, translation, and commentary of this book, and in this article we present what is known about the author, followed by salient features of the work and its later influences.

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[1] Marre 1865; Ibn al-Bannāʾ 1969.

[2] Ibn al-Bannāʾ 1994; Aballagh & Djebbar 2001, 59, 99-104. Lamrabet 1994, 82 mistakenly attributes the date of AH 701 to the *Talkhīṣ*.

[3] See: al-Ghurbī L #399, al-Mawāhidī L #414, Ibn Qunfudh L #425 RI #780, Ibn Zakariyyā RI #793 [M1], al-ʿUqbānī L #428 RI #781, Ibn al-Hāʾim RI #783, Ibn Haydūr L #429, Ibn Marzūq L #435, Ibn Majdī RI #815, al-Ḥabbāk L #445 RI #831, al-Qalaṣādī L #454 RI #865, Ibn Ghāzī L #468 RI #913, al-Wansharīsī L #475, Ibn al-Qāḍī L #514, Muḥammad al-Ghazzī RI #998. References: "L #428" refers to mathematician #428 in Lamrabet 1994, and "RI #831" is mathematician #831 in Rosenfeld & İhsanoğlu 2003.

The author

Al-Hawārī’s full name is given in the manuscripts of his book as ʿAbd al-ʿAzīz ibn ʿAlī ibn Dāwud al-Hawārī al-Miṣrātī.[4] The *kunya* “al-Miṣrātī” refers to the tribe or the city of the Libyan Miṣrāta. Following Lamrabet and Cheddadi, this implies that al-Hawārī hailed from the Berber tribe named Hawārī which emigrated from Libya to Morocco before the ninth century.[5]

Al-Hawārī relates in his chapter on double false position that Ibn al-Bannāʾ dictated to him certain procedures “on Wednesday, the twenty-eighth of the month of Rajab in this year”. This can only be the year 704H, so the date corresponds to Gregorian February 24, 1305, and it places al-Hawārī in Marrakesh at the time. Less than four months later he finished writing his commentary. The Madina MS reports that it was completed on Saturday, 18 Dhū al-Qaʿda, 704H, which corresponds to Gregorian June 12, 1305.

Al-Hawārī is not known to have written any other books, and we know nothing more of his life or career.[6] Four modern sources give the date of his death as 744 or 745H, but we have not found any support for this in the manuscripts.[7]

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[4] Some manuscripts show minor variations due to copy errors, such as “al-Hawāzī” for “al-Hawārī” and “Dāwūd” for “Dāwud”. Also, some historians write “al-Huwārī”. The only MS we have seen that indicates the vowel is the title page of Escorial 953/1, reproduced in [Ibn al-Bannāʾ 1969]. There it is “al-Hawārī”. The vocalization “Hawārī” (as opposed to “Huwārī”) is indicated on the title page of Madrid, Escorial MS 953/1.

[5] [Lamrabet 1994, 97-8; Ibn Khaldūn 2006 vol. 3, 460]. The Moroccan historian al-Mannouni shows the *kunya* instead as al-Misrātī, with an “s”, not “ص” [Mannouni 1977, 28]. This orthography is used in [Ibn al-Bannāʾ 1969] and repeated by Lamrabet, and [Aballagh & Djebbar 2001]. All the MSS show it as “ص”.

[6] Lambrabet indicates that al-Hawārī later settled in the city of Sebta (Ceuta) on the Strait of Gibraltar where he served as a civil notary and gave courses to some known contemporaries [Lamrabet 2008, 34]. Ibn al-Khaṭīb gives the full name of the notary as Abū Fāris ʿAbd al-ʿAzīz al-Hawārī. Ibn al-Qāḍī gives another name: ʿAbd al-ʿAzīz ibn Ibrāhīm ibn ʿAbd al-ʿAzīz ibn Aḥmad Abū Fāris al-Hawārī al-Jazīrī al-Sibtī (1220-1301) [Ibn al-Qāḍī 1977 vol. 2, 133-4]. A well known notary (*ʿadl*) in Sebta, al-Hawārī taught many students around 689H (1290-1): Qāsim Ibn Yūsuf al-Tujībī (1271-1329), Shaykh Abū al-Ḥasan ibn Sulayman al-Qurṭubī (d. 1329) and Muḥammad Ibn Jābir (1274-1348). These two characters do not match our al-Hawārī since their father and grand-father differ.

[7] In their catalogs of manuscripts in Madina both Tashkandy and ʿUmar Riḍā Kaḥḥāla give the year of al-Hawārī’s death as 745H (1344-5). The Madina MS is dated the following

The manuscripts

We consulted five manuscripts for our edition:

Madina, MS Ḥikmat 21 ḥisāb. 63 ff, 16 lines per page, 16 x 21 cm. This is followed by a short work of one page commencing "What was said about subtraction by seven". The title is given in the explicit as: *al-Lubāb fī sharḥ Talkhīṣ aʿmāl al-ḥisāb li Ibn al-Bannāʾ* ("The Essence of Commentary on Condensed [Book] on the Operations of Arithmetic by Ibn al-Bannāʾ")[8]. This manuscript is the oldest of the four. The copyist completed it on 18 Rabīʿ I, 746H (Gregorian July 19, 1345).

Istanbul, Süleymaniye Library, MS Şehit Ali Paşa 1977/2, ff. 54a-103b. Copied 20 Ramaḍān 880H/11 January 1476 in Constantinople. This MS carries the title *Sharḥ al-Talkhīṣ fī al-ḥisāb lī al-Hawārī* ("Commentary on the Condensed [Book] on Arithmetic by al-Hawārī"). The copyist distinguished Ibn al-Bannāʾs text from al-Hawārī's comments by placing in front of Ibn al-Bannāʾs extracts the letter "ص" (*ṣād*), which stands for *muṣannaf* (original text),[9] while passages of al-Hawārī are preceded by the letter "ش" (*shīn*), which is the first letter of the word *shārḥ* (commentary). The London, India Office MS (see below) employs the "ص" and "ش" similarly.[10]

Oxford, MS Marsh 378/3, ff. 109a-162a. Dated 1444 according to Woepcke.[11] Here the title is *Kitāb al-lubāb, wa huwa sharḥ al-Talkhīṣ li Ibn al-Bannāʾ fī ʿilm*[12] *al-ḥisāb* ("Essential Book, which is

year, but there is nothing in it to indicate whether the author was still living or not. Lamrabet gives the date "vers 744H/1345", and Bābānī writes "745" [Riḍā Kaḥḥāla 1973 vol. 4, 894; Tashkandy 1974, 110; Lamrabet 1994, 97; Bābānī 1951 vol. 1, 582].

[8] This is the same title of the Cairo manuscript (see below).

[9] [Ben Cheneb 1920].

[10] [Loth 1877, 225].

[11] [Ibn al-Bannāʾ 1969, 8].

[12] MS has "*al-ʿilm*" in error.

a Commentary on the Condensed [Book] of Ibn al-Bannāʾ on the Science of Arithmetic").

Tunis, National Library of Tunis MS 9940. 32 ff., 22 x 26 cm, 29 lines per page. Copied 4 Jumādā II, 1082H/October 8, 1671 in Damascus. The title is given here merely as *Kitāb sharḥ al-Talkhīṣ* ("Book of Commentary on the *Talkhīṣ*"). In this manuscript Ibn al-Bannāʾs passages are preceded by the letter "م" (*mīm*), which stands for *matn* or *muʾallaf*, both of which mean "[original] text" and al-Hawārī's comments start with a "ش" (*shīn*) for *sharḥ* (commentary).

