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Stochastic processes and applications to countably additive restrictions of group-valued finitely additive measures

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Abstract

As an application of a theorem concerning a general stochastic process in a finitely additive probability space, the existence of non-atomic countably additive restrictions with large range is obtained for group-valued finitely additive measures.

1. Introduction

This paper investigates a further generalization of a problem already studied in [5], [2], [3], [9], [1]. More precisely, given a finitely additive measure m on a set Ω, we seek a countably additive restriction preserving some "nice" properties of m.

We refer to the Introduction of [9] and to [4] and [10] for a detailed description of the intermediate steps toward the general solution of the problem presented here.

We mention that the techniques adopted in [3] and [9] cannot be transported to the present setting, differently to that of [1]. Nevertheless the proof that we exhibit here is a new, and in some sense more concrete one.

2. Results concerning stochastic processes

Let (S, Σ, P) be a finitely additive probability space, and let T be an infinite set of indexes.

DEFINITION 1. Let V denote the class of finite subsets of T. For each $v \in V$, $v = \{t_1, \dots, t_n\}$, let $\mathcal{R}_v$ denote the family of rectangles $R_v \subset \mathbb{R}^n$, of the form $R_v =]a_1, b_1] \times]a_2, b_2] \times \dots \times]a_n, b_n]$, where $a_i \le b_i$ for each $i, a_i \ge -\infty, b_i < +\infty$ for each i.

Let also $\mathcal{E}_v$ denote the algebra on $\mathbb{R}^v$ generated by $\mathcal{R}_v$, and $\mathcal{B}_v$ the σ-algebra generated by $\mathcal{R}_v$ namely the Borel σ-algebra. For every $R_v \in \mathcal{R}_v$, let $\widetilde{R}_v = R_v \times \mathbb{R}^{T-v} \subset \mathbb{R}^T$, and let us denote by $\widetilde{\mathcal{R}}_v$ the family

$$\widetilde{\mathcal{R}}_v = \{\widetilde{R}_v : R_v \in \mathcal{R}_v\}.$$

In a similar fashion we will define the algebra $\widetilde{\mathcal{E}}_v$ and the σ-algebra $\widetilde{\mathcal{B}}_v$. Finally we will set $\widetilde{\mathcal{R}} = \bigcup_{v \in V} \widetilde{\mathcal{R}}_v$, $\widetilde{\mathcal{E}} = \bigcup_{v \in V} \widetilde{\mathcal{E}}_v$,$\widetilde{\mathcal{B}} = \bigcup_{v \in V} \widetilde{\mathcal{B}}_v$.

Let now a family $\{X_t : S \to \mathbb{R}\}_{t \in T}$ of random variables (r.v.) be assigned. We shall denote with $F_t : \mathbb{R} \to [0, 1]$ the *distribution function* of X_t defined as

$$F_t(x) = P(X_t \le x) = P(X_t^{-1}(] - \infty, x]))$$

for every $x \in \mathbb{R}$. We shall finally denote by $\mathbf{X} : S \to \mathbb{R}^T$ the r.v defined as $\mathbf{X}(\mathrm{s})(\mathrm{t}) = \mathrm{X_t(s)}$.

Proposition 1

Let $\{X_t\}_{t \in T}$ be a family of r.v. on S, and suppose that each distribution function F_t is right-continuous at each x, and such that

$$\lim_{x \to -\infty} F_t(x) = 0 = 1 - \lim_{x \to +\infty} F_t(x) \,.$$

Then there exists a countably additive probability measure P_X, defined on $\widetilde{\mathcal{B}}$ such that

$$P_{\mathbf{X}}(\widetilde{E}) = P(\mathbf{X}^{-1}(\widetilde{E})) \tag{1}$$

for every $\widetilde{E} \in \widetilde{\mathcal{E}}$.

