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On weighted James' spaces

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Abstract

In this note we study the topological structure of weighted James spaces $J(h)$. In particular we prove that $J(h)$ is isomorphic to J if and only if the weight h is bounded. We also provide a description of $J(h)$ if the weight is a non-decreasing sequence.

Díaz and the author [3] have recently introduced weighted James spaces $J(h)$ to construct a Fréchet counterexample to a question of Valdivia. In this note we start a systematic study of the topological structure of these spaces. In particular we characterize when a weighted James space $J(h)$ is isomorphic to the classical quasirreflexive James space J. If the weight h is non-decreasing then we get a detailed description of $J(h)$.

The quasireflexive James space J is defined as

$$J := \left\{ y = (y_n);\ \|y\| := \sup\ \left(\sum_{i=0}^{k-1} \Big(\sum_{j=n_i+1}^{n_{i+1}} y_j \Big)^2 \right)^{1/2} < \infty \right\},$$

where the sup is taken over all increasing sequences of integers $0 = n_0 < n_1 < \ldots < n_k$ (see [5]). Given a map $h : \mathbb{N} \to [1, +\infty)$, we define the following *weighted* James space:

$$J(h) := \left\{ y = (y_n);\ \|y\| := \sup\ \Big(\sum_{i=0}^{k-1} \big(\sum_{j=n_i+1}^{n_{i+1}} y_j \big)^2 h(n_i+1) \Big)^{1/2} < \infty \right\},$$

where the sup is taken as above. In our proposition below we characterize when $J(h)$ is isomorphic to J. To handle $J(h)$ it is convenient to use also the increasing map $h' : \mathbb{N} \to [1, +\infty)$ defined as $h'(n) := \sup\{h(i);\ i \leq n\}$. Note that $\|e_j\| = h'(j)$, where e_j is the sequence taking the value 1 in the j-th place and 0 elsewhere, for every $j \in \mathbb{N}$. It is an open problem whether $J(h')$ is isomorphic or not to $J(h)$.

We recall some definitions and results about bases. Let X denote a Banach space. A sequence $(e_n)_{n\in\mathbb{N}}$ is said to be a basis of X if every element $x \in X$ can be written as $x = \sum_{i=1}^{\infty} a_i e_i$ for a unique sequence of scalars $(a_i)_{i\in\mathbb{N}}$. In such a case we define e_i^* in X' as follows: $e_i^*(x) := a_i$ for every $i \in \mathbb{N}$. The basis $(e_n)_{n\in\mathbb{N}}$ is said to be *shrinking* if $(e_n^*)_{n\in\mathbb{N}}$ is a basis of X' and is said to be *boundedly complete* if for every sequence of scalars $(a_n)_{n\in\mathbb{N}}$ the series $\sum_{i=1}^{\infty} a_i e_i$ converges provided that $(\|\sum_{i=1}^{n} a_i e_i\|)_{n\in\mathbb{N}}$ is bounded. It is a known result of James [4] that a Banach space X with basis is reflexive if and only if the basis is shrinking and boundedly complete. With the notation given above $(e_n)_{n\in\mathbb{N}}$ is a basis of $J(h)$ and it is known to be a boundedly complete basis of J.

The following technical lemma could be of independent interest. If $(y_i)_{i\in I}$ is contained in X then $\mathrm{sp}\{y_i;\ i \in I\}$ denotes the linear subspace spanned by $(y_i)_{i\in I}$.

Lemma 1

Let X be a Banach space with a basis $(e_n)_{n\in\mathbb{N}}$ that satisfies the following property: $\exists\ \delta > 0\ \forall\ n \in \mathbb{N}\ \exists\ r(n) > n\ \forall\ x \in \mathrm{sp}\{e_i;\ i \leq n\}\ \forall\ y \in \mathrm{sp}\{e_i;\ i \geq r(n)\}$,

$$\|x\| = \|y\| = 1 \Longrightarrow\ \|x + y\| < 2 - \delta.$$

Then the basis $(e_n)_{n\in\mathbb{N}}$ is shrinking.