Tehran, Library of Parliament MS 2672/2 ff. 10a-56b, copied before 972H/1564. The title page shows *Kitāb al-lubāb fī sharḥ Talkhīṣ aʿmāl al-ḥisāb* ("The Essence of Commentary on Condensed [Book] on the Operations of Arithmetic").

Other known manuscripts, which we could not access, are:
Madrid, Escorial MS 948/2, incomplete, copied 867H/1462
Madrid, Escorial MS 953/1, ff. 2b-79a, copied before 957H/1550
London, India Office MS 770/3 ff. 19b-69b copied 856H/1452
Rich (Morocco), Ḥamzāwiyya Library MS 145/2, 117 ff
Tamegroute (Morocco), al-Khizāna al-Nāsiriya MS 3080
Rabat, Bibliothèque Générale MS Q846, 69 ff.
Rabat, Bibliothèque al-Ḥasaniyya MS 2186/2, 97 ff, incomplete and mixed up
Damascus, al-Ẓāhirīya Library MS 6666/1, ff. 1b-112b, copied 1002H/1594

Cairo, Central Library of Islamic Manuscripts of the ministry of *waqf*, MS 1077, 80 ff. Copied 1270H/1853

Two manuscripts are reported to contain al-Hawārī's commentary but do not: Istanbul, SM Laleli 2780 and Cairo, Falak 6829/1. ff. 1a-53b, copied 1050H. The second of these is probably Ibn Majdī's commentary. The new digital catalogue of the Cairo Library confirms that MS 6829 is Hawārī's *sharḥ*.

Comparing al-Hawārī's *al-Lubāb* with his masters' *Talkhīṣ* and *Rafʿ al-ḥijāb*

In *Rafʿ al-ḥijāb* Ibn al-Bannāʾ expounded on his *Talkhīṣ* with further explanations, linguistic justifications, as well as some numerical examples. By contrast, al-Hawārī wrote in his introduction that his main goal was to provide numerical examples lacking in both the *Talkhīṣ* and *Rafʿ al-ḥijāb*. In fact, the book contains very few elaborations on Ibn al-Bannāʾs rules. Al-Hawārī reproduces the entire text of the *Talkhīṣ* and places his numerical examples after each of Ibn al-Bannāʾs explanations.

We have found that 32 of the 284 numerical examples in al-Hawārī's commentary are taken from *Rafʿ al-ḥijāb*. The following table gives the distribution by type of the number of these borrowed examples. Ibn al-Bannāʾs book is not mentioned in any of them:

	Whole numbers	Fractions	Roots	Scales	Algebra	Total
al-Hawārī's own	96	32	54	6	64	252
from *Rafʿal-ḥijāb*	1	13	10	3	5	32

In addition, eight problems on summing numbers from *al-Lubāb* are also in Ibn al-Yāsamīn's (d. 1204) *Talqīḥ al-afkār fī'l-ʿilm bi-rushūm al-ghubār* ("Grafting of opinions of the work on dust-figures"). [13] In an unrelated passage al-Hawārī quotes Ibn al-Yāsamīn, so of course he was familiar with that mathematician's work. Most of these summation problems are standard examples that are found in every book that explains the rules, such as adding the consecutive numbers from one to ten, adding the consecutive cubes of one to ten, etc.

The numerical examples in al-Hawārī that are not in *Rafʿ al-ḥijāb* are also not in any other known source, including Ibn al-Bannāʾs *al-*

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[13] [Ibn al-Yāsamīn 1993, 136-144].

Maqālāt al-arbaʿ,[14] his algebra book *al-Uṣūl wa'l-muqaddimāt fī'l-jabr wa'l-muqābala* (henceforth *Algebra*),[15] or in the arithmetic books by al-Ḥaṣṣār, Ibn al-Yasamīn, or Ibn Munʿim.[16] It thus appears that these examples are al-Hawārī's own. The vast majority of al-Hawārī's problems are purely arithmetical. Very few of them are framed in the *muʿāmalāt* (business) style.

Al-Hawārī includes seven passages from *Rafʿ al-ḥijāb* that are indicated as coming from the *Talkhīṣ*, but which are not in Souissi's edition.[17] It seems that in the course of dictating his *Talkhīṣ* Ibn al-Bannāʾ inserted material from his commentary. In three other places al-Hawārī mentions explanations given by Ibn al-Bannāʾ in which he cites *Rafʿ al-ḥijāb* explicitly.[18]

One passage in al-Hawārī's book belongs to the *Talkhīṣ* but is absent in Souissi's edition. In the section on summing number we find "If the disparity of the numbers is a known number other than doubling, then multiply the disparity by the number of numbers less one. Adding the first number to the result gives the last number. Add it to the first, and multiply it by half of the number of numbers. It yields the required answer."[19] This passage is also attributed to the *Talkhīṣ*

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[14] This book was edited by A. S. Saidan with the title *Māqālat fī'l-ḥisāb* [Ibn al-Bannāʾ 1984].

[15] Edited under the title *Kitāb al-jabr wa'l-muqābala*, again by Saidan [Saidan 1986, vol 2], and translated into French by Djebbar in his unpublished doctoral thesis: "Mathématiques et Mathématiciens dans le Maghreb médiéval (IXe - XVIe s.)", volume II. Université de Nantes, 1990.

[16] Specifically, al-Ḥaṣṣār's *Kitāb al-bayān wa'l-tadkhār fī ṣanʿat ʿamal al-ghubār* (latter 12th c.), Ibn al-Yasamīn's *Talqīḥ al-afkār* mentioned above, and Ibn Munʿim's *Fiqh al-ḥisāb* (12th-13th c.).

[17] Paragraphs taken from *Rafʿ al-ḥijāb* but absent from *Talkhīṣ*: how to add successive numbers starting with a number greater than 1 (p. 228.13-16 in [Ibn al-Bannāʾ 1994]), repeated subtractions (p. 245.8-15), two definitions of division (p. 263.4-9), other lesser known methods of denomination (p. 267.11-14), subtraction of fractions with repeated "and less" (also in *Maqālat*) (p. 275.8-9), two kinds of fraction conversions (p. 279.18-280.20), and rational vs. surd roots (p. 283.3-7).

[18] Madina MS, ff. 21b.-6, 50a.-3, 51a.-2.

[19] Madina MS fol. 8b.-3. This should be placed just after [Ibn al-Bannāʾ 1969, 42.12]. The disparity is the difference between consecutive terms. In the example given by al-Hawārī six numbers, starting with 10, have a disparity of three. So the sequence is 10, 13, 16, 19, 22, 25. So 3 (the disparity) multiplied by 5 (the number of number less one) gives 15,

in al-Qalaṣādī's commentary.[20] This section on addition of numbers, including the added paragraph, is nearly identical word-for-word to the treatment in Ibn al-Yāsamīn's *Talqīḥ al-afkār*. Ibn al-Bannāʾ seems to have copied it from this book .[21]

A little before this al-Hawārī ascribes the following passage to the *Talkhīṣ*: "If the situation is different, then multiply the remainder by the first <square> to get the required number. A different situation is when the first square is not one." The first sentence is in Ibn al-Yāsamīn's *Talqīḥ al-afkār*, Souissi's edition of the *Talkhīṣ*, and al-Qalaṣādī's commentary, but the second sentence is not in any of them.