Proof. For every $v \in V, v = (t_1, \ldots, t_n)$, we set $X_v = (X_{t_1}, \ldots, X_{t_n})$. Furthermore, let

$$g_v(x_1, \ldots, x_n) = P\big(X_v^{-1}(] - \infty, x_1] \times \ldots \times] - \infty, x_n])\big)\,.$$

It is straightforward to verify that g_v is monotonic with respect to each variable, and that $\lim_{x_j \to -\infty} g_v(x_1, \ldots, x_j, \ldots, x_n) = 0$ for every j.

We shall now verify the marginalization properties. Let $k < n$ be fixed and let X_k denote the vector $(X_{t_1}, \ldots, X_{t_k})$, and let g^k be the corresponding distribution

$$g^k(x_1, \ldots, x_k) = P\big(X_k^{-1}(] - \infty, x_1] \times \ldots \times] - \infty, x_k])\big).$$

Let $\varepsilon > 0$ be fixed, and choose a $\bar{x}$ in such a way that

$$1 - F_{t_j}(x) < \frac{\varepsilon}{n} \tag{2}$$

for every $j = k+1, \ldots, n$ and for every $x \geq \bar{x}$. Relationship (2) is equivalent to

$$P\big(X_{t_j} > x\big) < \frac{\varepsilon}{n} \tag{3}$$

for every $j = k+1, \ldots, n$ and for every $x > \bar{x}$. Therefore, corresponding to $\varepsilon > 0$ there exists a $\bar{x}$ such that

$$P\Big(\bigcup_{j=k+1}^{n} (X_{t_j} > x)\Big) < \varepsilon \tag{4}$$

for every $x \geq \bar{x}$. Therefore, choosing $x_{k+1}, x_{k+2}, \ldots, x_n$ greater than $\bar{x}$ and without varying $x_1, \ldots, x_k$ we shall find from (4)

$$\begin{aligned} &g^k(x_1, \ldots, x_k) - g_v(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = \\ &= P\left((X_{t_1} \leq x_1) \cap (X_{t_2} \leq x_2) \cap \ldots \cap (X_{t_k} \leq x_k) \cap \Big(\bigcup_{j=k+1}^{n} (X_{t_j} > x_j)\Big)\right) \leq \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{(x_{k+1}, \ldots, x_n) \to +\infty} g_v(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = g^k(x_1, \ldots, x_k)$ namely the marginalization property of g_v holds.

Furthermore, as it is easily verified, each g_v is right- continuous, with respect to every variable as well as globally, at every point, since each F_t is right-continuous.

To prove that each g_v generates a unique countably additive measure P_v on $\mathcal{B}_v$ it is enough to show that condition (4) of [8] (page 219-220) is satisfied. This condition can be expressed as:

Setting $P_v(]-\infty, x_1] \times \ldots \times]-\infty, x_n]) = g_v(x_1, \ldots, x_n)$ *and making use of the usual procedure (namely adding and subtracting) to extend* P_v *to the rectangles* $R_v \in \mathcal{R}_v$*, one necessarily finds* $P_v(R_v) \geq 0$*, for every* $R_v \in \mathcal{R}_v$*.*

Then, by making use of this standard procedure to get $P_v(R_v)$ we shall find in this case $P_v(R_v) = P(X_v^{-1}(R_v))$ for every R_v, and therefore Levine's condition is satisfied.

Thus, for every $v \in V$ a unique countably additive measure $P_v : \mathcal{B}_v \to [0,1]$ can be determined in such a way that, for every $R_v \in \mathcal{R}_v$

$$P_v(R_v) = P(X_v^{-1}(R_v))$$

and therefore, by finite summation, for every $E_v \in \mathcal{E}_v$

$$P_v(E_v) = P(X_v^{-1}(E_v)).$$

Since the measures P_v obviously verify, when v ranges in V, the compatibility conditions of Kolmogoroff (see [6]), it is possible to find a (unique) probability measure $P_{\mathbf{X}}$ on $\mathcal{B}$, admitting the P_v's as finite-dimensional distributions, that is

$$P_{\mathbf{X}}(\widetilde{B}_v) = P_v(B_v)$$

for all $B_v \in \mathcal{B}_v$ and $v \in V$.

