A survey on uncertainty principles related to quadratic forms

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ABSTRACT

Hardy’s Uncertainty Principle may be seen as a characterisation of all tempered distributions $f$ on $\mathbb{R}^d$ such that $e^{\pm xq}f$ and $e^{\pm xq'}\hat{f} \in S' (\mathbb{R}^d)$ are also tempered, with $q$ and $q'$ two positive definite quadratic forms. After proving this, we consider the same problem for general non degenerate quadratic forms $q$ and $q'$. A special attention is given to the case when $q(x, y) = q'(x, y) = \langle x, y \rangle$ on $\mathbb{R}^d \times \mathbb{R}^d$, for which the results that we describe here may be seen as a generalization of Beurling’s Uncertainty principle. We also consider other kinds of uncertainty principles related to quadratic forms, which describe strong or weak annihilating pairs in the sense of Havin and Jöricke.

1. Introduction

Our starting point is Hardy’s Uncertainty Principle that asserts that, when a function $f$ satisfies the two inequalities

$$ |f(x)| \leq C(1 + |x|)^N e^{-\pi |x|^2} \quad \text{and} \quad |\hat{f}(\xi)| \leq C(1 + |\xi|)^N e^{-\pi |\xi|^2}, $$

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then there exists a polynomial $P$, which is of degree at most $N$, such that $f(x) = P(x)e^{-\pi|x|^2}$.

We will see that the analogous statement for distributions is very easy to prove. The same is valid for Beurling’s Uncertainty Principle, and for Morgan’s Uncertainty Principle. In this last one, the second members of the two inequalities above are replaced by

$$e^{-2\pi \beta p |x|^p} \text{ and } e^{-2\pi \beta p' |\xi|^p'},$$

with $1 < p \leq 2$ and $p'$ its conjugate exponent.

One is lead to the following problem: for two non degenerate quadratic forms $q, q'$ on $\mathbb{R}^d$, consider the distributions $f \in S'(\mathbb{R}^d)$ for which

$$e^{\pm \pi q} f \in S'(\mathbb{R}^d) \text{ and } e^{\pm \pi q'} \hat{f} \in S'(\mathbb{R}^d).$$

We are interested in uncertainty principles, that is, pairs of quadratic forms for which the space of such distributions is small (that is, either identically 0, or with a simple description of all its elements in terms of one of them). We will answer this question when $q$ and $q'$ are positive definite. A special attention will be given to other non degenerate quadratic forms. We will describe all such distributions when $d = 2$ and $q(x, y) = q'(x, y) = 2xy$. Roughly speaking, all solutions are obtained from Gaussian functions that are solutions through a small number of simple operations, as it is the case in Hardy’s Uncertainty Principle. When we have such a complete description of the space of solutions, we say that we have a Strong Uncertainty Principle related to the two quadratic forms. If we add some integrability conditions on $f$ (basically, conditions that a Gaussian function does not satisfy) and prove that $f$ vanishes, we speak of a Weak Uncertainty Principle. We will give weak uncertainty principles, not only for this particular pair of quadratic forms, but also for a family of pairs of non degenerate quadratic forms. As an example, let us mention the following result, which may be seen as the analogue of Hardy’s Theorem.

**Theorem**

If a function $f$ on $\mathbb{R}^2$ satisfies the two inequalities

$$|f(x, y)| \leq Ce^{-2\pi a |xy|} \text{ and } |\hat{f}(\xi, \eta)| \leq Ce^{-2\pi b |\xi\eta|},$$

with $ab > 1$, then $f = 0$.

Note that the two functions are not a priori in $L^1$ or $L^2$, so that the Fourier transform has to be taken in the distribution sense.

The main tool is classical complex analysis, which appears naturally once one has performed an integral transform that has already been used by Bargmann in representation theory.
As a byproduct, we will see that for some pairs \((q, q')\), the two sets 
\(\{x \in \mathbb{R}^d; |q(x)| < A\}\) and \(\{\xi \in \mathbb{R}^d; |q'(\xi)| < A\}\) are a weakly annihilating pair, following the terminology of the book of Havin and Jöricke.

We will start with a long introduction on annihilating pairs and the uncertainty principle. This last one, which is fundamental in quantum mechanics and in signal processing, has been the object of many studies. But there are still simple questions which remain unanswered. Even if the subject has a long history, a certain number of recent results are fascinating. Most of them are based on complex methods, but there are exceptions, like the theorem of Shubin, Vakilian and Wolff [35]. Apart from the book of Havin and Jöricke [25], we would like to recommend the expository papers of Folland & Sitaram [15], and Havin [24], as well as the fundamental paper of Nazarov [33]. We chose not to report on the considerable work which has been done on Lie groups and concentrate on the Euclidean space. Some of the new material described here has already been generalized on Lie groups, see [34] for instance.

Except in the next section, which is not really original, the results are essentially due to the second author and are part of his thesis. They simplify and generalize completely the uncertainty principles obtained in the paper called *Hermite functions and uncertainty principles for the Fourier and the windowed Fourier transforms* [7], which was a joint work of the authors with Philippe Jaming.

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### 2. Examples of annihilating pairs of sets

We start with the following definitions, which we borrow from Havin and Jöricke.

**Definition 2.1** Let \(E\) and \(\Sigma\) be two measurable sets in \(\mathbb{R}^d\).

We call \((E, \Sigma)\) a weakly annihilating pair if the function \(f \in L^2(\mathbb{R}^d)\) vanishes as soon as \(f\) is supported in \(E\) and \(\hat{f}\) is supported in \(\Sigma\).

We call \((E, \Sigma)\) a strongly annihilating pair if there exists some constant \(C\) such that, for every function \(f \in L^2(\mathbb{R}^d)\),

\[
\|f\|_2^2 \leq C \left( \int_{E^c} |f|^2 dx + \int_{\Sigma^c} |\hat{f}|^2 dy \right). \tag{1}
\]

These notions are invariant through translations and dilations. More precisely, if \((E, \Sigma)\) is a weakly (resp. strongly) annihilating pair, the same
holds for all pairs \((a + E, b + \Sigma)\), as well as \((\delta E, \delta^{-1} \Sigma)\). Here \(a, b\) are elements of \(\mathbb{R}^d\), and \(\delta\) is a positive number.

There are many examples of weakly annihilating pairs given by strict subsets of \(\mathbb{R}^d\). For instance, in dimension one, a half-line and a set of positive measure give a weakly annihilating pair. In fact, it is difficult to produce pairs of subsets of \(\mathbb{R}^d\) that are non weakly annihilating, since it is difficult to have non vanishing functions whose support and spectrum both have a hole. The following example may look surprising.

**Example 2.2** In \(\mathbb{R}^{d+1}\), for \(d > 1\), we define the light cone \(C\) as the set of \(x\) such that \(x_1^2 + \cdots + x_{d+1}^2 > x_1^2 + \cdots + x_d^2\). Then, for \(d \geq 4\), there exists a non zero function \(f \in L^2(\mathbb{R}^{d+1})\) that vanishes inside \(C\), as well as its Fourier transform. So \((C, C)\) is not a weakly annihilating pair for \(d \geq 4\).

The function \(f\) is obtained as follows. If we note \(q(x) := x_1^2 + \cdots + x_{d+1}^2 - x_d^2\), then we note \(d\mu(x) := e^{-\pi q(x)} d\delta(x_1 - x_{d+1})\) (with \(\delta\) the Dirac mass at 0). Its Fourier transform is equal to \(d\nu(x) := e^{-\pi q(x)} d\delta_0(x_1 + x_{d+1})\). We now define \(f\) as the integral of the measures \(d\mu(Rx)\), where the integral is taken over all rotations \(R\) which preserve the vector \((0, \cdots, 0, 1)\). A direct computation allows to prove that, if we note \(x' := (x_1, \cdots, x_d)\), then

\[f(x) = c_d \left| x' \right|^d \left(1 - \frac{x_{d+1}^2}{|x'|^2} \right)^{(d-3)/2} e^{-\pi q(x)}.
\]

The function \(f\) is its own Fourier transform. It belongs to \(L^p(\mathbb{R}^{d+1})\) for \(\frac{d-1}{d-2} < p < d + 1\).

Our definitions stand for \(L^2\) functions. We could as well define \(p\)-weakly annihilating pairs. We will say that \((E, \Sigma)\) is a \(p\)-strongly annihilating pair whenever there exists some constant \(C\) such that, for all functions \(f \in L^p(\mathbb{R}^d)\) with spectrum in \(\Sigma\), one has the inequality

\[\|f\|_p^p \leq C \int_{E^c} |f|^p dx. \quad (2)
\]

It is elementary to see that this notion coincides with the previous one for \(p = 2\). Indeed, the existence of such a constant \(C\) for functions \(f \in L^2(\mathbb{R}^d)\) with spectrum in \(\Sigma\) implies easily that (1) is valid for all \(f \in L^2(\mathbb{R}^d)\) with the constant \(4C + 2\) (a discussion on the best constant may be found in [33]).

While it is not easy to prove that a pair is not weakly annihilating, there are simple necessary conditions on strongly annihilating pairs. Let
us first give a definition. Denote by $Q$ the unit cube centered at 0. We say that the set $F$ is $\gamma$-thick at scale $a > 1$ if, for almost all $x \in \mathbb{R}^d$,

$$|F \cap (x + aQ)| \geq \gamma a^d. \quad (3)$$

**Proposition 2.3**

1. If the set $\Sigma$ contains balls of arbitrary size, then there is no subset $E$ of positive measure such that $(E, \Sigma)$ is a strongly annihilating pair.
2. If the set $\Sigma$ is not negligible and $(E, \Sigma)$ is a strongly annihilating pair, then $E^c$ is $\gamma$-thick at scale $a > 1$ for some $\gamma > 0$ and $a > 1$.
3. Let $(E, \Sigma)$ be a strongly annihilating pair. Then there exists a constant $a > 1$ such that, whenever $\Sigma$ contains a parallelepiped with sides of length $(\alpha_1, \alpha_2, \cdots, \alpha_d)$, then $E$ cannot contain a parallelepiped with sides of length $(a\alpha_1^{-1}, a\alpha_2^{-1}, \cdots, a\alpha_d^{-1})$.

**Proof.** Assume that $(E, \Sigma)$ is a $p$-strongly annihilating pair, and that $\Sigma$ contains arbitrarily large balls. Then, by invariance by translation, (2) holds for $f \in L^p(\mathbb{R}^d)$ whose Fourier transform is compactly supported. By density, it is valid for all $f$. So $E$ is negligible.

Now, let us fix a function $f \in L^p(\mathbb{R}^d)$ with spectrum in $\Sigma$ and norm 1. We choose $a$ large enough so that

$$\int_{(aQ)^c} |f(t)|^p dt \leq \frac{C^{-1}}{2},$$

where $C$ is the constant in (2). It implies that

$$\int_0^{[aQ \cap E^c]} f^*(s) ds \geq \int_{[aQ \cap E^c]} |f(t)|^p dt \geq (2C)^{-1}. \quad (4)$$

Here $f^*$ stands for the decreasing rearrangement of $f$. The inequality above implies that $[aQ \cap E^c] > \gamma$, for some $\gamma$ which depends only on $f^*$. Let us translate $f$ and $aQ$ by $x$. Let us note by $f_x$ the function $f(\cdot - x)$. Then (4) holds with $f$ replaced by $f_x$ and $aQ$ by $x + aQ$. Moreover, $f_x$ has the same decreasing rearrangement $f^*$. This implies the thickness of $E^c$.