Still other passages in *al-Lubāb* attributed to the *Talkhīṣ* are not in Souissi's edition, or in any other known text, including the commentaries by al-ʿUqbānī and al-Qalaṣādī. Here are the translations, with the location in Souissi's edition of the *Talkhīṣ* where they would have gone:

> "And by 'power of ten' (*ʿaqd*) we mean that the first <non-zero> digit is equal to a ten or a hundred or the like." *Place after [Souissi 1969, 50.10].*[22]

> "When looking for roots fractions come in four types. In one of them the numerator has a rational root and the denominator does too. Work it out as explained above. In the second type neither of them has a rational root, so work it out by the first rule. In the third type the denominator has a rational root but the numerator does not have a rational root. For this type one can work it out by the first rule or by the second. In the fourth type the numerator has a rational root and the denominator does not have a rational root, so work it out by the first rule". *Place after [Souissi 1969, 64.18]. The rules, with al-Hawārī's numerical examples (omitted here), are also covered in [Ibn Ghāzī 1983, 155ff], but with no direct quotations.*

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which added to 10 (the first number) gives 25 (the last number). The rule for the sum is evident in the example, too.

[20] [al-Qalaṣādī 1999, 52].

[21] [Ibn al-Yāsamīn 1993, 136.6-12]. See [Djebbar 2002, 221].

[22] This is paraphrased in [Ibn Ghāzī 1983, 90.14-15].

"Another way to do this is to divide one of the numbers by the other. Then take the difference between the result and one, and then multiply by the divisor to get the required root. If the subtrahend or the minuend is more than one root or less, or if the ranks of their roots are different, then it is necessary to write them as one root or make them the same rank, like in addition." *Place after [Souissi 1969, 65.15]. Again, al-Hawārī includes numerical examples after each rule.*

Symbols and notations in al-Hawārī's book

Al-Hawārī's book shows Indian (i.e. "Arabic") notation for numbers, including fractions. The particle *illā* ("less") is the only other symbol used in the notation. As is typical of Arabic books, the notation for numbers appears only in figures and not as part of the running text. Unlike most of the later commentaries of the *Talkhīṣ*, al-Hawārī does not use the Maghrebian algebraic notation.

Al-Hawārī explains the shapes of the nine figures by quoting a poem:

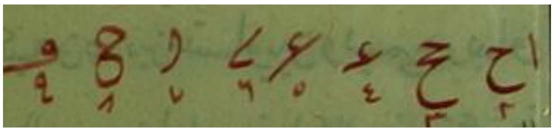
Someone has written a poem about them:
Alif and *ḥā*ʾ then *hajja* followed by *ʿuw*
and after the *ʿuw* by an *ʿayn*. Draw
*Hā*ʾ followed by an evident figure
looking like an anchor, and also you position
two zeros for eight with an *alif* between them
and *wāw* is the ninth <digit>; so understand it.[23]

This poem gives a mnemonic way to remember the Western forms of the digits. "Alif" stands for 1; *ḥā*ʾ ح () for 2; *ḥajja* is the superposition of the letter ح over the letter ج () for 3, *ʿū* عو () for 4, *ʿayn* ع () for 5, a *hā*ʾ ه () for 6, and an anchor () for 7,

23 Madina MS, ff. 3b-4a.

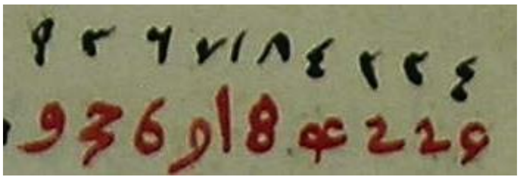
two zeros and an *alif* between them () for 8, and the letter *wāw* () for 9.[24] .

The writer of the Madina MS seems to have been an Eastern Arab who was not familiar with these forms, for he stricly follows the shapes described in the poem, below which he writes the Eastern forms (f. 4a):

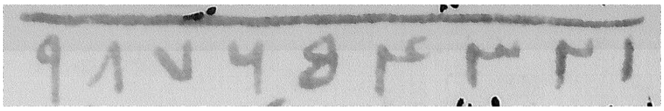


The Eastern forms are likewise written alongside the Western numerals for the next three folios, but after that only the Western forms are shown.

The Istanbul MS (f. 59a) shows the Western figures. Below is the number 9,367,184,225. In this one instance someone has written the Eastern forms above:



The Oxford MS shows the Eastern forms (f. 111b):



The Tunis MS (f. 3b) shows the Western forms with the poem, but reverts to the Eastern forms elsewhere:

[24] The figures for these numerals are taken from the Istanbul MS.


Numerals with the poem


The number 9,367,184,225

The notations for fractions in al-Hawārī's book originated before the twelfth century in Andalus or in the Maghreb. They are found in extant books written by al-Ḥaṣṣār, Ibn al-Yāsamīn, Ibn Munʿim and in Ibn al-Bannāʾs *al-Maqālāt al-arbaʿ*. Ibn al-Bannāʾ describes each type in the *Talkhīṣ*, but shows no examples. The following examples are given by al-Hawārī:

Simple (*basāʾiṭ*) fractions

$\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{9}, \frac{1}{10}$ and also $\frac{1}{11}$ and $\frac{1}{17}$.

Related (*muntasib*) fractions[25]

$\frac{2\ \ 4\ \ 5}{3\ \ 5\ \ 6}$ (in modern notation $\frac{5}{6}+\frac{4}{5\cdot 6}+\frac{2}{3\cdot 5\cdot 6}$.)

Distinct (*mukhtalif*) fractions

$\frac{5}{6}\ \frac{4}{5}$ (in modern notation $\frac{4}{5}+\frac{5}{6}$.)

Partitioned (*mubaʿʿiḍ*) fractions

$\frac{5 \bullet 3}{6 \bullet 4}$ (in modern notation $\frac{3\cdot 5}{4\cdot 6}$.)

Like his professor Ibn al-Bannāʾ, the author of *al-Lubāb* considered finding the numerator of a fraction to be an important task. Most numerical examples on fractions are aimed at finding the numerator (*basṭ*). For example:

- The numerator of the simple fraction $\frac{7}{9}$ is 7.
- The numerator of the related fraction $\frac{2\ \ 3\ \ 4\ \ 5}{3\ \ 5\ \ 7\ \ 8}$ is 596.

[25] All fractions are read right to left.

- The numerator of the distinct fractions $\frac{4}{6}\frac{1}{2}\frac{5}{7}$ is 122.
- The numerator of the partitioned fraction $\frac{3\bullet 5\bullet 7}{10\bullet 6\bullet 9}$ is 105.
- The numerator of the diminished fraction of disconnected type $\frac{1}{9}$ إلا $\frac{6}{8}$ is 46. (in modern notation $\frac{6}{8}-\frac{1}{9}$).

Mixed fractions require explanation. If the integer is on the right, like $\frac{3\ \ 5}{4\ \ 6}5$, it is added to the fraction. In modern notation this example is $5+\frac{5}{6}+\frac{3}{4\cdot 6}$. If the integer is on the left, it is multiplied by the fraction. Al-Hawārī's $10\frac{6}{8}\frac{4}{7}$ is to be read as $\left(\frac{4}{7}+\frac{6}{8}\right)\cdot 10$. When the integer is between two fractions it can be read three different ways. The example $\frac{3}{6}5\frac{4}{9}$ might mean $\left(\frac{4}{9}\cdot 5\right)+\frac{3}{6}$, $\frac{4}{9}+5+\frac{3}{6}$, or $\frac{4}{9}+\left(5\cdot\frac{3}{6}\right)$.

Operations on numbers

The algorithms presented by al-Hawārī are for use on a dust-board or other erasable surface. The layout of the numbers is the same as in the books of his predecessors. He typically writes the numbers in two parallel lines and the result above them as in these two examples:

To subtract 4968 from 5035, the work is shown as $\begin{array}{c}0067\\5035\\4968\end{array}$.