Proof. Given $f \in X'$ we denote $f(e_j)e_j^*$ by $f_j,\ j \in \mathbb{N}$. Note that $f = \sum_{n=1}^{\infty} f_j$ where the series converges in the topology $\sigma(X', X)$. We have to show that $\Big(\sum_{j=n}^{\infty} f_j\Big)_{n\in\mathbb{N}}$ converges to 0 in norm. We first check the following property:

$$\forall f \in X'\ \exists n_0 \in N\ \forall n \geq n_0 \quad \Big\| \sum_{j=n}^{\infty} f_j \Big\| < \Big(1 - \frac{\delta}{3}\Big) \|f\|. \tag{1}$$

By contradiction let us assume that there is $f \in X'$ such that for every $n \in \mathbb{N}$ there is $m(n) > n$ with $\left\| \sum_{j=m(n)}^{\infty} f_j \right\| \geq (1 - \frac{\delta}{3})\|f\|$. We fix $x \in \mathrm{sp}\{e_i;\ i \in \mathbb{N}\}$ with $\|x\| = 1$ and such that $f(x) > (1 - \frac{\delta}{3})\|f\|$. Let $n \in \mathbb{N}$ be such that $x \in \mathrm{sp}\{e_i;\ i \leq n\}$. We select $r(n) > n$ according to our statement. Then there is $m(n) > r(n)$ such that $\left\| \sum_{j=m(n)}^{\infty} f_j \right\| \geq (1 - \frac{\delta}{3})\|f\|$. Now given $\varepsilon > 0$ we take $y \in \mathrm{sp}\{e_i;\ i \geq m(n)\}$, with $\|y\| = 1$ and such that $\sum_{j=m(n)}^{\infty} f_j(y) > (1 - \varepsilon)\left\| \sum_{j=m(n)}^{\infty} f_j \right\|$. By applying the property of the basis we get

$$
\begin{aligned}
(2 - \delta)\|f\| > \|x + y\|\|f\| &\geq f(x + y) = f(x) + f(y) \\
&= f(x) + \Big(\sum_{j=m(n)}^{\infty} f_j \Big)(y) > \Big(1 - \frac{\delta}{3} + (1 - \varepsilon)\big(1 - \frac{\delta}{3}\big)\Big)\|f\|.
\end{aligned}
$$

Since ε is arbitrary we get a contradiction that settles (1). To finish given any $f \in X$ there is $n_1 \in \mathbb{N}$ such that $\left\| \sum_{j=n}^{\infty} f_j \right\| < (1 - \frac{\delta}{3})\|f\|$ for every $n \geq n_1$. We apply (1) to $\sum_{j=n_1}^{\infty} f_j$ and get $n_2 \in \mathbb{N}$ such that

$$
\Big\| \sum_{j=n}^{\infty} f_j \Big\| < \Big(1 - \frac{\delta}{3}\Big) \Big\| \sum_{j=n_1}^{\infty} f_j \Big\| < \Big(1 - \frac{\delta}{3}\Big)^2 \|f\|,\ \forall n \geq n_2.
$$

By induction we can find n_r such that $\left\| \sum_{j=n}^{\infty} f_j \right\| < (1 - \frac{\delta}{3})^r \|f\|$, $\forall n \geq n_r$ and this already implies that $\sum_{j=1}^{\infty} f_j$ converges in norm. □

Our main result is the following:

Theorem 2

Let $J(h)$ be a weighted James space. The following conditions are equivalent:
(i) $J(h)$ is not isomorphic to J.
(ii) $\sup\{h(i);\ i \in \mathbb{N}\} = \infty$.
(iii) $J(h)$ is reflexive.

Proof. Only (ii) implies (iii) needs a proof. Let $(e_n)_{n\in\mathbb{N}}$ denote the canonical basis in $J(h)$. We shall check that this basis is boundedly complete and shrinking.

(a) $(e_n)_{n\in\mathbb{N}}$ *is boundedly complete.* This is clear if $(h(n))_{n\in\mathbb{N}}$ is non-decreasing, i.e. if $h \equiv h'$. For the general case let $(a_i)_{i\in\mathbb{N}}$ be a sequence of scalars such that $(\| \sum_{i=1}^{n} a_i e_i \|)_{n\in\mathbb{N}}$ is bounded. We have to check the following condition,

$$
\forall \varepsilon\ \exists n \in \mathbb{N},\ \forall q > p \geq n \quad \Big\| \sum_{j=p}^{q} a_j e_j \Big\| < \varepsilon. \tag{2}
$$