Finally, assume that $(E, \Sigma)$ is a $p$-strongly annihilating pair. The same is valid for $(E_\delta, \Sigma_{\delta^{-1}})$, with the same constant $C$, where $E_\delta$ is obtained from $E$ by the linear transformation $\delta$, which is obtained by composing dilations in each variable. Let us take for $\delta$ the composition of dilations in each variable that transforms the parallelepiped into a unit cube on the Fourier side. Then there exists $a$, which depends only on the constant $C$, such that $E_\delta$ does not contain any translate of $aQ$. The statement is obtained when using the inverse transformation.

It is natural to mention first the following theorem, which is due to Benedicks [6].
Theorem 2.4 (Benedicks)

Assume that, for almost every $x \in (0, 1)^d$, the set $(x + \mathbb{Z}^d) \cap E$ is finite, and, for almost every $y \in (0, 1)^d$, the set $(y + \mathbb{Z}^d) \cap \Sigma$ is finite. Then the pair $(E, \Sigma)$ is weakly annihilating.

Proof. The main tool for this is Poisson Summation Formula, which implies that

$$e^{-2i\pi (x,y)} \sum_j f(x+j)e^{-2i\pi (j,y)} = \sum_k \hat{f}(y+k)e^{2i\pi (k,x)}. \quad (5)$$

Assume first that there exists $S \subset (0,1)^d$ of positive measure such that, for $x \in S$, the set $(x + \mathbb{Z}^d) \cap E$ is empty. Then the left hand side vanishes on $S$. Since the right hand side is a trigonometric polynomial in the $x$ variable for almost every $y$, it is identically $0$. So, for almost every $y \in (0,1)^d$ we know that $\hat{f}(y+k)$ is equal to $0$ for all $k \in \mathbb{Z}^d$. The conclusion that $f = 0$ follows easily.

Let us go back to the general case. It is easy to find $S$ of positive measure in $(0,1)^d$ and $N$ such that $(x+N\mathbb{Z}^d) \cap E$ is the empty set for $x \in S$. We will use the previous case after a change of scale. We are reduced to prove that, for almost every $y \in (0,N^{-1})^d$, the set $(y+N^{-1}\mathbb{Z}^d) \cap \Sigma$ is finite: just write it as a union of sets $(y_j+\mathbb{Z}^d) \cap \Sigma$, with $y_j = y + \frac{j}{N}$. □

Benedicks’ Theorem gives many examples of weakly annihilating pairs. Note that the assumptions of Theorem 2.4 are preserved through finite union. Let us give the following example.

Example 2.5 In $\mathbb{R}^2$, $(E, \Sigma)$ constitute an annihilating pair if $E$ and $\Sigma$ are obtained from the set $\{|xy| < C\}$ using any rotation and translation.

We could replace these regions by any region of the same type, with a finite number of asymptotes, and the approach to the asymptotes may be arbitrarily low. It is not the case when considering strongly annihilating pairs, as it may be seen when using the third assertion of Proposition 2.3. The pair obtained with the same equilateral hyperbola for $E$ and $\Sigma$ is critical as far as the property of strongly annihilating pair is involved. We shall come back to this property later, and prove that it is a strongly annihilating pair for $C$ small.

Benedicks’ Theorem is usually referred to when proving that $(E, \Sigma)$ is a weakly annihilating pair for $E$ and $\Sigma$ of finite measure, while one refers to Amrein-Berthier [1] to prove that $(E, \Sigma)$ is a strongly annihilating pair. In fact, it may also be deduced from Benedicks’ Theorem using some abstract nonsense. This leads to the following proposition.
Proposition 2.6

Let $E$ and $\Sigma$ be of finite measure. Then the pair $(E, \Sigma)$ is $p$-strongly annihilating for $1 \leq p \leq 2$, that is, there exists a constant $C$ such that, for $f \in L^p(\mathbb{R}^d)$ with spectrum in $\Sigma$, the following inequality is valid.

$$||f||_p^p \leq C \int_{E^c} |f|^p dx.$$  

Proof. Assume that there is no such constant $C$. Then there exists a sequence $f_n$ in $L^p(\mathbb{R}^d)$ of norm 1 and with spectrum in $\Sigma$, such that $f_n \chi_{E^c}$ tends to 0. Moreover, we may assume that $\hat{f}_n$ is weakly convergent in $L^{p'}(\mathbb{R}^d)$, with limit $f$. So, $f_n(x)$, which may be obtained as a scalar product with $e^{2\pi i \langle x, \xi \rangle} \chi_{\Sigma}$, tends to $f(x)$. As a consequence, $f_n \chi_E$ converges to $f$ in $L^p$ (remark that $|f_n|$ is bounded by $|\Sigma|^{\frac{1}{p}}$, so that we can apply Lebesgue’s Theorem). The limit $f$ has norm 1. But the function $f$ has support in $E$ and spectrum in $\Sigma$, so $f = 0$ by Benedicks’ Theorem (which is also valid for $L^p$ functions). We get a contradiction. □

The pair $(E, \Sigma)$ is also $p$-weakly annihilating for $p > 2$ whenever $E$ and $\Sigma$ are of finite measure. The same is not valid as far as $p$-strongly annihilating pairs are concerned: for $p > 2$, Nazarov gives in [33] an example of a set $\Sigma$ of finite measure such that, for all measurable sets $E$ of finite measure, it is not possible to find a constant $C$ for which the inequality (2) holds.

It is much more difficult to have a sharp estimate of the constant in Proposition 2.6. Let us mention the theorem of Nazarov, which states that in one dimension, and for $0 < p \leq 2$, one can take

$$C = C_0 e^{C_1 |E||\Sigma|}.$$  

The same problem in higher dimension is not yet solved in a completely satisfactory way.

For particular $\Sigma$ the range of possible $E$ may be widened. For $\Sigma$ a bounded set, the necessary condition (3) that we have given on $E$ in order that $(E, \Sigma)$ is a strongly annihilating pair is also sufficient: this is due to Logvinenko-Sereda, see [25] or [33]. Let us mention the following theorem, which improves their estimate.

Theorem 2.7 (Kovrizhkin)

Let $p \in [1, +\infty]$. There exists a constant $C$ such that, for $f$ which has its spectrum supported in the unit cube $Q$ and for $E$ which is $\gamma$-thick at scale $a > 1$, one has the inequality

$$||f||_p^p \leq C \gamma^{-C_0} \int_E |f|^p dx.$$  

(6)
As we have seen, Identity (5), a variant of Poisson summation, may be used to have sufficient conditions for having annihilating pairs. Let us mention the following elementary lemma, which allows to connect these notions on the torus with the notions that we have defined. Let us first say that the pair \((E, \Lambda)\), with \(E \subset \mathbb{T}^d\) and \(\Lambda \subset \mathbb{Z}^d\), is strongly annihilating for the torus \(\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d\) if there exists some constant \(C\) such that, for every \(f \in L^2(\mathbb{T}^d)\) with spectrum in \(\Lambda\), we have the inequality
\[
\|f\|^2 \leq C \int_{\mathbb{T}^d \setminus E} |f|^2 \, dx. \tag{7}
\]
Using (5), it is elementary to prove the following lemma, where we have identified \(E\) with a subset of \((0, 1)\).

**Lemma 2.8**

If the pair \((E, \Lambda)\) is strongly annihilating, with \(E \subset \mathbb{T}^d\) and \(\Lambda \subset \mathbb{Z}^d\), then the pair \((E + \mathbb{Z}^d, \Lambda + (0, 1))\) is strongly annihilating in \(\mathbb{R}^d\).

There is nearly a characterization of subsets \(\Lambda \subset \mathbb{Z}^d\) such that \((E, \Lambda)\) is a strongly annihilating pair for the torus for each \(E\) of positive measure: this is due to Miheev, 1975 (see [30]), and the complete proof may be found in the book of Havin and Jörjicke. Before giving its statement, let us recall the definition of \(\Lambda(p)\) sets.

**Definition 2.9** Let \(p > 1\). A set \(\Lambda \in \mathbb{Z}^d\) is called a \(\Lambda(p)\) set if there exists a constant \(C\) such that, for every function \(f \in L^p(\mathbb{T}^d)\) with spectrum in \(\Lambda\), one has the inequality
\[
\|f\|_p \leq C \|f\|_1. \tag{8}
\]

The link between this notion and strongly annihilating pairs is very simple.

**Lemma 2.10**

A set \(\Lambda \in \mathbb{Z}^d\) is a \(\Lambda(p)\) set if and only if, for some \(\delta > 0\), there exists \(\gamma > 0\) such that, for every function \(f \in L^p(\mathbb{T}^d)\) with spectrum in \(\Lambda\),
\[
|E| < \gamma \Rightarrow \int_E |f|^p \, dx \leq (1 - \delta)\|f\|_p^p. \tag{9}
\]

**Proof.** By density, it is sufficient to deal with trigonometric polynomials. If (9) holds, and if we choose \(\lambda = \gamma^{-1}\|f\|_1\), we obtain that
\[
\|f\|_p^p \leq (1 - \delta)\|f\|_p^p + \int_{|f| \leq \lambda} |f|^p \, dx.
\]
The second term is bounded by \( \gamma^{-(p-1)}\|f\|_p^p \). Putting the first one into the left hand side, we obtain (7) with \( C_p = \gamma^{-(p-1)}\delta^{-1} \).

Conversely, let us deduce (9) from (7). It is equivalent to prove that

\[
|E| > 1 - \gamma \Rightarrow \|f\|_p \leq \delta^{-1/p} \left( \int_E |f|^p \, dx \right)^{1/p}.
\]

Using (7), cutting the integral into two parts and using Hölder’s inequality, we write

\[
\|f\|_p \leq C\|f\|_1 \leq C(1 - |E|)^{1/p'}\|f\|_p + C \left( \int_E |f|^p \, dx \right)^{1/p}.
\]

We choose \( \delta = (2C)^{-p} \) and \( \gamma = (2C)^{-p} \).

We now give the definition of a \( \Lambda^*(p) \) set, which is central in the theorem of Miheev (see also the paper of Hare [22] for general Abelian groups).

**Definition 2.11** Let \( p > 1 \). A set \( \Lambda \in \mathbb{Z}^d \) is called a \( \Lambda^*(p) \) set if, for all \( \delta > 0 \), there exists \( \gamma > 0 \) such that, for every function \( f \in L^p(\mathbb{T}^d) \) with spectrum in \( \Lambda \),

\[
|E| < \gamma \Rightarrow \int_E |f|^p \, dx \leq \delta\|f\|_p^p.
\]  
(10)

It is clear from the definition that \( \Lambda^*(p) \) sets are \( \Lambda(p) \) sets, and that, for \( q > p \), \( \Lambda(q) \) sets are \( \Lambda^*(p) \) sets. Let us mention that one does not know whether \( \Lambda(2) \) sets are \( \Lambda^*(2) \) sets (there is no example of a \( \Lambda(2) \) set that is not a \( \Lambda(2 + \varepsilon) \) for some \( \varepsilon > 0 \)). The analogue for \( p < 2 \) is true since in this case a \( \Lambda(p) \) set is a \( \Lambda(q) \) set for some \( q > p \) (see [3]). The analogue for \( p > 2 \) is false: this last fact is an easy corollary of Bourgain’s construction of a \( \Lambda(p) \) set which is not a \( \Lambda(q) \) set for \( q > p \), see [8]. One can show that this particular example is not a \( \Lambda^*(p) \) set as well.