To multiply $\frac{1\ \ 4\ \ 3}{5\ \ 6\ \ 9}$ by $\frac{1}{3}\frac{3}{4}$ one writes $\begin{array}{c}\frac{1}{3}\frac{3}{4}\\ \frac{1\ \ 4\ \ 3}{5\ \ 6\ \ 9}\end{array}$, with the result $\frac{1\ \ 0\ \ 0\ \ 4}{4\ \ 5\ \ 6\ \ 9}$.

Roots (Madina MS ff. 40b-50a).

Al-Hawārī's treatment of roots is illustrated by not less than sixty numerical examples, most of them not known from prior works. In many instances he borrows passages from *Rafʿ al-ḥijāb*:

• the definition of rational and irrational numbers (Ibn al-Bannāʾ 1994, 283.3-7)

• the list of sufficient conditions for a number not to be a perfect square (pp. 284.14-285.1)

• conditions sufficient for a number to be a perfect square (pp. 285.2-6)

• applying the algorithm for computing the square root of 625 (pp. 284.4-9) and 729 (pp. 283.13-284.14).

• the arithmetical definition of binomials and apotomes (pp. 287.9-288.19), though the numerical examples are absent from *Rafʿ al-ḥijāb*.

Al-Hawārī's work is similar to the chapter on roots in Ibn al-Bannāʾs *Algebra* but the numerical examples in the two books are different except for the calculations of the roots of the binomial $8+\sqrt{60}$ and its apotome $8-\sqrt{60}$.[26]

The method of scales (Madina MS ff. 50a-56a)

Double false position was called the "method of scales (*kiffāt*)" by many Arabic mathematicians because of the shape of the diagram used in working out the problems. Ibn al-Bannāʾ wrote in the *Talkhīṣ* "the method of scales is from the art of geometry".[27] Randy Schwatrz wrote about the confusion that this statement has caused. Al-Qalaṣādī suggested that Ibn al-Bannāʾ linked double false position with geometry because of the drawing of the scales, and Franz Woepcke tried to read هندسة (geometry) as هند (Indian) to make the association instead with Indian numerals.[28] Al-Hawārī finally clarifies the statement: "It comes from the art of geometry because the ratio of the error of each scale to the difference between the scale and the unknown number is as the ratio of the assigned

[26] [Saidan 1986 vol 2, 510-533]. The common examples are on pp. 526-7.
[27] [Ibn al-Bannāʾ 1969, 88].
[28] [Schwartz 2005, 278, al-Qalaṣādī 1999, 233; Woepcke 1863, 510-513]

number to the unknown.[29] -This comes from switching and separating. Switching the ratio of the <calculated> portion to its scale to the ratio of the assigned number to the unknown, as was made clear in *Rafʿ al-ḥijāb*." There, in his own commentary, Ibn al-Bannāʾ wrote "the ratio of the error of each scale to the difference between the scale and the unknown number is as the ratio of the assigned number to the unknown".[30] So by "geometry" Ibn al-Bannāʾ was referring to the *geometric ratio* of numbers. Later Ibn Ghāzī would write "The above shows that the scales are one of the two kinds of geometrical proportions".[31]

After illustrating the method with five numerical examples, al-Hawārī writes that Ibn al-Bannāʾ dictated to him three more procedures used in the method of scales. These procedures are not found in Ibn al-Bannā''s other known published works:[32]

> Regarding this, my professor, the jurist and great erudite Abū al-ʻAbbās <Ibn al-Bannāʾ>, God bless him, dictated to me while I was studying with him on Wednesday, the twenty-eighth of the month of *Rajab* in this year:[33]
>
> There are three procedures. One of them is that you multiply the difference between the scales by one of the errors. If the two errors both exceed or both fall short, you divide the product by their difference. If one of them exceeds and the other falls short, you divide the product by their sum. If the selected error falls short, multiply the result of the division by the scale associated with the error. You add the result of this to the scale which you multiplied by its error if it falls short, and you subtract it from it if it exceeds. This gives the required <number>.
>
> The second procedure is that you multiply the difference between the scales by the sum of the errors if they both exceed or

[29] The "scale" is the presumed (false) solution that is placed in the scale in the diagram. The "error of each scale" is the difference between the value calculated with the scale and the given "assigned" result.

[30] [Ibn al-Bannāʾ 1994, 297.17-18].

[31] [Ibn Ghāzī 1983, 209.16].

[32] Madina MS, ff. 53b-54a.

[33] Gregorian February 24, 1305.

both fall short, and you divide by their difference. If one of them exceeds and the other falls short, you multiply the difference between the scales by the difference between the errors, and you divide by the sum of the errors. Keep this result in mind. If you want, you can add this remembered number to the difference between the scales and take half the sum. You add it to the scale with the larger error if it falls short, and you subtract it from it if it exceeds, to get the required <number>.

And if you wanted, take the remembered number and the difference between the scales, and subtract the smaller from the larger, and take half of the remainder. Add it to the scale with the smaller error if it falls short, and subtract it from it if it exceeds, to get the required <number>.

The third way is that you multiply the difference between the scales by your assigned number. If the errors both exceed or both fall short, you divide the product by their difference, and if one of them exceeds and the other falls short, you divided the product by their sum, to get the required <number>. So know it.

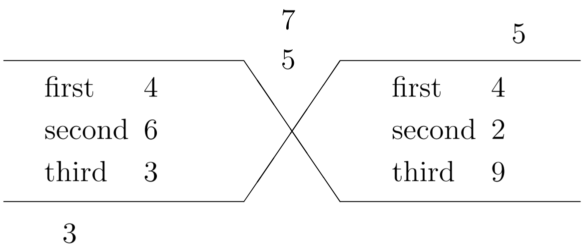
While all of the problems solved by double false position in Ibn al-Bannā’’s *Talkhīṣ* are based on proportion, al-Hawārī adds two intricate problems and their solutions from *Rafʿ al-ḥijāb* in which “scales are used to find the unknown in cases without proportions”.[34] No figures are found in extant copies of *Rafʿ al-ḥijāb*, but in his *al-Lubāb* al-Hawārī adds one figure for each problem. Below are translations of the problems with al-Hawārī’s diagrams:[35]

Three men want to buy a horse. The first says to the second, give me half of what you have, and together with what I have, I will have the price of the horse. The second says to the third: give me a third of what you have, and together with what I have, I will have the price of the horse. And the third says to the first, give me a fourth of what you have, and together with what I have, I will have the price of the horse.

[34] [Ibn al-Bannāʾ 1994, 299.3-300.22].
[35] Starting at Madina MS, fol. 54a.18.

We choose one scale for the three men. We assign to the first man whatever we want, so we make it four. The second also has whatever we want, so we make it two. Then we calculate the price of the horse, which is five. We put it above the dome,[36] which is what we will confront. The third man has, by calculation, nine. So if we added a fourth of what the first has, it comes to ten. The error of the first scale with these three numbers is five, which exceeds.

Then we turn to the other scale. We fix four for the first man, the same as we chose for him in the first scale, and we make the second whatever we want. We see that it should not be eight or more, since it would leave the third with nothing. So know it. So we make it six. This makes the price of the horse, which we confront, seven. We put it above the dome also, and from this the third necessarily has three. Adding this to a fourth of what the first has gives four. The error of the second scale[37] is then three, falling short. This is the figure:



So we multiply the error of each scale by what each one has in the other scale, and we divide the sum of the two products by the sum of the errors, as mentioned above, to get what each one of them has and the price of the horse. This gives us what the first has, which is the four, and the second four and a half, the third five and a fourth, and the price of the horse is six and a fourth.