Since $(h(n))_{n\in\mathbb{N}}$ is unbounded we can find an increasing sequence of integers $(p(n))_{n\in\mathbb{N}}$ such that $h(p(n)) = h'(p(n))$ for every $n \in \mathbb{N}$. It is readily checked by contradiction that (2) holds whenever p is taken to coincide with $p(i)$ for some $i \in \mathbb{N}$. (Indeed observe that given $s \in \mathbb{N}$ with $h(s) = h'(s)$ and given $y = (y_n) \in J(h)$ such that $y_i = 0$ if $i \leq s-1$, then we can take the supremum over all increasing sequences $s-1 = n_0 < n_1 < \ldots < n_k$ to estimate $\|y\|$.) Then, given $n \in \mathbb{N}$ we fix $p(n) \geq n$ and if $q > p \geq p(n)$ put

$$\Big\|\sum_{j=p}^{q} a_j e_j\Big\| \leq \Big\|\sum_{j=p(n)}^{p} a_j e_j\Big\| + \Big\|\sum_{j=p(n)}^{q} a_j e_j\Big\|,$$

from where (2) follows.

(b) $(e_n)_{n\in\mathbb{N}}$ *is shrinking.* We are going to check the condition in Lemma 1. Given $n \in \mathbb{N}$ we take $r(n) > n$ such that $h(r(n)) \geq 2h(i)$ for all $i \leq n$. Let $x \in \mathrm{sp}\{e_i;\ i \leq n\}$, $y \in \mathrm{sp}\{e_i;\ i \geq r(n)\}$ with $\|x\| = \|y\| = 1$ and let us estimate $\|z\|$ where $z = x + y$. To do this we take $0 = n_0 < n_1 < \ldots < n_k$. We assume that $n_k > r(n)$. Two cases may happen:

(1) There is $s \in \mathbb{N}$ such that $n \leq n_s < r(n)$. Then we can write:

$$\sum_{i=0}^{k-1}\Big(\sum_{j=n_i+1}^{n_{i+1}} z_j\Big)^2 h(n_i+1) = \sum_{i=0}^{s-1}\Big(\sum_{j=n_i+1}^{n_{i+1}} x_j\Big)^2 h(n_i+1) + \sum_{i=s}^{k-1}\Big(\sum_{j=n_i+1}^{n_{i+1}} y_j\Big)^2 h(n_i+1) \leq \|x\|^2 + \|y\|^2.$$

(2) There is $s \in I\!N$ such that $n_s < n$ and $n_{s+1} \geq r(n)$. In this case we have:

$$\begin{aligned}\Big(\sum_{j=n_s+1}^{n_{s+1}} z_j\Big)^2 h(n_s+1) &\leq 2\Big(\Big(\sum_{j=n_s+1}^{n} x_j\Big)^2 h(n_s+1) + \Big(\sum_{j=n}^{n_{s+1}} y_j\Big)^2 h(n_s+1)\Big)\\ &\leq 2\Big(\sum_{j=n_s+1}^{n} x_j\Big)^2 h(n_s+1) + \Big(\sum_{j=r(n)}^{n_{s+1}} y_j\Big)^2 h(r(n)),\end{aligned}$$

whence

$$\begin{aligned}\sum_{i=0}^{k-1}\Big(\sum_{j=n_i+1}^{n_{i+1}} z_j\Big)^2 h(n_i+1) =&\sum_{i=0}^{s-1}\Big(\sum_{j=n_i+1}^{n_{i+1}} x_j\Big)^2 h(n_i+1)\\ &+\Big(\sum_{j=n_s+1}^{n_{s+1}} z_j\Big)^2 h(n_s+1)+\sum_{i=s+1}^{k-1}\Big(\sum_{j=n_i+1}^{n_{i+1}} y_j\Big)^2 h(n_i+1)\\ &\le 2\Big(\sum_{i=0}^{s-1}\Big(\sum_{j=n_i+1}^{n_{i+1}} x_j\Big)^2 h(n_i+1)+\Big(\sum_{j=n_s+1}^{n} x_j\Big)^2 h(n_s+1)\Big)\\ &+\Big(\sum_{j=r(n)}^{n_{s+1}} y_j\Big)^2 h(r(n))\\ &+\sum_{i=s+1}^{k-1}\Big(\sum_{j=n_i+1}^{n_{i+1}} y_j\Big)^2 h(n_i+1) \le 2\|x\|^2+\|y\|^2.\end{aligned}$$

In both cases we get $\|z\|^2 \le 3$. We conclude from Lemma 1. $\square$

In our following result we provide a description of $J(h)$ if $(h(n))_{n\in\mathbb{N}}$ is non-decreasing. For every $n \in \mathbb{N}$, J_n denotes the subspace of J spanned by $\{e_i;\ 1 \le i \le n\}$, and $J_0 := \{0\}$.