Let us now state the theorem.

**Theorem 2.12** (Miheev)

Let \( \Lambda \) be a \( \Lambda^*(2) \) set in \( \mathbb{Z}^d \). Then, for \( E \) a measurable subset of \( \mathbb{T}^d \) of positive measure, there exists a constant \( C \) such that, for \( f \in L^2(\mathbb{T}^d) \) with spectrum in \( \Lambda \), one has the inequality

\[
\|f\|_2^2 \leq C \int_E |f|^2 \, dx.
\]
In particular, every pair \((F, \Lambda)\), with \(\Lambda\) a \(\Lambda^*(2)\) set and \(F\) of positive measure, is strongly annihilating. The proof contains difficult parts, based on combinatorial properties of \(\Lambda(p)\) sets, which may be also found in [31] in one variable, and in [23] for its generalization in all locally compact Abelian groups. One uses the fact that there are large gaps in \(\Lambda\). We refer to [25] for the complete proof, and content ourselves of a particular case, where it is a direct generalization of Zygmund’s original theorem on finite unions of lacunary sets (see [36]). Let us first give some definitions.

Let us define the lag of \(\Lambda\) as \(\min_{\lambda, \lambda' \in \Lambda} |\lambda - \lambda'|\). We say that the lag of \(\Lambda\) tends to infinity if \(\min_{\lambda, \lambda' \in \Lambda} |\lambda - \lambda'|\) tends to infinity for \(\lambda \in \Lambda\) tending to infinity. We will prove the theorem under the following additional assumption: \(\Lambda\) is a finite union of sets in \(\mathbb{Z}^d\) whose lag tends to infinity.

Proof of Miheev’s Theorem under the additional assumption. We write \(\Lambda\) as the union of \(\Lambda_j\), for \(j = 1, \cdots, k\). Up to a finite number of points, we may assume the lag of each \(\Lambda_j\) is larger than \(N\), with \(N\) arbitrarily large. Because of the next lemma, we may assume that each \(\Lambda_j\) itself has this property.

Lemma 2.13

Assume that \((F, \Lambda)\) is a \(p\)-strongly annihilating pair for each measurable set \(F\) with positive measure. Let \(\lambda_0 \not\in \Lambda\). Then \((F, \Lambda \cup \{\lambda_0\})\) is also a \(p\)-strongly annihilating pair for each measurable set \(F\).

Proof of Lemma 2.13. After a translation on \(\mathbb{Z}^d\), we may assume that \(\lambda_0 = 0\), and \(\Lambda\) does not contain 0. Let us assume that there exists a sequence of functions \(a_n + f_n\), with the spectrum of \(f_n\) in \(\Lambda\), such that \(\int_{\mathbb{T}^d} |f_n + a_n|^p dx = 1\) while \(\int_{\mathbb{T}^d \setminus F} |f_n + a_n|^p dx\) tends to zero. Eventually extracting a subsequence, we may assume that the sequence \(a_n\) has limit \(a\). It is easily seen that \(f_n\) is a Cauchy sequence in \(L^p(\mathbb{T}^d)\), which converges to \(f\). We know that \(f + a = 0\) on \(E\), with \(f\) having its spectrum in \(\Lambda\), and have to prove that \(f = 0\). For \(h \in \mathbb{T}^d\), the function \(f(\cdot + h) - f\), which has its spectrum in \(\Lambda\), vanishes on \(F \cap (-h + F)\), whose measure is positive when \(h\) is sufficiently small. Using the fact that \((F \cap (-h + F), \Lambda)\) is a \(p\)-strongly annihilating pair, we conclude that \(f(\cdot + h) = f\) for small \(h\), which implies that \(f = -a\). Since 0 does not belong to \(\Lambda\), it means that \(f = 0\), as desired. □

We will also use the following lemma, which is due to Zygmund [36] and has been improved by Nazarov with an estimate of the constant in terms of \(|E|\) and \(k\) (see [33]). We do not give its proof, which is in the same spirit as the previous one.
Lemma 2.14

Let \( E \) be a measurable set with positive measure, \( k \) a positive integer and \( p \geq 1 \). There exists a constant \( C \) such that, for all functions \( f \) with at most \( k \) non zero Fourier coefficients,

\[
\int_{T^d} |f|^p \, dx \leq C \int_E |f|^p \, dx.
\]

Let us go on with the proof of Miheev’s Theorem under the additional assumption. Using Lemma 2.13, we assume that \( |\lambda - \lambda'| \geq 2d(k+1)N \) for \( \lambda \neq \lambda' \) in \( \Lambda_j \), with \( N > 1 \) depending on \( E \) to be fixed later. Then, one can decompose \( \Lambda \) as the union of \( \Lambda(\lambda) \), where each \( \Lambda(\lambda) \) contains at most \( k \) elements, and the distance between two different \( \Lambda(\lambda) \)'s is larger than \( 2dN \). More precisely, we choose \( N \) large enough so that \( \| \chi_E - \sigma_N(\chi_E) \|_2 < \varepsilon \), with \( \sigma_N \) a partial sum of De La Vallée Poussin (obtained by using a product of kernels of De La Vallée Poussin in each variable). Its kernel is uniformly bounded in \( L^1(\mathbb{T}^d) \), so that \( \sigma_N(\chi_E) \) is bounded by some constant \( K \), and \( \hat{\sigma}_N(\chi_E)(n) = 0 \) when \( |n| < 2dN \). Let us write

\[
f = \sum f_l,
\]

with \( f_l \) with spectrum in \( \Lambda(l) \). Then \( |f|^2 = \sum |f_l|^2 + g \), with \( \hat{g}(n) = 0 \) when \( |n| < 2dN \). So, we can write

\[
\int_E |f|^2 \, dx = \int_E \sum |f_l|^2 \, dx + \int_{T^d} (\chi_E - \sigma_N(\chi_E))g \, dx.
\]

From Lemma 2.14, we deduce the existence of some constant \( c \) such that the first term is larger than \( c\|f\|_2^2 \). We will choose \( \varepsilon \) small enough so that the second term is half of the first one. To do this, we write

\[
\left| \int_E (\chi_E - \sigma_N(\chi_E))g \, dx \right| \leq \eta \int |g| \, dx + (K + 1) \int_{|\chi_E - \sigma_N(\chi_E)| > \eta} |g| \, dx.
\]

It remains to choose \( \eta \) and \( \varepsilon \) so that both terms are bounded by \( c\|f\|_2^2/4 \). It is sufficient to fix \( \eta = (K + 1)\delta = c/8 \), and then \( \varepsilon \) so that \( \eta^{-2}\varepsilon^2 \leq \gamma \), with \( \gamma \) corresponding to \( \delta \) in (10). Indeed, \( |g| \leq |f|^2 + \sum |f_l|^2 \), so that the integral of \( |g| \) is bounded by \( 2\|f\|_2^2 \), and we can use (10) for \( f \) and each \( f_l \).

When dealing with \( p > 2 \) and \( p \)-strong annihilating pairs, the situation is simpler than for \( p = 2 \). Indeed, one has the following proposition.

Proposition 2.15

For \( d = 1 \) and \( p > 2 \), the two properties are equivalent:

1. \( \Lambda \) is a \( \Lambda(p) \) set.
2. Foreach measurable set \( E \) of positive measure, the pair \((E, \Lambda)\) is \( p \)-strongly annihilating.
**Proof.** The fact that (1) implies (2) is very simple: the $L^p$ norm of a function with spectrum in $\Lambda$ is bounded by its $L^2$ norm, since $\Lambda$ is a $\Lambda(p)$ set. Remember that, for $p > 2$, a $\Lambda(p)$ set is also a $\Lambda^*(2)$ set. So this $L^2$ norm is bounded by the $L^2$ norm on $E$. Using Hölder’s inequality, it is bounded by the $L^p$ norm on $E$.

Conversely, assuming that the implication (9) does not hold for any $\gamma$ and $\delta$, one finds subsets $E_k$ whose union $E$ has measure less than $1/2$, and functions $f_k$ with spectrum in $\Lambda$ that have norm 1 in $L^p$, such that $\int_{E_k} |f_k|^p dx \leq \int_{E_k^c} |f_k|^p dx$ tends to 0. This contradicts (2) for the pair $(E^c, \Lambda)$. □

Remark that the implication $(2) \Rightarrow (1)$ is valid for all values of $p > 0$.

Using Lemma 2.8, it is easy to describe a large family of strongly annihilating pairs in $\mathbb{R}^d$ starting from these families of examples on the torus. It has been remarked in [28] in a particular case. Let us give the following example, related to quadratic forms.

**Example 2.16** Let $E$ be of positive measure in $(0,1)^2$ and $\Lambda := \{(n,n^2); n \in \mathbb{Z}\}$. Then $(E + \mathbb{Z}^2, \Lambda + (0,1)^2)$ is a strongly annihilating pair.

Indeed, it is well known that this set $\Lambda$ is $\Lambda(4)$ since it is lacunary in some sense, or satisfies the $B_2$ property: there is at most one possibility to write an element of $\mathbb{Z}^2$ as the sum of two elements of $\Lambda$ (up to a permutation).

Let us mention that Kovrizhkin proves that there are examples of thick non periodic sets, which form strongly annihilating pairs with such lacunary sets. There are also deep results in this direction in [33].

We turn to different results and methods. We rely on a paper of Shubin, Vakilian and Wolff [35], which contains new families of examples of strongly annihilating pairs.

**Definition 2.17** A set $E$ in $\mathbb{R}^d$ is called $\varepsilon$-thin (where $0 < \varepsilon < 1$) if, for all points $x \in \mathbb{R}^d$,

$$\frac{|E \cap B(x)|}{|B(x)|} < \varepsilon,$$

where $B(x)$ is the ball centered at $x$ of radius $\min(1,1/|x|)$.

Their theorem is the following.

**Theorem 2.18** (Shubin, Vakilian and Wolff)

Two $\varepsilon$-thin sets form a strongly annihilating pair, provided $\varepsilon$ is small enough, depending only on the dimension $d$.