From here, following Ibn al-Bannāʾ in *Rafʿ al-ḥijāb*, al-Hawārī discusses the way to adjust the answer if integers are required or if

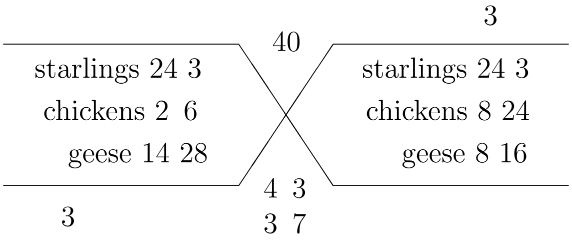
[36] "The dome" is the place above the center of the diagram.
[37] That is, after confronting this 10 with the 5.

the price of the horse is given. Also Ibn al-Bannāʾ solved this very problem by algebra in his *Algebra.* [38] Similar problems with algebraic solutions can be found in al-Karājī's *al-Fakhrī.*[39]

The second problem is:[40]

> Forty birds, among which are geese, chickens, and starlings, <all> for forty dirhams. The starlings are eight for a dirham, the chickens are one for two dirhams, and the geese are one for three dirhams. How many were chosen of each kind of bird?
>
> This type <of problem> may not have a solution. Two conditions must be satisfied. One is that the solution must consist of whole numbers, not fractions. Second, the price of the cheapest one, if multiplied by the number of birds, must be less than the total price; and the price of the most expensive bird multiplied likewise must be greater than the <total> price.
>
> It is obvious in this problem that the number of starlings must be eight or sixteen or twenty-four or thirty-two, and no other number. If there were eight, then thirty-two birds remain, and thirty-nine remains of the cost. If we check the second condition, the product of the remaining birds by the price of the cheapest one is larger than the number of the price, which is not valid. If we make the starlings sixteen, and we again check the remaining price, we find that it is also not valid. If we make the starlings twenty-four, and we again check what remains, the two conditions are met.
>
> So we suppose there are twenty-four starlings, and we assign the chickens whatever we want. Suppose they are eight. Then this leaves eight geese. The error in the price is three dirhams, which exceeds. Then we turn to the other scale. We suppose there are twenty-four starlings, the same as before, since it is a condition for this work that a number is repeated in the two scales. And we make the chickens whatever we want, but not <the same as> the first, so let it be fourteen. Then the number of geese is two. The error is three dirhams falling short. Here is the figure:

[38] [Saidan 1986 vol. 2, 570-1].
[39] [Saidan 1986 vol. 1, 224-8].
[40] Starting at Madina MS, fol. 55a.10.



> We work it out as before to get the required number. If we want we can first find the number of each type of bird, or the price of each type. The price of the starlings is three, and there are twenty-four of them. There are five geese and their price is fifteen, and there are eleven chickens and their price is twenty-two. If we suppose there are thirty-two starlings, it is not valid since the condition on the remaining <birds> is not met. Thus there is only one way to solve this problem.

Problems concerning the purchase of different kinds of birds date back at least to Abū Kāmil (d. 930), who solved them by algebra.[41]

We have found no earlier examples of indeterminate problems solved by scales. The only subsequent author we know to include them is Ibn Ghāzī in his *Bughyat al-ṭullāb fī sharḥ munyat al-ḥussāb* ("Aim of the Students in Commentary on Desire of Reckoners", 1483), but he writes that he took the problems from *Rafʿ al-ḥijāb*.[42]

A different use of *muʿādala* in algebra

The verb used to equate the two sides of an algebraic equation in medieval Arabic is *ʿadala* ("equal"), and the associated noun *muʿādala* is the word for "equation".[43] Ibn al-Bannāʾ naturally uses both words with these meanings in his *Talkhīṣ*, but he also introduces a different meaning of *muʿādala* with language that is not at all clear. After briefly presenting the meanings of *al-jabr*

41 [Sesiano 1999, 79-83].
42 [Ibn Ghāzī 1983, 218-222].
43 See [Oaks 2010] for a study of *ʿadala* in Arabic mathematics.

(restoration) and *al-muqābala* (confrontation), he writes "And *muʿādala* is that you restore the deleted to the appended, and you subtract the appended from the appended and the deleted from the deleted of things of the same kind."[44] His explanation of this passage in *Rafʿ al-ḥijāb* does little to help: "And restoring the deleted to the appended is *al-muʿādala* with restoration, and subtracting the appended from the appended and the deleted from the deleted of things of the same kind is *al-muʿādala* with confrontation."[45] At least here he indicates that somehow one performs "*al-muʿādala* with restoration" and "*al-muʿādala* with confrontation". Al-Qalaṣādī, in his commentary, does not explain *muʿādala* at all. He only describes the simplification of equations with *al-jabr* and *al-muqābala*.[46]

Al-Hawārī and al-ʿUqbānī clear up the confusion in their commentaries. Their specific examples are worth reviewing because there may be a link between this new meaning of *muʿādala* and the phrase *ugagliare le parti* (equalize the parts) in medieval Italian algebra. Al-Hawārī gives two examples. He does it not in the context of equations, but for the subtraction of one algebraic expression form another in which one of them has a deleted term (in an expression like "a *māl* less three things" ($x^2 - 3x$) the "*māl*" is a *diminished* quantity from which "three things" have been *deleted*[47]). Subtraction problems are like equations in that any amount added to or subtracted from one part must also be done to the other part.

The first example is to "subtract two and a thing ($2 + x$) from a *māl* less three things ($x^2 - 3x$)". He transforms the problem into an equivalent subtraction by restoring (*al-jabr*) the diminished *māl*, which amounts to making the $x^2 - 3x$ a full x^2, and by "balancing" or "equalizing" (*muʿādala*) the problem by adding $3x$ to the other part:

---

[44] [Ibn al-Bannāʾ 1969, 73.9].
[45] [Ibn al-Bannāʾ 1994, 309.7].
[46] [al-Qalaṣādī 1999, 248].
[47] [Oaks & Alkhateeb 2007, ğ3.5].

> We begin with the things in the diminished minuend, which is three. We add it to the two parts at once. This is what is meant by equalization (*al-muʿādala*) with restoration (*al-jabr*), since we add to the *māl* what it is associated with, which is three things. Thus we restore <the *māl*> by addition, which removes the deletion. It becomes larger, as we want for the minuend. Then we equalize (*nuʿādilu*, from the verb *ʿādala* for *muʿādala*), by which we restore the subtrahend by the amount we added in the minuend, which is by the three things. This works because subtracting two numbers gives the same result as subtracting them after we added a number to both of them, or after subtracting a number from both of them. Once this is done, the problem becomes: If someone said "subtract two and four things from a *māl* ". Then work it out as before. The remainder is the required number, which is a *māl* less four things and less two. So know it.

The second problem is to "subtract fifty-two dirhams less five things ($52-5x$) from two cubes and thirty dirhams ($2x^3+30$)". Here al-Hawārī first does a "restoration with equalization", followed by a "confrontation with equalization":

> We begin with the things in the diminished subtrahend, which are five. We add it to the two sides at once, as before. The problem becomes: if someone said, subtract fifty-two dirhams from two cubes and five things and thirty dirhams.
>
> Now consider the dirhams on the two sides. We take away the smaller quantity from the two sides at once. The problem reduces to: if someone said, "subtract twenty-two dirhams from two cubes and five things". This is confrontation (*al-muqābala*) with equalization (*al-muʿādala*), since we took thirty away from the subtrahend. It becomes smaller than the original subtrahend. So we equalize, by which we take from the minuend the same amount we subtracted from the subtrahend. Work it out as before. We work it out with the remainder as before. So remove what remains. This is the required amount, which is two cubes and five things less twenty-two dirhams.