In particular, if $E_v \in \mathcal{E}_v$ it follows

$$P_{\mathbf{X}}(\widetilde{E}_v) = P_v(E_v) = P(X_v^{-1}(E_v)) = P(\mathbf{X}^{-1}(\widetilde{E}_v)) :$$

by the arbitrariness of v condition (1) follows. □

DEFINITION 2. Besides the space $\mathbb{R}^T$ it will be convenient to consider the space L defined as follows: $L = \{f \in \mathbb{R}^T : f(t) = 0$ for every $t \in T - F$ where F is an at most countable set, depending upon $f\}$.

Observe that the cardinality of L is exactly equal to $\max\{\mathrm{card}(T), c\}$.

Over the space L we shall introduce the families $\mathcal{R}_v^L, \widetilde{\mathcal{R}}_v^L, \widetilde{\mathcal{R}}^L$ defined as

$$\begin{aligned} \mathcal{R}_v^L &= \{R_v \cap L : R_v \in \mathcal{R}_v\}, \\ \widetilde{\mathcal{R}}_v^L &= \{\widetilde{R}_v \cap L : \widetilde{R}_v \in \widetilde{\mathcal{R}}_v\}, \\ \widetilde{\mathcal{R}}^L &= \{\widetilde{R} \cap L : \widetilde{R} \in \widetilde{\mathcal{R}}\}, \end{aligned}$$

and in an analogous fashion we shall define the families $\mathcal{E}_v^L$, $\widetilde{\mathcal{E}}_v^L$, $\widetilde{\mathcal{E}}^L$, $\mathcal{B}_v^L$, $\widetilde{\mathcal{B}}_v^L$, $\widetilde{\mathcal{B}}^L$. Obviously, $\widetilde{\mathcal{E}}^L$ is an algebra on L, and $\widetilde{\mathcal{B}}^L$ is a σ-algebra on L.

We shall give another definition in the space $\mathbb{R}^T$. A subset $A \subset \mathbb{R}^T$ will be called *σ-binding* if there are an at most countable set $F \subset T$ and a subset $C \subset \mathbb{R}^F$ such that

$$A = C \times \mathbb{R}^{T-F}.$$

It is well known, for example, that all the sets in $\widetilde{\mathcal{B}}$ are σ-binding.

Theorem 1

If $B \in \widetilde{\mathcal{B}}$ and $B \cap L = \emptyset$ then $B = \emptyset$.

Proof. Since B is σ-binding there will be a finite or countable set $F \subset T$ and a set $C \subset \mathbb{R}^F$, such that

$$B = C \times \mathbb{R}^{T-F} .$$

For $f \in B$ we set $f_0 = f 1_F$. Then $f_0 \in B \cap L$. Therefore, if B is non-empty, $L \cap B$ is non-empty. □

Corollary 1

If $P : \widetilde{\mathcal{B}} \to [0,1]$ is a countably additive probability measure, we set

$$P^L(B \cap L) = P(B) \tag{5}$$

for every $B \in \widetilde{\mathcal{B}}$. Then $P^L : \widetilde{\mathcal{B}}^L \to [0,1]$ is a countably additive probability measure.

Proof. Let $B_1, B_2 \in \widetilde{\mathcal{B}}$ be such that $B_1 \cap L = B_2 \cap L$. Then $(B_1 \Delta B_2) \cap L = \emptyset$ and therefore, from Theorem 1, $B_1 = B_2$. Hence (5) well defines P^L. Moreover, it is obvious that $P^L(\emptyset) = 0$ and $P^L(L) = 1$. Finally, if $(B_n)_n$ is a sequence in $\widetilde{\mathcal{B}}$ such that the sets $B_n \cap L$ are pairwise disjoint, it follows, for $m \neq n$, $(B_n \cap B_m) \cap L = \emptyset$ and thus $B_n \cap B_m = \emptyset$. Hence

$$P^L\Big(\bigcup_n (B_n \cap L)\Big) = P\Big(\bigcup_n B_n\Big) = \sum_n P(B_n) = \sum_n P^L(B_n \cap L) ,$$

and thus P^L is σ-additive. □

3. Countably additive restrictions

Throughout this section Ω will denote an infinite set with $\mathrm{card}(\Omega) \geq c$, G an abelian group and $m : \mathcal{P}(\Omega) \to G$ a finitely additive measure (f.a.m.).