Their proof uses Littlewood-Paley decompositions and is a real variable proof. As examples of $\varepsilon$-thin sets, they give the sets $\{x \in \mathbb{R}^d; q(x) \in$
$E$, with $q$ a non-degenerate quadratic form and $E$ a set of finite measure, under the assumption that $|E|$ is small enough. A detailed proof is given in [10]. To illustrate this example when $E$ and $F$ are intervals as well as the proof of Theorem 2.18, we will give a slightly different example, whose proof is in the same spirit. We will consider strongly annihilating pairs $(E, \Sigma)$ with

\[ E = \{(x, y) \in \mathbb{R}^2; |xy| < 1\} \text{ and } \Sigma \subset \{((\xi, \eta) \in \mathbb{R}^2; |\xi\eta| < 1\}. \]

It is easy to see that one can as well replace $E$ and $\Sigma$ by unions of rectangles, and that it is sufficient to consider one branch of the hyperbolae. More precisely, we note

\[ Q_k := [2^{k-1}, 2^k] \times [-2^{-k}, 2^{-k}] \text{ and } Q_k^* := [-2^{-k}, 2^{-k}] \times [2^{k-1}, 2^k]. \]

We will prove the following.

**Proposition 2.19**

Let $f \in L^2(\mathbb{R}^2)$ be a function whose Fourier transform is supported in the union of $E_k \subset Q_k$, with $|E_k| < \varepsilon$. Then

\[
\int_{Q_k^*} |f(x, y)|^2 dx dy \leq C \varepsilon^{1/2} \|f\|_2^2.
\]

Let us write $f$ as $f = \sum_{k>0} f_k$, where $f_k$ has its spectrum in $E_k$ and $\|f\|_2^2 = \sum \|f_k\|_2^2 = 1$. Using Plancherel’s Formula, then Cauchy-Schwarz’ Inequality, it is sufficient to prove that

\[
\sum_{p,k,l} \left( \int_{(x,y) \in E_k, (x',y') \in E_l} |\hat{\chi}_{Q_k^*}(x-x', y-y')|^2 dx dy dx' dy' \right)^{1/2} \times \|f_k\|_2 \|f_l\|_2 \leq C \varepsilon^{1/2}.
\]

Using again Cauchy-Schwarz’ Inequality, it is sufficient to prove that

\[
\sum_{p,k,l} \left( \int_{(x,y) \in Q_k, (x',y') \in Q_l} |\hat{\chi}_{Q_k^*}(x-x', y-y')|^4 dx dy dx' dy' \right)^{1/4} \times \|f_k\|_2 \|f_l\|_2 \leq C.
\]

Let us note

\[ d_{p,k,l} := \left( \int_{(x,y) \in Q_k, (x',y') \in Q_l} |\hat{\chi}_{Q_k^*}(x-x', y-y')|^4 dx dy dx' dy' \right)^{1/4}. \]
We are lead to prove that the operator given on $\ell^2$ by the matrix $\sum_p d_{p,k,l}$ is bounded. It is sufficient to have uniform bounds for $\sum_{p,k} d_{p,k,l}$. The computation is tedious, but elementary. It is based on the inequality
\[
\int_{|x| < a, |x'| < b} \frac{|\sin(x - x')|^4}{|x - x'|^4} dx dx' \leq C \min\{a, b, ab\},
\]
from which we deduce that $d_{p,k,l} \leq C 2^{-|k-p|/8} \times 2^{-|p-l|/8}$. The required estimates follow at once.

Remark that we could as well replace the rectangles $Q_k$ and $Q_k^*$ by translates of them.

This method has been considerably improved in [11], where new examples of strongly annihilating pairs are given.

Finally, we would like to mention the link of the fact that quadratic forms generate strongly annihilating pairs with Heisenberg type inequalities, and with the existence of spectral gaps for partial differential equations. Shubin, Vakilian and Wolff were lead to their theorem in order to prove the existence of such a spectral gap. We will show the link between these notions in a much simpler context than the one that they studied.

**Corollary 2.20**

Let $q$ and $q'$ two non degenerate quadratic forms on $\mathbb{R}^d$. Then, there exists some constant $c > 0$ such that, for all $f \in L^2(\mathbb{R}^d)$ with norm $1$, one has the inequality
\[
\int_{\mathbb{R}^d} |q(x)|^2 |f(x)|^2 dx \times \int_{\mathbb{R}^d} |q'(\xi)|^2 |\hat{f}(\xi)|^2 d\xi > c.
\]

**Proof.** It follows from homogeneity considerations, replacing $f$ by its dilates, that it is equivalent to prove that
\[
\int_{\mathbb{R}^d} |q(x)|^2 |f(x)|^2 dx + \int_{\mathbb{R}^d} |q'(\xi)|^2 |\hat{f}(\xi)|^2 d\xi > 2c.
\]
The link with strongly annihilating pairs is immediately seen when restricting the integrals to the sets
\[
\{x \in \mathbb{R}^d; |q(x)| > \varepsilon\} \quad \text{and} \quad \{\xi \in \mathbb{R}^d; |q'(\xi)| > \varepsilon\}.
\]
Let us remark that we could have replaced the power 2 by any power. □

In particular, for $q(x, y) = \langle x, y \rangle$ in $\mathbb{R}^d \times \mathbb{R}^d$, we find that
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |\langle x, y \rangle|^2 |f(x, y)|^2 dxdy
\]
\[
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} |\langle \xi, \eta \rangle|^2 |\hat{f}(\xi, \eta)|^2 d\xi d\eta > \varepsilon \|f\|^2.
\]
If we consider the function $g$ obtained after a partial Fourier transform in the second variable, we obtain the following.

**Corollary 2.21**

There exists some constant $c > 0$ such that, for all $g \in S(\mathbb{R}^d \times \mathbb{R}^d)$ one has the inequality

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \sum_{j=1}^{d} x_j \partial_{y_j} g(x, y) \right|^2 dxdy$$

$$+ \int_{\mathbb{R}^d \times \mathbb{R}^d} \left| \sum_{j=1}^{d} y_j \partial_{x_j} g(x, y) \right|^2 dxdy \geq c \|g\|_2^2.$$

This means that the second order operator

$$- \sum_j \sum_k (x_j x_k \partial_{y_j} \partial_{y_k} + y_j y_k \partial_{x_j} \partial_{x_k})$$

has a spectral gap. Other links between Heisenberg inequalities and spectral gaps may be found in [2].

There are possible generalizations of all results to the Radar Ambiguity functions instead of Fourier transforms. These last ones are defined by

**Definition 2.22** Let $u, v$ be two functions of $L^2(\mathbb{R}^d)$. The Radar-Ambiguity function associated to $u$ and $v$ is defined for $x, y \in \mathbb{R}^d$ by

$$A(u, v)(x, y) = \int_{\mathbb{R}^d} u \left( t + \frac{x}{2} \right) v \left( t - \frac{x}{2} \right) e^{-2i\pi (t, y)} dt.$$

One can as well define strongly (resp. weakly) annihilating sets. The previous results have their counterpart in this context, see [10, 20] and [18]. These generalizations cover as well Wigner transforms or windowed Fourier transforms, which may be obtained from the Radar Ambiguity function through elementary changes.

### 3. Hardy’s uncertainty principle for distributions

Let us first recall the classical theorem of Hardy, see [21] or [25].
Theorem 3.1 (Hardy)

Let $f \in L^2(\mathbb{R}^d)$ be a function such that the two following inequalities hold, for some constant $C$ and some non negative integer $N$:

$$|f(x)| \leq C(1 + |x|)^N e^{-\pi|x|^2} \quad \text{and} \quad |\hat{f}(\xi)| \leq C(1 + |\xi|)^N e^{-\pi|\xi|^2}.$$

Then there exists a polynomial $P$, which is of degree at most $N$, such that $f(x) = Pe^{-\pi|x|^2}$.

This theorem has been extended in many directions and many contexts. Let us particularly mention the extension given by Cowling and Price [9], where uniform conditions have been replaced by integrability conditions. A generalization of this has been given in [7], and may be stated as follows.

Theorem 3.2 (Cowling-Price type)

Let $N \geq 0$. Assume that $f \in L^2(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} |f(x)| \frac{e^{\pi|x|^2}}{(1 + |x|)^N} dx < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} |\hat{f}(\xi)| \frac{e^{\pi|\xi|^2}}{(1 + |\xi|)^N} dy < +\infty.$$

Then $f(x) = P(x)e^{-\pi|x|^2}$ for some polynomial $P$.

Let us remark that once one knows that $f$ is of the form $Pe^{-\pi|x|^2}$, it is easy to find necessary and sufficient conditions on the degree of $P$, depending on the integrability assumptions.

We shall again generalize this last statement, and write weaker assumptions, which can be expressed in terms of distributions.

Theorem 3.3

Let $f \in S'((\mathbb{R}^d)$. We assume that

$$e^{\pi|x|^2} f \in S'((\mathbb{R}^d) \quad \text{and} \quad e^{\pi|\xi|^2} \hat{f} \in S'((\mathbb{R}^d).$$

Then $f(x) = P(x)e^{-\pi|x|^2}$ for some polynomial $P$.

Proof. In order to understand why we introduce an auxiliary function, let us start from the solution. We want to prove that $f$ may be written as $f = Pe^{-\pi|x|^2}$ for some polynomial $P$, or, equivalently, that $\hat{f}e^{-\pi|\xi|^2}$ may be written as $Qe^{-2\pi|\xi|^2}$ for some polynomial $Q$. Again, taking Fourier transforms, it is equivalent to prove that $f * e^{-\pi|x|^2}$ may be written as $Re^{-\frac{\pi}{2}|x|^2}$ for some polynomial $R$. We have gained the property that $f * e^{-\pi|x|^2}$ extends a priori into an entire function for an arbitrary
tempered distribution \( f \). So, we are lead to prove that the entire function \( F \), defined in \( \mathbb{C}^d \) by

\[
F(z) = e^{\pi/2z^2} \left\langle f(x), e^{-\pi(x-z)^2} \right\rangle
\]

is a polynomial. Here, for \( z \in \mathbb{C}^d \), we note

\[
z^2 = z_1^2 + z_2^2 + \cdots + z_d^2,
\]

and the brackets stand for the action of the distribution \( f \) on a test function in the Schwartz class.

The auxiliary function \( F \) will be the main ingredient of our proofs. It is called the Bargmann transform of \( f \), given by the following definition.

**Definition 3.4** Let \( f \in S'({\mathbb{R}}^d) \). Then its Bargmann transform \( \mathcal{B}f \) is defined for \( z \in \mathbb{C}^d \) by

\[
\mathcal{B}f(z) := e^{\pi/2z^2} \left\langle f(x), e^{-\pi(x-z)^2} \right\rangle.
\]

While the Bargmann transform has been widely used in representation theory (see [14, 4, 5]) as well as in signal processing (see [19]), it does not seem to have been used for uncertainty principles.

It is elementary to see that \( \mathcal{B} \) is injective, a point that will be used later. We remark that, by definition of the Fourier transform for tempered distributions (or Parseval’s Identity), we have also that

\[
\mathcal{B}f(z) = e^{-\pi/2z^2} \left\langle \hat{f}(\xi), e^{-\pi(x-i\xi)^2} \right\rangle.
\]

Let us come back to the proof of Theorem 3.3. Remember that \( F = \mathcal{B}f \). Let us use the first assumption on \( f \). Then \( F(z) \) may as well be written as

\[
\left\langle f(x)e^{\pi|x|^2}, e^{-2\pi(x-z)^2} \right\rangle.
\]

By assumption, both distributions \( e^{\pi|x|^2} f \) and \( e^{\pi|\xi|^2} \hat{f} \) are of order less than \( N \) for some integer \( N \). The order of a tempered distribution is defined as follows.