Today we perform a subtraction like $(2x^3 + 30) - (52 - 5x)$ by distributing the minus sign and combining like terms. But in medieval algebra there was no equivalent of "distributing the minus sign". The $52 - 5x$ was regarded as a deficient or diminished 52, so the rule for subtracting it required some explanation. Al-Khwārizmī gave proofs with geometric diagrams to justify the results of adding or subtracting diminished quantities, while most later algebraists, al-Hawārī included, worked them by "restoring" the diminished term(s) and adding the same amounts to the other side. We have not seen the "confrontation" (*al-muqābala*) of like terms in subtractions like this in other books.

In his commentary on the *Talkhīṣ* al-ʿUqbānī illustrates this meaning of *muʿādala* in the context of equations. He simplifies the equation $10 - 2x = 5x - 3$ by removing the exceptions "less two things" and "less three" (the restoration), and adding the $2x$ to $5x$ and adding the 3 to 10 (the equalization) to get $13 = 7x$:

> And if you wanted to remove the troublesome exceptions from the two sides, then take them away from them, and add to each side what had been deleted from the other side. So you add two things to the five things and three to the ten. So your equation becomes: thirteen equal seven things. All of this is called restoration (*jabara*), which is what is meant by saying *al-jabr* and *al-muʿādala*, though the term *al-jabr* wa'*l-muqābala* (restoration and confrontation) is on the whole the generally accepted phrase for this manipulation of both parts. This term became the way to name the entire process, or to <name> the process without the long explanation <required by> the other term.[48]

Al-ʿUqbānī then illustrates Ibn al-Bannāʾs statement "you subtract the appended from the appended and the deleted from the deleted of things of the same kind" by simplifying $10 - 4x = 16 - 6x$ to $2x = 6$. Although al-ʿUqbānī does not say so explicitly, Ibn al-Bannāʾ called this "equalization with confrontation" in *Rafʿ al-ḥijāb*.

[48] [Harbili 1997, 381.2].

We now have a clearer understanding of Ibn al-Bannāʾs meaning of *muʿādala* (equalization). In "restoration with equalization" the restoration (*al-jabr*) of a diminished quantity only affects that quantity. To equalize or balance (*al-muʿādala*) the problem or equation, one must also add the same amount to the other side. One applies "confrontation with equalization" when there are like terms in the two parts. The confrontation is the subtraction of the smaller term from the larger, and the equalization is the removal the smaller from the other side. Equalization, then, is the compensating operation to balance the problem or equation after restoring or confronting.

Curiously, Ibn al-Bannāʾ does not use *muʿādala* for this purpose in his *Algebra*. In fact, we have not found the word used this way outside these two commentaries. It may have been invented by Ibn al-Bannāʾ while he wrote the *Talkhīṣ* to emphasize to students the need for operating on *both* sides of an equation or subtraction. Whatever its origin, it never came into common use.

Al-ʿUqbānī's remark that *al-jabr wa'l-muqābala* is "the generally accepted phrase" for restoring/confronting and equalizing is born out in many Arabic books.[49] Al-Hawārī, for example, uses the phrase *al-jabr wa'l-muqābala* this way: "a *māl* less three things equal twenty-four dirhams less five things ($x^2 - 3x = 24 - 5x$). So you restore and confront (*tajbir wa tuqābil*) as before, and the problem becomes: a *māl* and two things equal twenty-four dirhams".[50]

In place of the phrase *al-jabr wa'l-muqābala* most medieval Italian algebraists wrote either *ristora le parti* (restore the parts) or *ugagliare le parti* (equalize the parts). The earliest known use of the latter phrase is in Maestro Dardi's 1344 *Aliabraa Argibra*. In problem 44, for example, he arrives at the equation $x^4 + 11x^2 + 25 = 100$ and writes "Now equalize the parts (*raghuaglia*

[49] See [Oaks & Alkhateeb 2007, section 4.1].
[50] Madina MS, f. 59b.16.

le parte): subtract 25 numbers from each part, and you will have that $x^4 + 11x^2$ will be equal to the other part, which is 75 numbers.[51]

The meanings of the words *ristorare* and *uguagliare* in Italian algebra are modified from the Arabic. Now "restoration" (*ristorare,* from the Arabic *al-jabr*) no longer serves to restore a diminished quantity, and "equalization" (*uguagliare*, from *muʿādala*) is no longer the compensating operation. Both now take the meaning of the Arabic phrase *al-jabr wa'l-muqābala*. We do not know if the Italian *uguagliare* descends directly from Ibn al-Bannāʾs *muʿādala*, but both terms refer to a "balancing" or "equalization" of the two sides when simplifying an equation or subtraction.

Three problems dictated to al-Hawārī by Ibn al-Bannāʾ

Al-Hawārī concludes his book by presenting three recreational problems in arithmetic that were dictated to him by Ibn al-Bannāʾ. In each problem a person is asked to think of a secret number and to perform a succession of operations. At some point the person is asked to announce the value of the number resulting from the operations, from which the hidden number can be recovered.

Similar problems are found in other arithmetic books. Seven problems of this type are in Ibn al-Yāsamīn's *Talqīḥ al-afkār*, three problems are in the last chapter of *al-Maʿūna* by Ibn al-Hāʾim (1389), and fourteen such problems are in Ibn Haydūr's commentary of the *Talkhīṣ* (ca. 1400). Yaʿqūb al-Kindī (d. ca. 873) wrote a book on hidden numbers, the *Risālā fī istikhrāj al-aʿdād al-muḍmara* (*Treatise on Finding Hidden Numbers*)[52]. We have not examined it so we do not know if his problems are of the same sort that are found in the later arithmetic books.

The problems posed in the books of Ibn al-Yāsamīn, Ibn al-Hāʾim, and Ibn Haydūr are different from the problems related to

[51] [Maestro Dardi 2001, 119.24]. We have replaced Dardi's notation with modern notation. In other examples Dardi writes *raghuaglia le parti* when there is only a restoration, and also where both a restoration and confrontation are performed.

[52] MS Istanbul, Aya Sofya 4830, ff. 81a-86a. See [Djebbar 2004; Djebbar 2007].

al-Hawārī by Ibn al-Bannāʾ.[53] Because they have not been published before, we include a translation here:

> We conclude this work with three problems of witty reckoning. I do this following other arithmeticians, who continue to write similar problems at the end of their compositions.
>
> One of them is that we tell someone to subtract his <secret> number from ten, then to subtract the square of the remainder from the square of his number.
>
> If the <square of the> remainder is smaller, <we subtract it from the square of the secret number and> we ask for the remainder. We divide it by ten, and we add to the result half of its remainder from ten to get the secret number.
>
> And if the square of the remainder is larger, we subtract the square of the secret number from it, and we ask for the remainder. We divide it by the ten, and we subtract the result from ten. Half the remainder is the secret number.
>
> We can also ask the person to subtract his secret number from something other than ten. Working it all out still gives the required number.
>
> The second problem. We tell someone to partition ten into two parts secretly. Then we tell him to divide the square of one of them by their surface [i.e. product], and we ask for the outcome. If we knew this, it is the ratio of one of the two parts to the other. So we partition the ten according to this ratio.
>
> And similarly we can work with any number other than ten and partition it into two secret parts and we get them similarly.
>
> The third problem. A secret number is partitioned into two secret numbers. How much is it? And how much are the two parts?
>
> We tell him to multiply one of the parts by the other, square each one of them, then subtract the square of the smaller from the surface. Then we ask for the remainder. And he then subtracts the surface from the square of the larger, and we ask for the remainder. We then take the root of the difference between the

[53] These three problems are not included in the latest book written on Ibn al-Bannāʾ's works : [Aballagh and Djebbar 2001].

two asked numbers, which is the difference between the parts. Then we divide the sum of the asked numbers by it to get the secret number, which is the sum of the two parts. If we add to it the root of the difference between them, it gives double the larger of them, and if we subtract it from their sum, it leaves double the smaller of them. So know it.

