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Nonlinear dynamics and bifurcation analysis in two models of sustainable development

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Summary

We show in this document two mathematical models of development. Namely, we state two systems of nonlinear differential equations with state variables which regard to ecosystem, social and economic dimensions. We analyze nonlinear dynamics through bifurcation theory and state space simulations. Complex nonlinear systems including variables from ecosystem, social and economic dimensions show that, depending on the parameter values and initial conditions, different patterns can be obtained. Some of these patterns are related to fast decay of the exhaustible resources. Thus, also fast sustainable actions must be taken into account to prevent ecological disasters.

Keywords: Sustainability modeling, Bifurcations, Nonlinear dynamics

1 Introduction

Nontrivial conclusions can be stated from the analysis of systems of nonlinear differential equations which model realistic processes. Several development models have been studied from the system dynamics framework[1]. Usually, bifurcation theory cannot be applied to these models and many specific nonlinear phenomena remain hidden.

Development can be studied in the framework of indicators, variables, components, and dimensions. In this paper we regard to key variables regarding ecosystem, social and economic dimensions. We analyze two models. In the first (oversimplified) one, we find the equilibrium points and their stability, and we study codimension-one and codimension-two bifurcations. As it will be shown, equilibrium points coalescence and disappearance, and nonlinear oscillations are found through limit point and Hopf bifurcations. In the second mathematical model we study a more complex system with five state variables related to the three dimensions mentioned above.

1.1 Bifurcation theory

Simulations of systems of nonlinear differential equations can be performed with numerical methods such as Runge-Kutta, or more elaborated schemes. When the numerical results must be shown, several options are available: time waveforms (spatial variables versus time), phase diagrams (plots in the coordinates space, or projections on a lower-dimensional state space), or bifurcation diagrams. Bifurcation diagrams can be computed regarding variation of one parameter (codimension-one bifurcations) or with the variation of two parameters (codimension-two bifurcations). Codimension-one bifurcation diagrams usually include one meaningful state variable and one parameter, while codimension-two bifurcation diagrams show bifurcation curves and bifurcation points in a two-dimensional parameter space.

Generic codimension-one bifurcations are the saddle-node (denoted by LP in the figures of this paper) and Hopf (denoted by H). Basically, the saddle-node bifurcation regards to disappearance of two equilibrium points as a parameter is varied. Instead, Hopf bifurcation regards to the appearance of sustained oscillations (also called periodic orbits or limit cycles) as a parameter is varied.

Generic codimension-two bifurcations can be classified into five different classes: Bautin, Cusp, Bodganov-Takens, Zero-Hopf and Hopf-Hopf. Specific characteristics of these bifurcations can be found in the book of Kuznetsov[2]. These codimension-two bifurcation points act as organization centers in the sense that close to these parameter values, many sorts of dynamic behavior can be found.

2 Two-dimensional mathematical model

In this section we analyze a two dimensional model corresponding to a development system of a sort of primitive human community[3]. The authors realize that it is an oversimplified model, but it has been shown to explain the main development features of the inhabitants of the Pascua Island. Basically, it is assumed that the survival of the human community is due to an exhaustible resource (wood from trees in a forest) and an inexhaustible resource (land). For the growth of the forest, it is also assumed that the forest has a finite carrying capacity, and that below a certain threshold, the forest is unable to recover. Regarding the human population, it is assumed that it grows when the utility from wood and land exceeds some average value and it decays when the utility is below this average value. Taking into account these assumptions, we can formulate a two-dimensional system of nonlinear differential equations with the trees and human populations. Details can be found in the paper by D'Alessandro[4].

Concretely the system of nonlinear differential equations is

$$\begin{cases} \dot{S} = \left[\rho \left(S / \underline{K} - 1 \right) \left(1 - S / \overline{K} \right) - \alpha \beta L \right] S \\ \dot{L} = \gamma \left(\lambda \left(1 - \beta \right)^{\delta} L^{\delta - 1} + \overline{\phi} \alpha \beta S \quad - \overline{\sigma} \right) L \end{cases} \tag{1}$$

where variable S regards to tree population in the forest, L regards human population, and the rest of symbols are system parameters. Different stable and unstable equilibrium points can be computed. Its existence depends on the specific parameter values.