**Definition 3.5** The order of the tempered definition \( g \) is the smallest integer \( N \) for which there exists some constant \( C \) such that, for every test function \( \varphi \),

\[
|\langle g, \varphi \rangle| \leq C p_N(\varphi),
\]

where the semi-norm \( p_N \) is defined by

\[
p_N(\varphi) := \max_{|\alpha|+|\beta| \leq N} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta \varphi(x)|.
\]
The computation of the norm $p_N$ of the test function involved is elementary, and leads to the following estimate,

$$|F(z)| \leq C(1 + |z|)^N e^{\pi/2|\Im(z)|^2}. \quad (16)$$

If we write $F$ using the Fourier transform of $f$, that is using (14), we can use the second assumption, and prove as before that

$$|F(z)| \leq C(1 + |z|)^N e^{\pi/2|\Re(z)|^2}. \quad (17)$$

Recall that we want to prove that $F$ is a polynomial. This is given in the following lemma, which is an easy consequence of the Phagmèn-Lindelöf Principle.

**Lemma 3.6**

Assume that the entire function $F$ on $\mathbb{C}^d$ satisfies the two estimates (16) et (17). Then $F$ is a polynomial of degree at most $N$.

**Proof.** It is sufficient to prove that $F$ is separately a polynomial in each variable. So, it it sufficient to prove the lemma in one dimension. From now on, let us assume that $d = 1$, and prove that

$$|F(z)| \leq C(1 + |z|)^N, \quad (18)$$

from which we conclude by Liouville’s Theorem. This bound is clearly valid on the two real and imaginary axes. Let us consider $z$ in the first quadrant. Then $z$ lies inside the angular sector with boundary given by the real axis and the line $\Im(z) = \varepsilon - 1 \Re(z)$ for $\varepsilon > 0$ small enough.

We will use an elementary variant of Phagmèn-Lindelöf’s principle which we recall here, and which may be found in: [17]: let $G$ be an entire function of order 2 in the complex plane and let $\alpha \in [0, \pi/2]$; assume that $|G(z)|$ is bounded by $C(1 + |z|)^N$ on the boundary of some angular sector $\{r e^{i\beta} : r \geq 0, \beta_0 \leq \beta \leq \beta_0 + \alpha\}$. Then the same bound is valid inside the angular sector (when replacing $C$ by $2^N C$).

Let us note $G(z) = F(z) e^{\varepsilon z^2}$, which is of order 2, and take as angular sector the one that has been described above. The polynomial bound that is required is valid on the boundary of this sector. So, by Phagmèn-Lindelöf’s principle, there is also a polynomial bound inside the angular sector. Moreover, the constant does not depend on $\varepsilon$. Letting $\varepsilon$ tend to 0, we conclude that the required estimate (18) holds inside the first quadrant. The same proof is valid in the other quadrants.

This finishes the proof of the lemma. \qed

This allows us to conclude also for the proof of the theorem. Theorem 3.1 and Theorem 3.2 are easy corollaries. \qed

As it was pointed out in [7] for Theorem 3.2, one can weaken the assumptions of Theorem 3.3 in more than one dimension.
Theorem 3.7
Let \( f \in S'(\mathbb{R}^d) \). We assume that, for each \( j = 1, \cdots, d \)
\[
e^{-\pi|x_j|^2} f \in S'(\mathbb{R}^d) \quad \text{and} \quad e^{-\pi|\xi|^2} \hat{f} \in S'(\mathbb{R}^d).
\]
Then \( f(x) = P(x)e^{-\pi|x|^2} \) for some polynomial \( P \).

Proof. We proceed exactly as before, to find that \( F = Bf \) is a polynomial
in each of the variables \( z_j \), the other ones being fixed. □

Moreover, one directional assumption is sufficient to have a sharp
information on the distribution.

Theorem 3.8
Let \( f \in S'(\mathbb{R}^d) \). We assume that
\[
e^{-\pi|x_1|^2} f \in S'(\mathbb{R}^d) \quad \text{and} \quad e^{-\pi|\xi_1|^2} \hat{f} \in S'(\mathbb{R}^d). \tag{19}
\]
Then there exists an integer \( N \geq 0 \) and distributions \( f_k \in S'(\mathbb{R}^{d-1}) \)
such that \( f \) may be written as
\[
f(x) = \sum_{k=0}^{N} x_1^k e^{-\pi|x_1|^2} \otimes f_k(x_2, \cdots, x_d).
\]

Proof. Let us note \( x' = (x_2, \cdots, x_d) \in \mathbb{R}^{d-1} \). We prove as before that
\( F \) is a polynomial in the variable \( z_1 \). This means that \( F \) may be written as
\[
F(z) = \sum_{k=0}^{N} F_k(z')z_1^k,
\]
where the \( F_k \)'s are entire functions on \( C^{d-1} \).

Every function \( F_k e^{-\frac{\pi}{2}|x'|^2} \) defines a tempered distribution when re-
stricted to \( \mathbb{R}^{d-1} \). Indeed, \( F e^{-\frac{\pi}{2}|x|^2} \) is a tempered distribution as the
inverse Fourier transform of \( e^{-\pi|\xi|^2} \hat{f} \). Moreover, one can find \( N + 1 \) test functions \( \varphi_k \) with compact support in one variable, such that
\[
\left( x_1^k e^{-\pi/2|x_1|^2}, \varphi_j \right) = \delta_{k,j}.
\]
Then,
\[
\left( F_k e^{-\pi/2|x'|^2}, \psi(x') \right) = \left( F e^{-\pi/2|x|^2}, \varphi_k(x_1) \psi(x') \right).
\]
Taking the inverse Fourier transforms of \( F_k e^{-\frac{\pi}{2}|x'|^2} \), we can write
\[
e^{-\pi|\xi|^2} \hat{f} = \sum_{k=0}^{N} \xi_1^k e^{-2\pi|\xi_1|^2} \otimes g_k(\xi'),
\]
for some \( g_k \in S'(\mathbb{R}^{d-1}) \). If we proceed as above, we can extract each coefficient from the sum, and prove that \( e^{\pi |\xi'|^2} g_k \) is a tempered distribution. A last inverse Fourier transform allows to conclude. \( \square \)

This theorem has many corollaries. One is used to Uncertainty Principles which assert that functions that are too small at infinity, as well as their Fourier transforms, vanish. Let us mention, especially, these two direct corollaries.

**Corollary 3.9**

Let \( f \in S'(\mathbb{R}^d) \). We assume that, for some \( a > 1 \),

\[
e^{\pi|x|^2} f \in S'(\mathbb{R}^d) \quad \text{and} \quad e^{a\pi|\xi|^2} \hat{f} \in S'(\mathbb{R}^d).
\]

Then \( f = 0 \).

**Corollary 3.10**

Let \( f \in S'(\mathbb{R}^d) \). We assume that \( f \) and \( \hat{f} \) are supported in a band \( \{|x_1| < C\} \). Then \( f = 0 \).

This last corollary may be seen as a variant of the notion of weakly annihilating pairs of sets for distributions.

One can also conclude when conditions on \( f \) and \( \hat{f} \) involve two quadratic forms \( q \) and \( q' \). For a non degenerate quadratic form \( q \), we note \( q^* \) the dual quadratic form, such that

\[
q^*(\xi) := \langle A^{-1}\xi, \xi \rangle \quad \text{whenever} \quad q(x) := \langle Ax, x \rangle.
\]

Let us recall, in particular, that the Fourier transform of the function \( e^{-\pi q(x)} \) is, up to a constant, equal to \( e^{-\pi q^*(\xi)} \) whenever \( q \) is positive-definite. We have

**Corollary 3.11**

Let \( f \in S'(\mathbb{R}^d) \). Let \( q, q' \) be two positive quadratic forms. Assume that

\[
e^{\pi q(x)} f \in S'(\mathbb{R}^d) \quad \text{and} \quad e^{\pi q'(\xi)} \hat{f} \in S'(\mathbb{R}^d).
\]

Then \( f \) vanishes unless \( q^* - q' \) is semi-positive definite. For \( q^* = q' \), then \( f \) may be written as \( Pe^{-\pi q} \) for some polynomial \( P \).

**Proof.** After a change of coordinates, we can assume that

\[
q(x) := |x|^2 \quad \text{and} \quad q'(\xi) := a_1\xi_1^2 + \cdots + a_d\xi_d^2.
\]

We use Corollary 3.9 to conclude. We could easily go on in the description of \( f \) when \( q^* - q' \) is semi-positive definite without vanishing. \( \square \)

We will consider, from now on, non positive non degenerate quadratic forms.
4. Beurling’s uncertainty principle for distributions

There is a story about a theorem of Beurling that had been lost. Hörmander has given a proof of this theorem in [26]. It is not an easy proof, so that Beurling’s Theorem was considered as a mysterious isolated phenomenon. The proof was simplified in [7], and its statement was extended. We give here this generalized statement.

**Theorem 4.1** (Beurling type)

Let \( f \in L^2(\mathbb{R}^d) \) and \( N \geq 0 \). Then

\[
\iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(x)||\hat{f}(y)| (1 + |x| + |y|)^N e^{2\pi|x,y|} dxdy < +\infty
\]

if and only if \( f \) may be written as

\[
f(x) = P(x)e^{-\pi(Ax,x)},
\]

where \( A \) is a real positive definite symmetric matrix and \( P \) is a polynomial of degree \( < \frac{N-d}{2} \).

We will give a simple proof of a bilinear and distribution version, which implies directly Theorem 4.1.

**Theorem 4.2**

Let \( f, g \) be two non zero distributions, \( f, g \in S'((\mathbb{R}^d) \), such that

\[ e^{\pm 2\pi(x,y)} f \otimes \hat{g} \in S'(\mathbb{R}^d \times \mathbb{R}^d) \]

and

\[ e^{\pm 2\pi(x,y)} \hat{f} \otimes g \in S'(\mathbb{R}^d \times \mathbb{R}^d) \] (24)

Then there exists an orthogonal decomposition of \( \mathbb{R}^d \), that is \( \mathbb{R}^d = E' \oplus E'' \), such that \( f \) and \( g \) may be written as

\[
f(x) = P(x',\partial_{x''})e^{-\pi(Ax',x')} \quad \text{and} \quad g(x) = Q(x',\partial_{x''})e^{-\pi(Ax',x')}
\]

where \( A \) is a real non negative symmetric matrix and \( P, Q \) are polynomials. Here \( x' \) and \( x'' \) are the orthogonal projections of \( x \) on \( E' \) and \( E'' \).