These three problems were dictated to me by my professor, the great jurist Abū al-ʿAbbās <Ibn al-Bannāʾ>, may God be pleased with him.

Modern mathematical explanation of the first problem:

We let x be the hidden number. If $(10-x)^2 < x^2$, then set $y = x^2 - (10-x)^2$. This number is announced (given). Divide y by 10 then add to the result half of 10 less the result, i.e. calculate $\frac{y}{10} + \frac{1}{2}(10 - \frac{y}{10})$. This is the secret number x.

If $(10-x)^2 > x^2$, then set $y = (10-x)^2 - x^2$. This number is announced (given). Then the secret number $x = \frac{1}{2}(10 - \frac{y}{10})$.

Second problem

$10 = x + y$. Ask for $\frac{x^2}{xy}$. This number is announced (given), and is the ratio $\frac{x}{y}$. The solution can be obtained since we know that $x + y = 10$ and $\frac{x}{y}$ is given.

Third problem

$z = x + y$, where $x > y$ and all are hidden numbers.

Calculate $a = xy - y^2$ and announce the result, thus a is given.

Calculate $b = x^2 - xy$ and announce the result, thus b is given.

Then $x - y = \sqrt{b-a}$ and $x + y = \dfrac{a+b}{x-y}$ are now given.

Then $2x = (x+y) + (x-y)$ and $2y = (x+y) - (x-y)$.

Al-Hawārī's influence

Two biographers working in the extreme Maghreb (modern Morocco) include notices about Ibn al-Bannāʾ, his *Talkhīṣ*, and different commentators. Aḥmad ibn Muḥammad Ibn al-Qāḍī (d. 1616) and Aḥmad Bābā al-Tunbuktī (d. 1627) each name five commentators, but not al-Hawārī. Further, the modern historian Muḥammad Ḥājjī has no entry for al-Hawārī in his *Mawsūʿat aʿlām al-Maghrib*, a ten volume chronological list of well-known North African jurists and scholars. Ḥājjī compiled his list from the biographical works of Ibn Qunfudh (d. 1508), al-Wansharīsī (d. 1512), al-Shanshāwunī (d. 1578), and others.[54] The lack of any biographical information on al-Hawārī suggests that he never held any official post, and his book leaves little doubt that he himself was not a first-rank mathematician. He was apparently merely a student of Ibn al-Bannāʾ who was authorized by his master to complete the *Talkhīṣ* with his own book on arithmetic.

Despite al-Hawārī's lack of fame, *al-Lubāb* found its niche in mathematics education. Scrutiny of catalogues of libraries shows that copies of it were regularly made in the succeeding centuries and in different Islamic countries. In particular, *al-Lubāb* was used by some scholars in Istanbul. One of the extant manuscripts was copied there in 1476, and the only premodern biographer to mention al-Hawārī is the Ottoman historian Ḥājjī Khalīfa (d. 1657).[55] He makes a note of Ibn al-Bannāʾs *Talkhīṣ*, and the only commentary he mentions is *al-Lubāb*.

Apart from the extant manuscripts and the entry in Ḥājjī Khalīfa we have found three instances of al-Hawārī's influence on the teaching of mathematics: (1) the borrowings from *al-Lubāb* made by Ibn Ghāzī in extreme Maghreb in 1483, (2) the collection of problems borrowed directly from *al-Lubāb* in mid-eighteenth century Istanbul by the Ottoman mathematician Şeker Zāde, and (3) the inclusion of *al-Lubāb* in the curriculum of the Tunisian al-Zaytūna mosque-university in 1875.

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[54] [Ḥājjī 1996, vol. 2].

[55] Ibn al-Qāḍī (who also versified the *Talkhīṣ*) names Ibn Qunfudh, al-ʿUqbānī, Ibn al-Hāʾim, Ibn Haydūr, and al-Qalaṣādī [Ibn al-Qāḍī 1970], while al-Tunbuktī names Ibn Qunfudh, al-ʿUqbānī, Ibn Haydūr, al-Qalaṣādī, and Ibn Ghāzī [al-Tunbuktī 1989]. [Ḥājjī Khalīfa 2000 vol. 2, p. 400 (#3532)].

Al-Hawārī and Ibn Ghāzī

Muḥammad Ibn Ghāzī al-ʿUthmānī al-Miknāsī (1437-1513) condensed the *Talkhīṣ* into a poem of 461 verses titled *Munyat al-ḥussāb* (“Desire of Reckoners”). Later, in 890H/1483, he wrote a commentary on his poem titled *Bughyat al-ṭullāb fī sharḥ munyat al-ḥussāb* (“Aim of the Students in Commentary on Desire of Reckoners”). In addition to quoting Ibn al-Bannāʾs *Talkhīṣ* and *Rafʿ al-ḥijāb*, Ibn Ghāzī also cites many other mathematicians, including al-Ḥaṣṣār, Ibn al-Yāsamīn, Ibn Qunfudh, and Ibn Haydūr. In two places Ibn Ghāzī cites al-Miṣrātī (i.e. al-Hawārī):

> “… this <definition> has been explained in *al-Maqālāt*,[56] and this explanation is well-known; al-Miṣrātī and others based <their work> on it”. (Ibn Ghāzī 1983, 19.10-11)

> “When referring to this in *Rafʿ al-ḥijāb*, Abū Muḥammad ʿAbd al-ʿAzīz al-Miṣrātī said, …” (Ibn Ghāzī 1983, 210.3)

Ibn Ghāzī also includes a great number of paragraphs and examples from *al-Lubāb*. But while he credits his borrowings from other mathematicians, he never cites his source when it is al-Hawārī’s book. Below is a list of the numbers of problems of different types taken by Ibn Ghāzī from al-Hawārī’s book that are not in *Rafʿ al-ḥijāb*:

- Sum of numbers : 16
- Casting out nines or sevens : 3
- Types of multiplications : 4
- Computing the numerator of a fraction : 11
- Operations on fractions : 10
- Operations on roots : 26
- Double false position (scales) : 6
- Algebra : 27

[56] I.e. Ibn al-Bannāʾs book *Maqālāt fī’l-ḥisāb* (“Conversations on Arithmetic”).

Additionally, there are two places where Ibn Ghāzī quotes Ibn al-Yāsamīn and Ibn Ṭāhir, and both quotations are also in al-Hawārī. It is likely that Ibn Ghāzī took them from *al-Lubāb*.[57]

Al-Hawārī and Şeker Zāde[58] (d. 1787)

The second book we found to made use of *al-Lubāb* is *Amthilatun min Talkhīṣ Ibn al-Bannāʾ wa al-Ḥāwī li Ibn al-Hāʾim* ("Examples from Ibn al-Bannāʾs *Talkhīṣ* and *al-Ḥāwī*[59] of Ibn al-Hāʾim"). It was compiled in the mid-eighteenth century by al-Sayyīd Faydh Allah Sarmid, known as Şeker Zāde. Şeker's father, Muḥammad ibn ʿAbd al-Raḥmān al-Istanbūlī, was a renowned Ottoman calligrapher who taught him the Arabic language and the art of Arabic calligraphy. After a normal schooling in the Ottoman madrasas, Şeker Zāde was one of the best private students in mathematics and astronomy of Muṣṭafā Ṣidkī (d. 1769) and became specialized in these matters. Of his later career we know that in 1775 he was teaching mathematics and calligraphy as a *mullah* in the *madrasa* at Izmīr, and that he died in 1787.