Definition 3. We will say that m is *continuous* iff, for every neighborhood U of the neutral element 0 in G there exists a decomposition of Ω into finitely many sets $\{A_1, \ldots, A_n\}$, with $A_i \cap A_j = \emptyset$ when $i \neq j$ and such that $m(E) \in U$ for every $E \subseteq A_i$ and for all $i = 1, \ldots, n$. Let us denote by $\mathcal{I}(0)$ a neighborhood basis of 0 in G.

We will say that m is *absolutely continuous* with respect to a scalar f.a.m. $\nu : \mathcal{P}(\Omega) \to \mathbb{R}_0^+$ iff, for every $U \in \mathcal{I}(0)$ there exists $\delta > 0$ such that for every $E \subset \Omega$ with $\nu(E) \leq \delta$ it follows that $m(E) \in U$. In this case we will write $m \ll \nu$.

We will say that m is *singular* with respect to ν iff, for every $U \in \mathcal{I}(0)$ and for every $\varepsilon > 0$ a set $A \subset \Omega$ exists, such that $m(E) \in U$ for every $E \subset A$, and $\nu(A^c) < \varepsilon$. This definition is symmetric with respect to m and ν. We will write then $m \perp \nu$ or $\nu \perp m$.

We will say that ν is *absolutely continuous* with respect to m, and we will write $\nu \ll m$, provided for every $\varepsilon > 0$ there exists a neighborhood $U \in \mathcal{I}(0)$ such that the following implication holds:

$$\{m(E) : E \subset A\} \subset U \Rightarrow \nu(A) < \varepsilon,$$

for every $A \subset \Omega$.

Finally, we will say that m and ν are *equivalent* iff $m \ll \nu$ and $\nu \ll m$. In this case we will say that ν is a *control* for m.

We shall report a Theorem which will be needed in the sequel.

Theorem 2

(Lebesgue decomposition; [7]) Let $m : \mathcal{P}(\Omega) \to G$ and $\nu : \mathcal{P}(\Omega) \to \mathbb{R}_0^+$ be two f.a.m.'s. Then there are only two f.a.m. $\nu_1, \nu_2 : \mathcal{P}(\Omega) \to \mathbb{R}_0^+$ such that:
(i) $\nu_1 + \nu_2 = \nu$;
(ii) $\nu_1 \perp m, \quad \nu_2 \ll m$.

We mention that an analogous result holds for m with respect to ν, but we are not going to use it in this paper.

Corollary 2

Let $m : \mathcal{P}(\Omega) \to G$ be a f.a.m., and assume that there exists a scalar f.a.m. ν such that $m \ll \nu$. Then m admits a control.

When the assumptions of Corollary 2 are satisfied we will say that m is *controllable*.

Proof. Let (ν_1, ν_2) be the Lebesgue decomposition stated in Theorem 2. From (ii) we find $\nu_2 \ll m$ and $\nu_1 \perp m$. We will show that $m \ll \nu_2$: thus ν_2 is a control for m.

Let then $U \in \mathcal{I}(0)$. Since $\nu_2 \ll m$ there exists a $\delta > 0$ such that $\nu_2(E) < \delta \Rightarrow m(E) \in U_1$ for $E \subset \Omega$, where $U_1 \in \mathcal{I}(0)$ is such that $U_1 + U_1 \subset U$. Being $\nu_1 \perp m$, corresponding to U and δ there is a set $F \subset \Omega$ such that $\nu_1(F) < \frac{\delta}{2}$ and $m(E) \in U_1$ for every $E \subset F^c$.