**Proof.** Let us emphasize the fact that now, with distributions, non negative definite symmetric matrices are allowed, as well as derivatives of Dirac masses. We go back to the notations used in the proof of Theorem 3.3, except that we have now two distributions \( f \) and \( g \). So, we define \( F := Bf \) by (11), and define as well \( G := Bg \). We can write

\[
F(z)G(-iz) = e^{\pi z^2} \left\langle f(x) \otimes \hat{g}(y), e^{-\pi((x-z)^2+(y-z)^2)} \right\rangle.
\]
We will prove that $F(z)G(-iz)$ is a polynomial, using Lemma 3.6. Let $\varphi_0$ be a smooth function in one variable, which is supported in $[-2,2]$ and identically 1 on $[-1,1]$. We write $1 = \varphi_+ + \varphi_0 + \varphi_-$, with $\varphi_\pm$ supported on the positive real axis. We use the same trick, multiplying and dividing by $e^{2\pi i(x,y)}$, as in the proof of Theorem 3.3, and find that $F(z)G(-iz) = A(z) + A_0(z) + A_+(z)$, with
\[
A_{\pm}(z) = e^{-\pi z^2} \left(e^{\pm 2\pi i(x,y)} f(x) \otimes \widehat{g}(y), \varphi_{\pm}(x,y) e^{-\pi \langle |x|^2 + |y|^2 \pm 2(x+y)\rangle - 2(x+y,z)} \right),
\]
while
\[
A_0(z) = e^{-\pi z^2} \left(f(x) \otimes \widehat{g}(y), \varphi_0((x,y)) e^{-\pi \langle |x|^2 + |y|^2 - 2(x+y)\rangle} \right).
\]
For $z$ and $\zeta$ in $\mathbb{C}^d$, we have used the notation
\[
z \cdot \zeta := z_1\zeta_1 + z_2\zeta_2 + \cdots + z_d\zeta_d. \tag{25}
\]
In order to have the required estimate, if we note $u$ the real part of $z$, we conclude using the following lemma, where the variables $X$ and $Y$ stand for $x + y$ and $x - y$.

**Lemma 4.3**

There exists a constant $C$ such that, for every $u \in \mathbb{R}^d$,
\[
\begin{align*}
(i) \quad |X|^M e^{-\pi \langle |X|^2 - 2(X,u) \rangle} & \leq C(1 + |u|)^M e^{\pi |u|^2} \text{ for } |X| > |Y| \\
(ii) \quad |Y|^M e^{-\pi \langle |Y|^2 - 2(X,u) \rangle} & \leq C(1 + |u|)^M e^{\pi |u|^2} \text{ for } |X| < |Y| \\
(iii) \quad (|X| + |Y|)^M e^{-\pi \langle \frac{1}{2}(|X|^2 + |Y|^2) - 2(X,u) \rangle} & \leq C(1 + |u|)^M e^{\pi |u|^2} \text{ for } |X|^2 - 8 < |Y|^2 < |X|^2 + 8.
\end{align*}
\]

**Proof.** The first inequality is elementary: just write that $|X| \leq |X-u| + |u|$. For the second one, write that $|Y| \leq 2(|X-u| + |u|) + 2(|Y|^2 - |X|^2)^{\frac{1}{2}}$. For the last one, write that $|X| + |Y| \leq 2(|X-u| + |u|)$.

We have proved the estimate
\[
|F(z)G(-iz)| \leq C(1 + |z|)^N e^{\pi/2|\Im(z)|^2}.
\]
We obtain the other required estimate when exchanging the roles of $f$ and $g$. By Lemma 3.6, we know that $F(z)G(-iz)$ is a polynomial. The assumptions are invariant through the change of $f(x)$ into $f(-x)$, so $F(z)G(iz)$ is also a polynomial. To conclude, we will use the following lemma in several complex variables. A proof can be found in [7].

**Lemma 4.4**

Let $\varphi$ be an entire function of order 2 on $\mathbb{C}^d$ such that, on every complex line, either $\varphi$ is identically 0 or it has at most $N$ zeros. Then, there exists a polynomial $P$ with degree at most $N$ and a polynomial $R$ with degree at most 2 which vanishes at 0, such that $\varphi(z) = P(z)e^{R(z)}$.  

So, we know that $F(z) = P(z)e^{R(z)}$ and $G(z) = Q(z)e^{S(z)}$. Let us use again the fact that $F(z)G(-iz)$ and $F(z)G(iz)$ are polynomials to conclude that $S = R$ is a homogeneous polynomial of degree 2. So, we can write $R(z) = -\frac{\pi}{2}z^2 + \pi A \cdot z + \pi B z \cdot z$, with $A$ and $B$ real symmetric matrices. As above, we deduce from the fact that $f$ is a tempered distribution the inequality

$$|F(z)e^{\pi z^2}| \leq C(1 + |z|)^N e^{\pi|\Re z|^2}.$$ 

Taking $z = iu$, with $u \in \mathbb{R}^d$, it follows that $A$ and $I - A$ are non negative. From now on, the proof is tedious, but elementary. One has to invert the Bargmann transform. We refer to [7] when the eigenvalues of $A$ are all different from 0 or 1, and to [13] in the general case.

Let us note that the proof was divided into two parts. In the first one, we considered the distribution $h(x,y) := f(x) \otimes \hat{g}(y)$, and studied its Bargmann transform, which is given by $Bh(z,\zeta) = F(z)G(-i\zeta)$. The assumptions allowed us to prove that it is a polynomial on the diagonal, using the same proof as in Hardy’s Uncertainty Principle. All this part is easily generalized to distributions that are not given by a tensor product, and we will do it in the next sections.

As for Hardy’s Uncertainty Principle, it is easy to deduce from the theorem sufficient conditions for which all solutions are identically 0.

**Corollary 4.5**

Let $f, g \in S'(\mathbb{R}^d)$. Assume that

$$e^{\pm 2\pi \langle x,y \rangle} f \otimes \hat{g} \in S'(\mathbb{R}^d \times \mathbb{R}^d)$$

and

$$e^{\pm 2\pi \langle x,y \rangle} \hat{f} \otimes g \in S'(\mathbb{R}^d \times \mathbb{R}^d).$$

(26)

Assume, moreover, that $f$ and $\hat{g}$ are given by locally integrable functions such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x)||\hat{g}(y)|}{(1 + |x| + |y|)^N} e^{2\pi|\langle x,y \rangle|} dx dy < +\infty.$$ 

Then $f$ or $g$ vanishes when $N \leq d$. For $N > d$, then $f$ and $g$ may be written as

$$f(x) = P(x)e^{-\pi\langle Ax,x \rangle}, \quad g(x) = Q(x)e^{-\pi\langle Ax,x \rangle},$$

where $A$ is a real positive definite symmetric matrix and $P, Q$ are polynomials, with $\deg(P) + \deg(Q) < N - d$, when $N > d$. 

5. Weak uncertainty principles related to general non degenerate quadratic forms

Our generalization of Beurling’s Theorem may be seen as a description of particular distributions in \( \mathbb{R}^d \times \mathbb{R}^d \) that are sufficiently decreasing, as well as their Fourier transforms, in terms of the quadratic form \( \langle x, y \rangle \).

We shall, in the same time, generalize the problem to all non degenerate quadratic forms, and reduce to weak uncertainty principles, that is, sufficient conditions on a distribution \( f \) and its Fourier transform that imply that \( f = 0 \).

**Theorem 5.1** (Demange)

Let \( q \) be a non degenerate quadratic form and let \( q^* \) be its dual. Let \( f \in S'(\mathbb{R}^d) \) such that \( fe^{\pm \pi q} \) and \( \hat{f}e^{\pm \pi q^*} \) \( f \in S'(\mathbb{R}^d) \). Furthermore, we assume that \( f \) is an integrable function and satisfies the condition:

\[
\int_{\mathbb{R}^d} |f(x)|e^{\pi |q(x)|} dx < +\infty \tag{27}
\]

Then \( f \) vanishes identically.

**Proof.** Eventually making a linear change of variables, we may assume that \( q(x) = q^*(x) = x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_d^2 \).

We introduce the same auxiliary function \( F = Bf \) as in the proof of Theorem 3.3,

\[
F(z) = e^{\pi/2z^2} \left\langle f(x), e^{-\pi (\langle x-z^2 \rangle)} \right\rangle = e^{-\pi/2z^2} \left\langle \hat{f}(x), e^{-\pi (\langle x-iw \rangle)} \right\rangle.
\]

We will prove that \( F = 0 \). We use the explicit expression for \( q \) and \( q^* \) and write \( z = (u, v) \) with \( u \in \mathbb{C}^k \) and \( v \in \mathbb{C}^{d-k} \). The assumption on \( f \) and elementary computations lead to the estimate

\[
|F(u, v)| \leq Ce^{\pi/2R(u^2 + v^2)} \max_{|x| \geq |y|; x, y \in \mathbb{R}^d} e^{-2\pi(|x|^2 - 2(\langle x, Ru \rangle - 2(y, Rv)))} + \max_{|y| \geq |x|; x, y \in \mathbb{R}^d} e^{-2\pi(|y|^2 - 2(\langle x, Ru \rangle - 2(y, Rv)))}.
\]

These maxima are equal to \( e^{\frac{\pi}{2}(1 + |Ru| + |Rv|)^2} \). So, we have the estimate

\[
|F(u, v)| \leq Ce^{\pi/2(2|R_u||R_v| + 3|u|^2 + |v|^2)} \tag{28}
\]

If we use the other expression for \( F \), real and imaginary parts are exchanged. Since \( \hat{f} \) is only a distribution, the proof is a little more complicate, but proceeding as in the previous section we find that

\[
|F(u, v)| \leq C(1 + |u| + |v|)^N e^{\pi/2(2|u||v| + 3|u|^2 + |v|^2)} \tag{29}
\]
So, using Lemma 3.6, we find that $F(u, 0)$ and $F(0, v)$ are polynomials. Since they are bounded for $u \in \mathbb{R}^d$ (resp. $v \in \mathbb{R}^d$), they are constant. Moreover, for $u$ real tending to infinity and $v = 0$, we have that $F(u, v)$ tends to 0. This is obtained when seeing it as an integral of a function that tends to 0, and using Lebesgue Bounded Convergence Theorem. So, we have $F(u, 0) = F(0, v) = 0$. Let us now consider derivatives of $F$. A small modification of computations above leads to the same estimates as above, except for a polynomial growth.

\[
|\partial_\alpha F(u, v)| \leq C(1 + |u| + |v|)^{|\alpha|e^{\pi/2}(2|\Re u| + |3u|^2 + |3v|^2)}.
\]

So $\partial_\alpha F(u, 0)$ and $\partial_\beta F(0, v)$ are polynomials of degree at most $|\alpha| - 1$ (resp. $|\beta| - 1$). Indeed, the inequalities above prove that they are polynomials of degree at most $|\alpha|$ (resp. $|\beta|$), and the behavior at infinity when $u$ or $v$ are real proves that they cannot be of degree $|\alpha|$ (resp. $|\beta|$).

Finally, all derivatives of $F$ vanish at $0$:

\[
\partial_\alpha \partial_\beta F(0, 0) = \partial_\alpha (\partial_\beta F(u, 0))|_{u=0} = 0
\]

if $|\alpha| \geq |\beta|$. Exchanging the role of variables, this derivative vanishes also when $|\alpha| < |\beta|$. This finishes the proof of Theorem 5.1 under the first assumption. □

One may weaken the assumptions, and generalize both the theorem and the corollary to some pairs of quadratic forms that are not conjugate. We refer to [12] for this. A corollary of these generalizations is the following, where new weakly annihilating sets are given in terms of conjugate quadratic forms.