Trivial equilibrium points are:

$$\begin{aligned} &1.\quad L^* = 0, \qquad S^* = 0 \\ &2.\quad L^* = 0, \qquad S^* = \underline{K} \\ &3.\quad L^* = 0, \qquad S^* = \overline{K} \\ &4.\quad L^* = \left(\frac{\lambda \left(1 - \beta \right)^{\delta}}{\overline{\sigma}} \right)^{\left(1 / \left(1 - \delta \right) \right)}, \qquad S^* = 0 \end{aligned} \tag{2}$$

At each one, either the human or the tree population is zero. Moreover, two additional non-trivial points can exist for some parameter values. The algebraic expressions for the non-trivial equilibrium points are rather complicated and thus they are not included in this document. At these two additional points, both the tree population and the human population are non-zero.

Through bifurcation numerical simulations we can observe that these non-trivial points are created in a saddle-node bifurcation (LP). One of the equilibrium points is stable and the other is unstable. The stable one losses stability in a further (supercritical) Hopf bifurcation, where nonlinear oscillations are created (see Figure 1). We computed several bifurcation diagrams with all the available parameters, and we always observed the same mechanism.

Regarding codimension-two phenomena, we continued the Hopf bifurcation in two parameters and a curve of Hopf bifurcations was obtained. Some initial computations show that this curve can be a degenerated curve of codimension-two Bautin (GH) points.

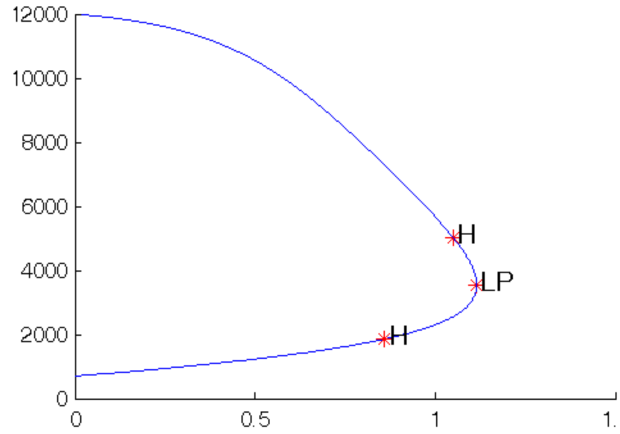

Figure 1: Codimension-one bifurcation diagram, when parameter α (technological parameter) is varied.

3 Five-dimensional mathematical model

In this section we consider a more complex model including three extra state variables α, λ and σ. They are related to technological development, land fertility and economic welfare respectively. Thus it can be considered as a better approximation to a development system.

$$
\begin{aligned}
\dot{S} &= (\rho(S/k_1 - 1)(1 - S/k_2) - \alpha\beta L)S \\
\dot{L} &= \gamma(\lambda(1-\beta)^{\delta} L^{\delta-1} + \phi\alpha\beta S - \sigma)L \\
\dot{\alpha} &= k_L \alpha L^{\delta_2}\left(\frac{L - L_{min}}{L_{min}^2 + (L - L_{min})^2}\right) \\
\dot{\lambda} &= \lambda(k_a(\alpha_0 - \alpha) - k_b L^{\delta_3}) \\
\dot{\sigma} &= (r_1(S - S_2) + r_2(\lambda - \lambda_2))(r_3\alpha + r_4\sigma) - r_5 L
\end{aligned}
\tag{3}
$$

Computing all equilibrium points in this new system is a heavy task, and some preliminary analysis shows that there are at least twelve different cases depending on the parameter values. Also, stability regards to a five-dimensional jacobian, and it must be computed through symbolic computation software.