Let now $A \subset \Omega$ be a set such that $\nu_2(A) < \frac{\delta}{2}$. We have

$$m(A) = m(A \cap F) + m(A \cap F^c).$$

Since $A \cap F \subset A$ it has to be $\nu_2(A \cap F) < \frac{\delta}{2}$, and since $A \cap F \subset F$ it has to be $\nu_1(A \cap F) < \frac{\delta}{2}$. Therefore it is $\nu(A \cap F) < \delta$ whence $m(A \cap F) \in U_1$. Furthermore, since $m(E) \in U_1$ for every $E \subset F^c$, we will have $m(A \cap F^c) \in U_1$. In conclusion $m(A) \in U_1 + U_1 \subset U$.

This shows that $m \ll \nu_2$. □

The following two Propositions concern the continuity.

Proposition 2

Let $\nu : \mathcal{P}(\Omega) \to \mathbb{R}_0^+$ and $m : \mathcal{P}(\Omega) \to G$ be two f.a.m.'s. If $m \ll \nu$ and ν is continuous, then m is continuous. If $\nu \ll m$ and m is continuous, then ν is continuous. If m is continuous and controllable, then there exists a continuous control for m.

Proof. Straightforward. □

Proposition 3

([2]) Let $\nu : \mathcal{P}(\Omega) \to \mathbb{R}_0^+$ be a continuous f.a.m. Then, for every $A \subset \Omega$ there exists a family $\{A(t)\}_{t \in [0,1]}$ of subsets of A such that:

(i) $A(0) = \emptyset,\ A(1) = A$;
(ii) $t < t' \Rightarrow A(t) \subset A(t')$;
(iii) $\nu(A(t)) = t\nu(A)$.

We shall assume from now on that $m : \mathcal{P}(\Omega) \to G$ is a continuous controllable f.a.m. with range $R \subset G$ infinite and $\mathrm{card}(R) \leq \mathrm{card}(\Omega)$. We shall denote by ν a continuous control for m, and we can assume, without loss of generality, that $\nu(\Omega) = 1$.

Since $\mathrm{card}(\Omega) \geq c$, and ν is defined on the whole $\mathcal{P}(\Omega)$ there exists a subset $H \subset \Omega$ such that $\nu(H) = 0$ and $\mathrm{card}(H) = \eta$ where $\eta = \max\{c, \mathrm{card}\ (R)\}$. In this way it is also true that $m(E) = 0$ for all $E \subset H$. Set $S = \Omega - H$. Obviously H can be chosen in such a way that S and Ω have the same cardinality. Furthermore R is the range of $m|_{\mathcal{P}(S)}$.

Let $\varphi : R \to \mathcal{P}(S)$ be a one-to-one map such that $\varphi(m(S)) = S$ and also $m(\varphi(r)) = r$ for each $r \in R$. Let T be the range of φ. In this way T is a set of subsets of S. Let us define a family of maps $\{X_A\}_{A\in T}$, $X_A : S \to \mathbb{R}$ in the following way: $X_A(s) = (1 - inf\{u \in [0,1] : s \in A(u)\})1_A$ where $\{A(u)\}_{u\in[0,1]}$ is the family described in Proposition 3.

Lemma 1

For every $A \in T$ let $F_A(x) = \nu(X_A \le x)$, $x \in \mathbb{R}_0^+$. Then

$$F_A(x) = \begin{cases} 0 & \textit{if} \quad x < 0 \\ \nu(A^c) + x\nu(A) & \textit{if} \quad 0 \le x \le 1 \\ 1 & \textit{if} \quad x > 1 \end{cases} \tag{6}$$

and therefore F_A is right-continuous at each point, and

$$\lim_{x\to-\infty} F_A(x) = 0 = 1 - \lim_{x\to+\infty} F_A(x)$$

for every $A \in T$.