**Corollary 5.2** (Demange)

Let $f \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq 2$ and $C > 0$. We assume that $f$ is supported by the set $\{x \in \mathbb{R}^d; |q(x)| < C\}$, and $\hat{f}$ is supported by the set $\{x \in \mathbb{R}^d; |q^*(x)| < C\}$. Then $f = 0$.

For $p = 1$, the corollary is a direct consequence of Theorem 5.1, using the fact that the assumptions on the supports imply that $f e^{\pm \pi q}$ and $\hat{f} e^{\pm \pi q}$ are tempered distributions. We refer to [12] for the other cases. For $p = 2$ and $C$ small, it is also a consequence of the theorem of Shubin, Vakilian and Wolff.

We are now in position to prove the theorem given in the introduction, related to Dimension 2. After a linear change of variables, we consider as well the quadratic form $q(x, y) = x^2 - y^2$. We shall prove a little more.

**Proposition 5.3**

Let $f \in S'(\mathbb{R}^2)$ such that $f e^{\pm \pi (x^2 - y^2)}$ and $\hat{f} e^{\pm \pi (\xi^2 - \eta^2)} \in S'(\mathbb{R}^2)$. Furthermore, we assume that $f$ is a locally integrable function and satisfies the condition:
\(f e^{\pi a|x^2 - y^2|}\) belongs to \(L^p(\mathbb{R}^2)\) for some \(a > 1, p \in (1, +\infty]\). \hfill (30)

Then \(f\) vanishes identically.

**Proof.** We only sketch the proof and use the notations above. The main point is to prove that \(F(u, 0)\) is zero, knowing already that it is a polynomial (and the same for \(F(0, v)\)). We consider this quantity for \(u\) real. We cut the integral into two parts, depending whether \(|x| > |y|\) or not, and use Hölder’s inequality. We will consider the first case, the proof being the same for the other one. It is sufficient to prove that

\[
\int \int_{|x| > |y|} e^{-p'(1+a)\pi |x|^2 - (a-1)\pi |y|^2 - 2\pi xu + \pi/2 |u|^2} dx dy
\]

tends to zero when \(u\) tends to infinity. A first integration in \(y\) allows us to reduce to

\[
\int e^{-p' [2\pi |x|^2 - 2\pi xu + \pi |u|^2]} dx,
\]

and this last quantity is bounded by \(C(1 + |u|)^{-1}\).

We do not know, at this moment, whether this proposition can be generalized in higher dimension.

6. Strong uncertainty principles related to non degenerate quadratic forms

We speak of strong uncertainty principle when we have a complete description of all distributions \(f\) such that

\(f e^{\pm \pi q} \in S'(\mathbb{R}^d)\) and \(\hat{f} e^{\pm \pi q^*} \in S'(\mathbb{R}^d)\).

Before proceeding with a particular case, let us mention the following invariance property, whose proof is direct.

**Lemma 6.1**

If \(f \in S'(\mathbb{R}^d)\) is such that \(f e^{\pm \pi q} \in S'(\mathbb{R}^d)\) and \(\hat{f} e^{\pm \pi q^*} \in S'(\mathbb{R}^d)\), then the same is valid for all derivatives of \(f\) and multiplication by polynomials.

We have seen that we could characterize all such \(f\) for \(q\) positive definite. In the general case, it is much more difficult. In particular,
there may exist distributions that are supported in the zero set of $q$ and such that $\hat{f}$ is supported in the zero set of $q^*$. We will reduce here to $\mathbb{R}^2$. Up to a change of coordinates, there is only one quadratic form to consider. It will be easier to deal either with $q(x, y) = q^*(x, y) = x^2 - y^2$, or with $2xy$ (as in Beurling’s Uncertainty Principle), depending on the issue. A simple change of variables allows us to go from one case to the other one.

First, one has a complete description of those distributions $f$ that are supported, as well as their Fourier transforms, by the zero set of the quadratic form. For this, it is easier to deal with the quadratic form $2xy$.

**Lemma 6.2**

Assume that $f \in S'(\mathbb{R}^2)$ is supported by the set where $xy = 0$, as well as its Fourier transform. Then $f$ is a finite linear combination of distributions $x^k \otimes \delta(k)(y)$ and $y^l \otimes \delta(l)(x)$.

The proof is classical. We use the fact that $\partial_x^N \partial_y^N f = 0$ for some integer $N$, and the same for its Fourier transform.

The next examples are Gaussian functions.

**Example 6.3** Let $f(x, y) = e^{-\pi (A(x,y),(x,y))}$, with $A$ some $2 \times 2$ positive definite matrix. Then, the two conditions $fe^{\pm 2\pi xy} \in S'(\mathbb{R}^2)$ and $\hat{f}e^{\pm 2\pi xy} \in S'(\mathbb{R}^2)$ (31)

are satisfied if and only if $A$ may be written as

$$
\begin{pmatrix}
t & 0 \\
0 & \frac{1}{t}
\end{pmatrix}
$$

for some $t > 0$.

As a consequence, all linear combinations of such Gaussian functions will satisfy the same assumptions.

Let us go on with some other examples, which look surprising, even though they are very simple.

**Example 6.4** Let $f_0(x, y) = \frac{e^{-\pi |x||y|}}{|y|}$. Then

$$
\hat{f}_0(x, y) = -i \frac{x}{|x|} e^{-2\pi |x||y|}.
$$

In particular, $f_0$ satisfies the conditions (31).

To prove this formula, it is sufficient to prove that the partial Fourier transform of the first function in the second variable coincides with the
inverse Fourier transform of the second one in the first variable. In both cases, we find the conjugate Poisson kernel, \( \frac{1}{\pi} \frac{x}{x^2 + y^2} \). Moreover, using the fact that

\[
\frac{1}{\pi} \frac{x}{x^2 + y^2} = \int_0^{+\infty} xe^{-\pi t(x^2 + y^2)} dt
\]

and taking the inverse Fourier transform in \( y \), we find that

\[
f_0(x, y) = \int_0^{+\infty} x e^{-\pi tx^2 - \pi/2 ty^2} t^{-1/2} dt
\]

for \( x \neq 0 \). We have written \( f_0 \) as a linear combination of the Gaussian functions that satisfy (31). Let us remark that the integral makes sense when \( xy \neq 0 \), and also in the distribution sense.

This example has the property to give an equality case in the analogue of Hardy’s Theorem. Multiplying by polynomials or taking derivatives, we can also find families of examples \( f \) for which \( f e^{2\pi |xy|} \) has an arbitrary polynomial growth. The following example is no more a function, but a distribution of order 1.

**Example 6.5** There exists a distribution \( f \in S'(\mathbb{R}^2) \) satisfying (31) that coincides with \( \frac{1}{|x|} e^{-2\pi |x||y|} \) for \( x \neq 0 \), and such that \( \hat{f} \) coincides with \( \frac{1}{|y|} e^{-2\pi |x||y|} \) for \( y \neq 0 \).

The distribution \( f \) may be obtained as the inverse Fourier transform in the second variable of the distribution \( p.v. \frac{1}{\pi (x^2 + y^2)} \). The formulas follow at once. To prove that \( f e^{\pm 2\pi xy} \) is a tempered distribution, we write \( e^{\pm 2\pi xy} = \varphi(xy) e^{\pm 2\pi xy} + (1 - \varphi(xy)) e^{\pm 2\pi xy} \), with \( \varphi \) smooth and supported in \([-1, +1]\). The first term gives rise to a tempered distribution since \( f \) is a tempered distribution. For the second one, we use the explicit formula for \( f \). Note that we can again write, outside the axes,

\[
f(x, y) = \int_0^{+\infty} e^{-\pi tx^2 - \pi/2 ty^2} t^{-1/2} dt.
\]

In fact, when multiplied by \( x \), we recognize the distribution \( f_0 \) given in the previous example.

We will see that all distributions that satisfy (31) are given by such integrals. It is the case of all our examples, even the distributions given by Lemma 6.2: if we consider the distribution \( \delta(y) \), for instance, its partial Fourier transform is identically 1, which may be written as

\[
1 = \pi (x^2 + y^2) \int_0^{+\infty} e^{-\pi t(x^2 + y^2)} dt.
\]

Taking the inverse Fourier transform, we get at least a formal expression.
Let us now state the main theorem. We first give some notations. For \( t > 0 \), we denote by \( u_t \) the exponential
\[
u_t(x, y) := e^{-\pi tx^2 - \pi ty^2},
\]
and, for \( \mu \) a bounded measure on \( (0, \infty) \), by \( u_\mu \) the bounded function
\[
u_\mu(x, y) := \int_0^{+\infty} e^{-\pi tx^2 - \pi ty^2} d\mu(t).
\]
Its Fourier transform is obtained when exchanging \( x \) and \( y \), and it satisfies the inequality
\[|\nu_\mu(x, y)| \leq \|\mu\|e^{-2\pi|xy|}.
\]
Moreover, \( u_\mu \) is real analytic outside the axes.

**Theorem 6.6** (Demange)

Let \( f \in S'(\mathbb{R}^2) \). Then \( f \) satisfies the conditions

\[
fe^{\pm 2\pi xy} \in S'(\mathbb{R}^2) \quad \text{and} \quad \hat{f}e^{\pm 2\pi xy} \in S'(\mathbb{R}^2)
\]

if and only if \( f \) may be written as

\[
f = \sum_{k=1}^{K} P_k(x, \partial_x, y, \partial_y) u_{\mu_k},
\]

where \( \mu_k \) are bounded measures. In particular, such a distribution is a real-analytic function outside the axes.

Taking derivatives in \( x \) and \( y \), let us remark that such a distribution may also be written outside the axes as

\[
\sum_{k=1}^{K} P_k(x, y) \int_0^{+\infty} e^{-\pi tx^2 - \pi ty^2} d\nu_k(t), \quad (32)
\]

where \( P_k \) are polynomials and \( \nu_k \) are Radon measures on \( (0, \infty) \) for which

\[
\int_0^{\infty} (1 + t + t^{-1})^{-m} d|\nu_k|(t) < \infty.
\]

We have already done half of the proof of the theorem. In order to prove the converse, it will be easier to consider the quadratic form \( x^2 - y^2 \), so that we can rely on part of the computations of the previous section. Before starting, we write the equivalent of Example 6.3 in this context.
Example 6.7 Let \( f(x, y) = e^{-\pi (A(x, y), (x, y))} \), with \( A \) some \( 2 \times 2 \) positive definite matrix. Then, the two conditions

\[
f_{e^{\pm \pi (x^2 - y^2)}} \in S'(\mathbb{R}^2) \quad \text{and} \quad \hat{f}_{e^{\pm \pi (x^2 - y^2)}} \in S'(\mathbb{R}^2)
\]

(33) are satisfied if and only if \( A \) may be written as

\[
\begin{pmatrix}
\alpha & \beta \\
\beta & \alpha
\end{pmatrix}
\]

for some \( \alpha, \beta \), with \( \alpha = \sqrt{\beta^2 + 1} \). Moreover, in this case, \( \mathcal{B}f(z, \zeta) = \sqrt{1 - t^2} e^{\pi \alpha \zeta} \), with \( t = \frac{\beta}{\alpha + 1} \).