While still a student Şeker Zāde made copies of many mathematics books by well-known authors such as Ibn al-Hāʾim, al-Qalaṣādī, and Ibn al-Bannāʾ. He also wrote his own books on the mathematics he had learned, including a sexagesimal multiplication table and an introduction to logarithms.[60] His *Amthilatun* falls into this latter category. The unique manuscript is an exercise book containing a collection of numerical examples illustrating the definitions and computational rules primarily from Ibn al-Bannāʾs *Talkhīṣ*, with some also from its derivative, Ibn al-Hāʾim's *al-Ḥāwī*.[61] Well over a hundred problems from *al-Lubāb* are included

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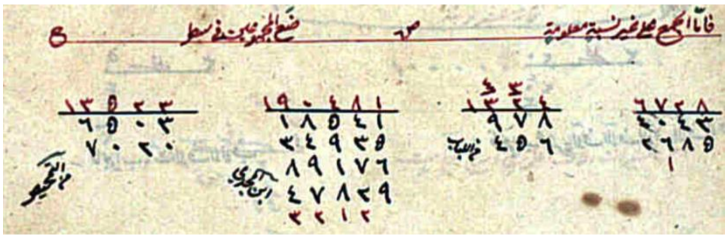
[57] Ibn al-Yāsamīn: Madina MS fol. 23b.12; [Ibn Ghāzī 1983, 90.2]. Ibn Ṭāhir: Madina MS fol. 51a.13; [Ibn Ghāzī 1983, 201.17].
[58] For more on Şeker Zāde see [Abdeljaouad 2011].
[59] This is Ibn al-Hāʾim's *al-Ḥāwī fī'l-ḥisāb* ("Gathering of Arithmetic"), an abridgement of the *Talkhīṣ*.
[60] For the complete list see [Abdeljaouad 2011, 10-13].
[61] Istanbul, Süleymaniye Library MS Esad Efendi 3150/2, ff. 10b-90a, ff. 97-98, and ff. 105-110.

in Şeker Zāde's *Amthilatun*, along with a scattering of problems taken from other books and some of the author's own invention.

Traditional Arabic arithmetic books explain rules and concepts rhetorically. Notation is only presented here and there to illustrate what one would write out when actually doing the calculations. Şeker Zāde's *Amthilatun* is remarkable in that it consists mainly of solutions to problems written entirely in notation. This is the same notation invented in the thirteenth century and popularized by al-Qalaṣādī and Ibn Ghāzī in the fifteenth century, including the notation for algebra that is absent in al-Hawārī. At the top of a typical section Şeker puts the title from the *Talkhīṣ* on the right and the similar one from *al-Ḥāwī* on the left, both written in red. Below this he puts the worked out examples in several vertical diagrams

For instance, when presenting the addition of numbers Ibn al-Bannāʾ writes a long paragraph detailing the steps of the addition scheme.[62] This paragraph is quoted by al-Hawārī[63] and is illustrated by two numerical examples. In the first example, concerning the addition of 4043 and 2685, the scheme starts from the units digit, while in the second example concerning the addition of 978 and 456, the scheme starts from the last digit (i.e. the highest power). All of al-Hawārī's explanations are rhetorical. In the excerpt shown below Şeker Zāde writes the title of the section *On adding numbers with no known relation* and under it he presents four problems taken from three different commentaries of the *Talkhīṣ*. The first two are the examples from *al-Lubāb*, the third is from Ibn Majdī's *Ḥāwī al-Lubāb* and the last is from Ibn Haydūr's *al-Tamḥīṣ*. Şeker Zāde indicates his source by writing "from *al-Lubāb*", "from Ibn Majdī" and "from *al-Tamḥīṣ*" next to the examples

[62] [Ibn al-Bannāʾ 1969, 44.13-24].
[63] Starting at Madina MS, f. 8.3.



Şeker Zāde, fol. 16b

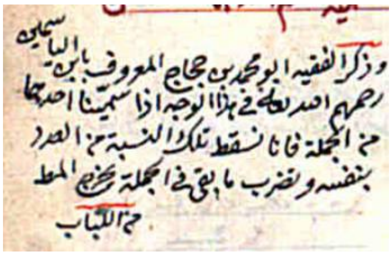
Occasionally Şeker Zāde will give a rhetorical rule or proposition taken either from the *Talkhīṣ* or from some other work on arithmetic or algebra. Here, too, each extract is attributed to the book from which it was borrowed by an expression such as "from the *Talkhīṣ*", "from *al-Lubāb"*, etc. Borrowed extracts and given numbers are usually written in black, and the results of operations are in red. The systematic alternate use of red and black inks makes the pages quite visually appealing.

The total numbers of example problems from different sources are given in the following list:

111 from al-Hawārī's *al-Lubāb*
14 from *Kitāb al-tamḥīṣ fī sharḥ al-Talkhīṣ* and 2 from *Tuhfat al-ṭullāb* by Ibn Haydūr al-Tādilī (d. 1413),
7 from *Sharḥ Talkhīṣ ʿamāl al-ḥisāb lī Ibn al-Bannāʾ* by Qalaṣādī (d. 1486),
7 from *al-Qawl al-mubdiʿ fī sharḥ al-muqniʿ* by Sibṭ al-Māridīnī (d. 1506),
6 from *Ḥāwī al-lubāb fī sharḥ al-Talkhīṣ* by Ibn Majdī (d. 1447),
16 from ten other authors.

Şeker also borrows (folio 27a) the same citation from Ibn al-Yāsamīn as Ibn Ghāzī, but he explicitly attributes it to *al-Lubāb*:[64]

[64] See footnote 53.



Şeker Zāde, fol. 27a

The Zaytūna reform

Among the 217 courses taught in 1871 at the Zaytūna mosque university in Tunis, only five were reserved for mathematics, and these were merely rhymed poems and their commentaries on arithmetic and *farāʾiḍ* (inheritance). In 1875 the Tunisian Prime Minister Khayreddine announced a new curriculum for the Zaytūna mosque structured into three cycles: lower, intermediate, and superior, to be composed of 28 disciplines with recommended textbooks for each.

Ten works were prescribed for mathematics: two for the lower cycle and four each for the intermediate and superior cycles. The superior cycle included a text on Euclidean geometry, a commentary of a work on astronomy, and two arithmetic books: Ibn Ghāzī's *Aim of the students* that we saw borrowed much from al-Hawārī, and al-Hawārī's *al-Lubāb*. It was the first time that this book appeared in the list of recommended textbooks.

This new curriculum was an attempt to impose some government control over the institution and to inject some professionalism among the faculty. Even so, the selection of books was highly traditional and outdated. Nevertheless the *ʿulamāʾ* running the mosque severely criticized it and hindered its implementation. It is likely that the new curriculum was never fully implemented, and that mathematics instruction continued to be based on rote learning of *urjūzas* (poems), and on commentaries of the linguistic and stylistic aspects of their wording, thus preventing any effective practice of mathematics.

Conclusion

Despite the obscurity of its author, *al-Lubāb* remained a popular textbook in the centuries after it was composed, making its way to Istanbul and into the curriculum of nineteenth century Tunisia.

In writing *al-Lubāb* al-Hawārī set out to explain with numerical examples all the rules expressed in Ibn al-Bannāʾs *Talkhīṣ*. He of course intended his book for Arabic speaking arithmetic students, but he has also succeeded in helping *us* better understand Ibn al-Bannāʾs book and medieval Arabic arithmetic in general. Al-Hawārī's book should be valuable today not just for historians, but also for mathematics educators as valuable testimony to the teaching of mathematics in fourteenth century North Africa.

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