Proof. For every $x \in]0,1]$ we have

$$(X_A < x) \subset (A(1-x))^c \subset (X_A \le x). \tag{7}$$

We are now going to prove that, for $x \in]0,1]$, we have $\nu(X_A = x) = 0$. First, for $x = 1$, $(X_A \le 1) = S$ and thus $F_A(1) = 1$. Since, for every $\varepsilon > 0$, $(X_A \le 1-\varepsilon) \subset (X_A < 1)$ from (7) it is also true that

$$(A(\varepsilon))^c = (A(1-(1-\varepsilon)))^c \subset (X_A < 1).$$

Having $\nu((A(\varepsilon))^c) = 1 - \varepsilon\nu(A)$ for every $\varepsilon > 0$, we obtain $\nu((X_A < 1)) = 1$. This shows that $\nu((X_A = 1)) = 0$.

We move now to the case $0 < x < 1$.

Fix $\varepsilon > 0$ in such a way that $0 < x - \varepsilon < x + \varepsilon < 1$.

Then $(X_A = x) \subset (X_A < x+\varepsilon) - (X_A \le x-\varepsilon)$. From (7) it follows that $(X_A < x+\varepsilon) \subset (A(1-x-\varepsilon))^c$, and $(X_A \le x-\varepsilon)^c \subset A(1-x+\varepsilon)$ whence

$$\nu((X_A = x)) \le \nu(A(1-x+\varepsilon) \cap (A(1-x-\varepsilon))^c) = \\ \nu(A(1-x+\varepsilon) - A(1-x-\varepsilon)) = 2\varepsilon\nu(A).$$

By the arbitrariness of ε it follows $\nu((X_A = x)) = 0$.

This yields that $\nu((X_A < x)) = \nu((X_A \le x))$ for every $x \in]0,1]$. Therefore, again from (7), we shall find

$$\nu((X_A \le x)) = \nu((A(1-x))^c) = \nu(A^c) + \nu(A - A(1-x)) = \nu(A^c) + x\nu(A).$$

Hence, relationship (6) is proven for $x \in]0,1]$. The relationship is obvious for $x < 0$ and for $x > 1$. It remains to prove it for $x = 0$, namely that $\nu((X_A \leq 0)) = \nu(A^c)$. But $(X_A \leq 0) = (X_A = 0)$ and, from what has just been proven,

$$\nu((X_A = 0)) \leq \lim_{\varepsilon \to 0} \nu((X_A \leq \varepsilon)) = \nu(A^c).$$

On the other side, if $s \in A^c$ then $X_A(s) = 0$, and hence $A^c \subset (X_A = 0)$ whence it is also $\nu((X_A = 0)) \geq \nu(A^c)$, which concludes the proof. □

We are now able to prove the main theorem.

Theorem 3

Let Ω and m be as in Lemma 1. Then there exists an algebra $\mathcal{A} \subset \mathcal{P}(\Omega)$ such that $m|_{\mathcal{A}}$ is continuous and countably additive and its range coincides with the range of m.

Proof. If T is the set previously described, T has the same cardinality as R and the last one is underneath $\mathrm{card}(\Omega)$. Let L be the space introduced in Section 2 corresponding to $\mathbb{R}^T$. We have

$$\mathrm{card}(L) = \max\{c, \mathrm{card}(R)\} = \eta = \mathrm{card}(H).$$

Then there exists a bijection α between L and H, which induces a complete isomorphism between $\mathcal{P}(L)$ and $\mathcal{P}(H)$. Consider now the family of random variables $\{X_A : A \in T\}$ defined as in the previous proof. From Lemma 1 and Proposition 1 there exists a countably additive probability measure $\nu_{\mathbf{X}} : \mathcal{B} \to [0,1]$ such that

$$\nu_{\mathbf{X}}(\widetilde{E}) = \nu(\mathbf{X}^{-1}(\widetilde{E})) \tag{8}$$

for every $\widetilde{E} \in \widetilde{\mathcal{E}}$. If we now define $\varphi : \widetilde{\mathcal{E}} \to \mathcal{P}(\Omega)$ as

$$\varphi(\widetilde{E}) = \mathbf{X}^{-1}(\widetilde{E}) \cup \alpha(\widetilde{E} \cap L),$$

we shall find that φ is a homomorphism between algebras, and that its range, which we shall denote by $\mathcal{A}$, is the required algebra.