The best is to parameterize such quadratic forms using \( t \), which belongs to \((-1, +1)\). We call \( e_t \) the corresponding exponential, that is,

\[
e_t(x, y) := e^{-\pi (1 + t^2)/(1 - t^2) (x^2 + y^2) + \pi t/(1 - t^2) xy},
\]

and

\[
e_{\mu}(x, y) = \int_{-1}^{+1} e_t(x, y) d\mu(t)
\]

for \( \mu \) a bounded measure on \((-1, +1)\). This is a real-analytic function outside the diagonals.

Theorem 6.8

Let \( f \in S'(\mathbb{R}^2) \). Then \( f \) satisfies the condition

\[
f_{e^{\pm \pi (x^2 - y^2)}} \in S'(\mathbb{R}^2) \quad \text{and} \quad \hat{f}_{e^{\pm \pi (x^2 - y^2)}} \in S'(\mathbb{R}^2)
\]

if and only if \( f \) may be written as

\[
f = \sum_{k=1}^{K} P_k(x, \partial_x, y, \partial_y) e_{\mu_k},
\]

where \( \mu_k \) are bounded measures on \((-1, 1)\). In particular, such a distribution is a real-analytic function outside the diagonals.

It remains to prove the necessary condition. Let \( F = \mathcal{B}f \) be as in the last section. We sketch the proof, and refer to [12] for the details. The same kind of computations allows us to obtain that

\[
F(z, \zeta) = \sum_{-N}^{+N} F_k(z, \zeta),
\]
where each of the functions that satisfy the homogeneity relations $F_k(tz, t^{-1} \zeta) = t^k F(z, \zeta)$ for $t \in \mathbb{C}^*$. Moreover, solving a linear system of equations for $2N + 1$ distinct values of $t$ allows to have for each $F_k$ the same kind of estimates that we have for $F$:

$$|F_k(z, \zeta)| \leq C(1 + |z| + |\zeta|) M e^{\pi/2(2|z| + (3|z|)^2 + (3|\zeta|)^2)},$$

$$|F_k(z, \zeta)| \leq C(1 + |z| + |\zeta|) M e^{\pi/2(2|\zeta| + (3|\zeta|)^2 + (3|z|)^2)}.$$

It is straightforward that each $F_k(z, \zeta)$ can be written as $z^k G_k(z\zeta)$ or $\zeta^{-k} G_k(z\zeta)$ depending on the sign of $k$. Moreover, it may be seen (and we refer to [12] for this) that each $G_k$ has exponential growth, and satisfies the conditions of Paley-Wiener Theorem, that is, may be written as the Laplace transform of a distribution $\nu_k$ with support in $[-1, 1]$ (with the normalization given below). We will use the following elementary lemma.

**Lemma 6.9**

A distribution $\nu$ with support in $[-1, 1]$ can be written as a finite sum of derivatives (of any order) of some measures $\theta_l$ that are also supported in $[-1, 1]$ and satisfy

$$\int \frac{d\theta_l(t)}{\sqrt{|1-t^2|}} < \infty.$$

Finally, we have written

$$F(z, \zeta) = \sum_{k=1}^{K} P_k(z, \zeta) \int_{-1}^{1} e^{\pi x t} d\mu_k(t),$$

with $\mu_k$ that satisfy the above integrability condition. It remains to see that each function

$$G(z, \zeta) = P(z, \zeta) \int_{-1}^{1} e^{\pi z \zeta t} d\mu(t)$$

may be written as $Bf$, for $f$ of the required form. It is clear when $P$ is identically 1. Indeed, using Example 6.7, we can write in this case that $G = B\nu$, with $\sqrt{1-t^2} d\mu(t) = d\nu(t)$.

To finish the proof, it is sufficient to use the following elementary lemma.

**Lemma 6.10** Let $f \in S'(\mathbb{R}^2)$. Then

$$B(2\pi x f - \partial_x f)(z, \zeta) = 2\pi z Bf(z, \zeta),$$

$$B(2\pi y f - \partial_y f)(z, \zeta) = 2\pi \zeta Bf(z, \zeta).$$
This completes the proof of the theorem.

As an immediate corollary, we have the following property, which is related to weakly annihilating pairs.

**Corollary 6.11**

Let \( f \in S'(\mathbb{R}^2) \) be supported in the set where \(|xy| \leq C\), as well as its Fourier transform. Then \( f \) is supported by the coordinate axes, as well as its Fourier transform.

Let us mention that B. Demange has generalized Theorem 6.8 to the quadratic form \( q(x, y) = |x|^2 - y^2 \) in \( \mathbb{R}^{d+1} \), with \( d > 1 \), see [12]. The main difference is the fact that those distributions \( f \in S'(\mathbb{R}^{d+1}) \) for which \( e^{\pm \pi q} f \in S'(\mathbb{R}^{d+1}) \) and \( e^{\pm \pi q} \hat{f} \in S'(\mathbb{R}^{d+1}) \) are only real-analytic inside the light cone \( C \), given by \( q < 0 \). Moreover, he has a complete description of those distributions that are supported in the complement of \( C \), as well as their Fourier transforms, in terms of integrals that are built from a different family of solutions.

It is much easier to generalize Theorem 6.6 to \( \mathbb{R}^d \times \mathbb{R}^d \), and tempered distributions such that

\[
\varphi(|x||y|) e^{2\pi|a|p} |x|^p f \in S'(\mathbb{R}^d) \quad \text{and} \quad \varphi(|x||y|) e^{2\pi|b|q} |y|^q \hat{f} \in S'(\mathbb{R}^d),
\]

with \( \varphi \) a \( C^\infty \) function that is identically 0 in a neighborhood of 0 and identically 1 outside a compact set. We refer to [13].

### 7. The Morgan uncertainty principle and one-sided estimates

There is a distribution version of the uncertainty principle of Morgan.

**Theorem 7.1** (Morgan)

Let \( 1 < p < 2 \), and let \( q \) be the conjugate exponent. Assume that \( f \in S'(\mathbb{R}^d) \) satisfies

\[
\varphi(x) e^{2\pi a/p} |x|^p f \in S'(\mathbb{R}^d) \quad \text{and} \quad \varphi(y) e^{2\pi b/q} |y|^q \hat{f} \in S'(\mathbb{R}^d),
\]

with \( \varphi \) a \( C^\infty \) function that is identically 0 in a neighborhood of 0 and identically 1 outside a compact set. Then \( f = 0 \) if \( ab > |\cos(\frac{\pi}{2})|^p \).

Morgan had considered the one dimensional case, and functions \( f \) that satisfy uniform estimates

\[
|f(x)| \leq Ce^{-2\pi a/p} |x|^p \quad \text{and} \quad |\hat{f}(y)| \leq Ce^{-2\pi b/q} |y|^q.
\]  \( (34) \)
Uncertainty principles related to quadratic forms

He has proved in [32] that such functions are identically 0 if \( ab > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}} \), and that this condition is sharp: he has constructed a family of examples in the equality case. Let us also mention that Gel’fand and Shilov have considered classes of functions that satisfy one of the inequalities of (34), as well as their derivatives, to generalize the notion of tempered distributions [16, 17].

Proof. As in Section 3, we will give the proof for functions. Moreover, the other variables do not play any role once one passes to the Bargmann transform, so that we will restrict to the dimension one. Finally, we assume that \( p < 2 \) since the statement has already been proved for \( p = 2 \), see Corollary 3.9. So, \( f \) is an integrable function such that

\[
e^{2\pi a^p/p|x|^p} f \in L^1(\mathbb{R}) \quad \text{and} \quad e^{2\pi b^q/q|y|^q} \hat{f} \in L^1(\mathbb{R}).
\]

After a dilation, we can assume that \( a = 1 \). We will use the elementary fact that for \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) such that, for \( t > s > 0 \)

\[
\frac{2}{p}(t^p - s^p) \leq C_\varepsilon + (t - s)^2 + \varepsilon^p.
\]

Indeed, the left hand side is bounded by \( 2(t - s)t^{p-1} \leq (t - s)^2 + t^{2p-2} \), which allows to conclude for \( s > t/2 \); otherwise, we write that \( \frac{2}{p}t^p \leq C + \frac{1}{4}t^2 \leq C + (t - s)^2 \).

Let us consider

\[ F(z) := e^{-\pi/2z^2} \mathcal{B} f(z). \]

Then it follows directly from (14) and from the assumption on \( \hat{f} \) that

\[
|F(z)| \leq Ce^{2\pi(b^p/p|\xi|^p}, \tag{35}
\]

using the inequality \( |\langle y, \eta \rangle| \leq \frac{|\eta|^p}{p} + \frac{|y|^q}{q} \). Now, using the first assumption on \( f \) and the inequality above, one proves that for \( z_1 = \xi \) real,

\[
|F(\xi)| \leq Ce^{-2\pi(1-z)^p/p|\xi|^p}, \tag{36}
\]

with \( \varepsilon > 0 \) arbitrarily close from 0. If we choose \( \varepsilon \) small enough so that \( (1 - \varepsilon)b > |\cos(\frac{p\pi}{2})|^{\frac{1}{p}} \), then one concludes that \( F \) is identically 0 using again Phragmen-Lindelöf’s Principle. Details on this last point may be found in [32, 33] or [7].

\[ \square \]

In the general case one has to compute norms \( p_N \), which may be done without real difficulty. This proof allows as well to deal with one-sided assumptions in Morgan’s or Hardy’s Theorem. Previous one-sided versions of Morgan’s Theorem are due to Nazarov [33].
Theorem 7.2 (One-sided Morgan type)

Let $1 < p < 2$, and let $q$ be the conjugate exponent. Assume that $f \in S'(\mathbb{R}^d)$ satisfies

$$\varphi(x_1) e^{2\pi a x_1^p / p} f \in S'(\mathbb{R}^d) \quad \text{and} \quad \varphi(y_1) e^{2\pi b y_1^q / q} \hat{f} \in S'(\mathbb{R}^d),$$

with $\varphi$ a $C^\infty$ function that is identically 0 in $(-\infty, 1]$ and identically 1 in $[2, +\infty)$. Then $f = 0$ if $ab > \sin(\pi / p)$.

Proof. As above, we give the proof in dimension one, when $f$ is an integrable function for which

$$\int_{1}^{\infty} e^{2\pi a x^p / p} |f(x)| dx < \infty \quad \text{and} \quad \int_{1}^{\infty} e^{2\pi b y^q / q} |\hat{f}(y)| dy < \infty.$$

We claim that we obtain (35) when $\Im z < 0$. Indeed, cutting the integral into two parts, we conclude as before for the integral from 0 to $\infty$, while the integral from $-\infty$ to 0 gives rise to a bounded quantity. For similar reasons, the estimate (36) is valid when $\xi > 0$, since the integral for negative $x$ is then bounded by $Ce^{-\pi |\xi|^2}$. We refer to [33] for the end of the proof, that is the use of Phragmen-Lindelöf principle in the quadrant $\Im z < 0, \Re z > 0$. Let us mention that the constant is also critical. \qed

One has also a one-sided version of Hardy’s Uncertainty Principle for distributions, that is, the last theorem is also valid for $p = 2$: it is sufficient to modify slightly the proofs of Section 2.

References