In this paper we only show several dynamical patterns depending on initial conditions of the state variables. Figure 2 shows that depending on the initial values of the state variables, there totally different dynamic behavior is obtained. The first pattern corresponds to initial people growth and forest decay, followed by a forest total recovery with decay of human people. The second pattern corresponds to chaotic oscillations after an initial growth and forest decay. The third pattern shows that the forest is finally lost although some human community survives. Transitory trajectories in the third pattern show the following evolution:

- Phase 1: Population growth and forest reduction.
- Phase 2: First (small) oscillations of population and forest.
- Phase 3: Seemingly convergence to an equilibrium point.
- Phase 4: Evolution to a new seemingly equilibrium point with forest reduction.
- Phase 5: Second (big) oscillations of population and forest.
- Phase 6: Forest total exhaustion.

Even more interesting, numerical simulations show that the two oscillations occur in a relative small time period. Specifically, the second oscillation leading to forest exhaustion is very fast. This means that when oscillations are detected, fast sustainability actions should become a priority. If not, the forest is inevitably lost.

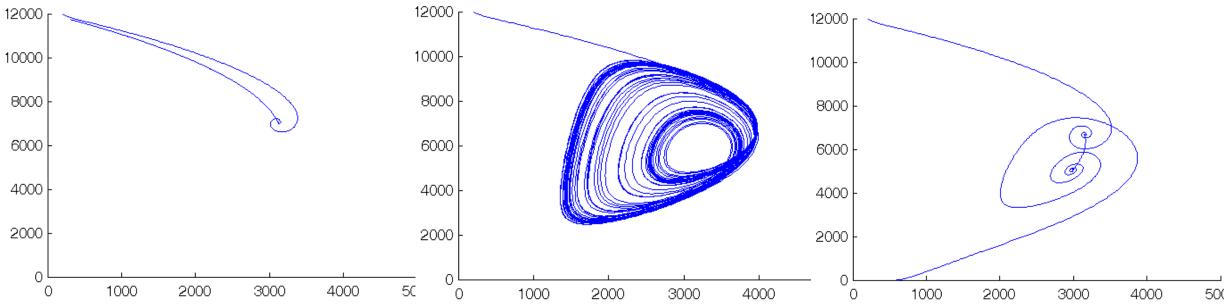

Figure 2: Different dynamic patterns corresponding to several choices of initial conditions.

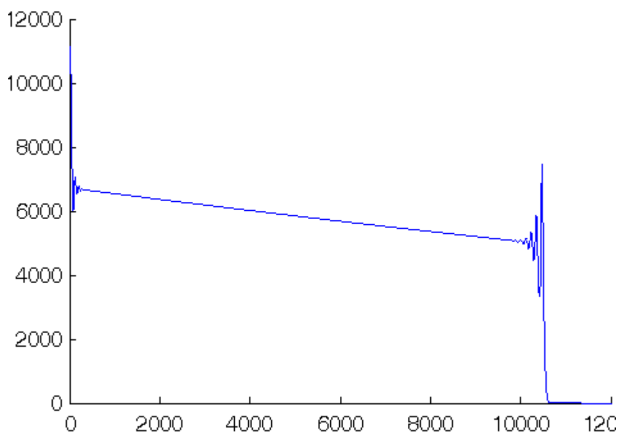

Figure 3: Time waveform of the number of trees in the forest corresponding to the third pattern in Figure 2

4 Conclusions

- Systems of nonlinear differential equations provide a good framework for studying sustainable development since nonlinear mathematical tools such as bifurcation theory can be applied, leading to non-evident results.
- For low-dimensional (simplified) systems modeling population growth and forest, it has been shown that, depending on parameter values, non-trivial equilibrium points can exist. This means that sustainable development can be achieved in low-developed human communities. Also, sustainable development can be achieved with oscillations of human population and forest.
- More complex nonlinear systems including variables from ecosystem, social and economic dimensions show that, depending on the parameter values and initial conditions, different patterns can be obtained. Some of these patterns are related to fast decay of the exhaustible resources. Thus, also fast sustainable actions must be taken into account to prevent ecological disasters.

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