In fact, we shall show that $m|_{\mathcal{A}}$ has for range R. Let $r \in R$, and let us pick an $A \in T$ such that $m(A^c) = r$.

We then set $\widetilde{E} = \{f \in \mathbb{R}^T : f(A) \leq 0\}$: here A is seen as a singleton in T, and hence $\widetilde{E} \in \widetilde{\mathcal{R}}$. Moreover it is

$$m(\varphi(\widetilde{E}) = m(\mathbf{X}^{-1}(\widetilde{E})) = m((X_A \leq 0)).$$

Since $\nu((X_A \leq 0)) = \nu(A^c)$ and since $A^c \subset (X_A \leq 0)$ from $m \ll \nu$ it follows also $m((X_A \leq 0)) = m(A^c) = r$. Hence $m|_{\mathcal{A}}$ ranges on the whole set R.

We are now going to prove that $m|_{\mathcal{A}}$ is continuous.

Let $\varepsilon > 0$ be fixed, and let $n \in \mathbb{N}$ be such that $n > \frac{1}{\varepsilon}$. Let also $A_1 = (X_S \leq \frac{1}{n})$, $A_2 = (\frac{1}{n} < X_S \leq \frac{2}{n})$, $\ldots$ $A_n = (\frac{n-1}{n} < X_S \leq 1)$. The sets A_j are pairwise disjoint, their union coincides with S and $\nu_j(A) = \frac{1}{n} < \varepsilon$ for every j, by (7).

Setting $\widetilde{A}_j = \{f \in \mathbb{R}^T : \frac{j-1}{n} < f(S) \leq \frac{j}{n}\}$ (considering again S as an element of T), we will have $\widetilde{A}_j \in \widetilde{\mathcal{E}}$ for every j, and the $\widetilde{A}_j$'s form a decomposition of $\mathbb{R}^T$. Moreover we have

$$\nu(\varphi(\widetilde{A}_j)) = \nu(\mathbf{X}^{-1}(\widetilde{A}_j)) = \nu(A_j) < \varepsilon$$

for every j. Therefore ν is continuous on $\mathcal{A}$, and hence m is continuous on $\mathcal{A}$, since $m \ll \nu$.

We are finally going to prove that $\nu|_{\mathcal{A}}$ is σ-additive. In order to do this, let us denote by $\lambda : \widetilde{\mathcal{B}}^L \to [0,1]$ the countably additive probability measure $\lambda = \nu^L_{\mathbf{X}}$ according to the definition given in Corollary 1.

Let now $(\widetilde{E}_n)_n$ be a sequence in $\widetilde{\mathcal{E}}$ such that $\varphi(\widetilde{E}_n) \downarrow \emptyset$. It is then

$$\nu(\varphi(\widetilde{E}_n)) = \nu(\mathbf{X}^{-1}(\widetilde{E}_n)) = \nu_{\mathbf{X}}(\widetilde{E}_n) = \lambda(\widetilde{E}_n \cap L),$$

for every n. Being $\varphi(\widetilde{E}_n) \downarrow \emptyset$ it has to be $\alpha(\widetilde{E}_n \cap L) \downarrow \emptyset$. Since α is a complete isomorphism it will also hold $\widetilde{E}_n \cap L \downarrow \emptyset$, and thus $\lambda(\widetilde{E}_n \cap L) \downarrow 0$. This yields $\nu(\varphi(\widetilde{E}_n)) \downarrow 0$, and hence the countable additivity of $\nu|_{\mathcal{A}}$. Being $m \ll \nu$, this will imply the countable additivity of $m|_{\mathcal{A}}$. The theorem is now completely proved. □

Remark 1. A version of Theorem 3 holds true also if the continuity condition (both in the hypotheses and in the thesis) is dropped. One can follow the same device, but the definition of X_A must be simplified: namely $X_A = 1_A$ for every A and therefore the functions F_A are just right-continuous Heaviside-type functions.

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