Theorem 3

Let $(h(n))_{n\in\mathbb{N}}$ be a non-decreasing and unbounded weight. Then there is a sequence of integers $(m(n))_{n\in\mathbb{N}}$ such that $J(h)$ is isomorphic to the l_2-sum

$$l_2(J_{m(n)}) := \left\{(x_n) \in \prod_n J_{m(n)};\ \Big(\sum_{n=1}^{\infty}\|x_n\|^2\Big)^{1/2} < \infty\right\}.$$

In particular $J(h)$ is isomorphic to a complemented subspace of J.

Proof. We choose $q(0) = 1 \le q(1) \le q(2) \le \dots$ such that $2^n \le h(i) < 2^{n+1}$, $\forall q(n) \le i < q(n+1), n = 0, 1, \dots$. If we define $h^*(i) := 2^n$, $\forall q(n) \le i < q(n+1), n = 0, 1, \dots$ then $h^* \le h \le 2h^*$, hence $J(h) \cong J(h^*)$. Thus we assume without loss of generality that $h(i) = 2^n$, $\forall q(n) \le i < q(n+1)$, $n = 0, 1, \dots$. Now for all $n \in \mathbb{N}$ we set $m(n) := q(n) - q(n-1)$, and define the following isomorphism

$$I_n : J_{m(n)} \to \mathrm{sp}\{e_i;\ q(n-1) \le i \le q(n)-1\} \subset J(h),\ \sum_{i=1}^{m(n)} a_i e_i \to \sum_{i=q(n-1)}^{q(n)-1} a_i e_i.$$

We consider the weighted space

$$l_2(J_{m(n)}, 2^n) := \left\{ (x_n) \in \prod_n J_{m(n)};\ \left(\sum_{n=1}^{\infty} \|x_n\|^2 2^n \right)^{1/2} < \infty \right\}.$$

$l_2(J_{m(n)}, 2^n)$ is clearly isomorphic to $l_2(J_{m(n)})$. On the other hand it is readily checked that the following map is an isomorphism (One should only use the following inequality to estimate the norms: $(a_1 + \ldots + a_k)^2 \le 2a_1^2 + \ldots + 2^k a_k^2,\ \forall k \in \mathbb{N},\ a_i \in \mathbb{R},\ i = 1, \ldots k.$):

$$I : l_2(J_{m(n)}, 2^n) \to J(h),\ (x_n) \to \sum_{n=1}^{\infty} I_n(x_n).$$

This already proves our first statement. The fact that $l_2(J_{m(n)})$ is isomorphic to a complemented subspace of J can be seen in [1] and its references. □

As a consequence, if the weight h is non-decreasing we prove that there are three kinds of non-isomorphic weighted spaces $J(h)$: The quasireflexive space J, the l_2-sum $(J_1 \oplus J_2 \oplus \ldots)_{l_2} = l_2(J_n)$ (this space is reflexive but not super-reflexive since J is finitely representable in $l_2(J_n)$) and the Hilbert space l_2.

Corollary 4

Let $(h(n))_{n\in\mathbb{N}}$ be a non-decreasing weight. Then $J(h)$ is isomorphic to one of the following three spaces:

(i) J,

(ii) $l_2(J_n)$,

(iii) l_2.

Proof. If $(h(n))$ is bounded then $J(h) \cong J$. If $h(n)$ is not bounded then we apply Theorem 3 and get $J(h) \cong l_2(J_{m(n)})$ for some sequence of integers $(m(n))_{n\in\mathbb{N}}$. Two cases may happen: If $(m(n))$ is bounded then $l_2(J_{m(n)})$ is clearly isomorphic to l_2; if $\sup\{m(n);\ n \in \mathbb{N}\} = \infty$ then $l_2(J_{m(n)})$ is isomorphic to $l_2(J_n)$ by [2, Corollary]. □

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Referee's comment. Several years ago Peter Casazza distributed an unpublished manuscript showing that the converse of Lemma 1 fails for every equivalent norm on Tsirelson's space. This relates to an old question of Vitali Milman ($\approx$ 1970, Russian Survey) which asks: Can every reflexive Banach space X be given an equivalent norm $|.|$ so that whenever $|x_n| = 1$ in X and $\lim_{n,m\to\infty} |x_n + x_m| = 2$, then (x_n) converges in X. This problem is open even in Tsirelson's space.

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