A theorem on derivations in semiprime rings

QING DENG

Department of Mathematics, Southwest China Normal University, Chongqing 630715, P.R. of China

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Abstract

Let $R$ be a semiprime ring with suitably-restricted torsion, $U$ a nonzero left ideal of $R$ and $D : R \to R$ a nonzero derivation. If for each $x \in U$, $[D(x), x]_n = [[\cdots [D(x), x] \cdots x], x] \in Z(R)$ with $n$ fixed, then $R$ must contain nonzero central ideals in case $D(U) \neq 0$.

In a recent paper, Lanski [3] gives an extension of a well-known theorem of Posner [4] by showing that a prime ring $R$ is commutative if $[D(x), x]_n = 0$ for all $x$ in a nonzero ideal of $R$. In [2], we studied the commutativity of semiprime rings with derivations and proved that a semiprime ring $R$ must contain nonzero central ideals if $[[D(x), x], x] \in Z(R)$, which yields generalization of a theorem of Bell and Martindale [1]. The purpose of this note is to extend the result of Lanski [3, Theorem 1] from prime rings to semiprime rings with suitably-restricted additive torsion.

Throughout this paper, $R$ denotes an associative ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$, and $I_a(b)$ for $[b, a]$, and $[x, y]_n = [[\cdots [x, y], \cdots], y]$.

For easy reference we state two lemmas.

Lemma 1 ([2, Lemma 1]).

Let $n$ be a positive integer, $R$ be an $n!$-torsion-free ring, and $f$ be an additive map on $R$. For $i = 1, 2, \cdots, n$, let $F_i(X, Y)$ be a generalized polynomial which is homogeneous of degree $i$ in the non-commuting indeterminates $X$ and $Y$. Let $a \in R$ and $(a)$ the additive subgroup generated by $a$. If

$$F_n(x, f(x)) + F_{n-1}(x, f(x)) + \cdots + F_1(x, f(x)) \in Z(R)$$

for all $x \in (a)$, then $F_i(a, f(a)) \in Z(R)$ for $i = 1, 2, \cdots, n$. 

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Lemma 2 ([5, Theorem]).

Let \( R \) be a ring and \( P \) a prime ideal such that \( \text{char}(R/P) = 0 \) or \( \text{char}(R/P) \geq n \). If \( a_1, a_2, \cdots, a_n+1 \) are elements of \( R \) such that \( a_1a_2a_3 \cdots a_nxa_{n+1} \in P \) for all \( x \in R \), then \( a_i \in P \) for some \( i = 1, 2, \cdots, n+1 \).

Theorem

Let \( n \) be a fixed positive integer, let \( R \) be a \( (n+1)! \)-torsion-free semiprime ring, and let \( U \) be a nonzero left ideal of \( R \). If \( R \) admits a nonzero derivation \( D \) such that \( [D(x),x]_n \in Z(R) \) for all \( x \in U \), then \( R \) contains nonzero central ideals or \( D(U) = 0 \).

Proof. For the proof, we need three steps.

Lemma A

\([D(x),x]_n = 0 \) for all \( x \in U \).

Proof. Linearizing the conditions \([D(x),x]_n \in Z(R)\) and using Lemma 1, we get

\[
I^n_x(D(y)) + I^{n-1}_x([D(x),y]) + \cdots + I_x([D(x),x]_{n-2},y) + [D(x),x]_{n-1},y \in Z(R).
\]

Replacing \( y \) by \( x^2 \), noting that each term in the relation is \( 2x[D(x),x]_n \), we then have

\[
2(n+1)x[D(x),x]_n \in Z(R), \quad \text{and} \quad ([D(x),x]_n)^2 = ([D(x),x]_{n-1},x[D(x),x]_n) = 0.
\]

Since the center of a semiprime ring contains no nonzero nilpotent elements, we obtain \([D(x),x]_n = 0 \). \( \square \)

Lemma B

\[[[D(x),x]_{n-1}]^2 x = 0 \) for all \( x \in U \).

Proof. From \([D(x),x]_n = 0 \), we have \( I^n_x([D(x),x]_i) = 0 \) for \( i + j \geq n \). Linearizing \([D(x),x]_i = 0 \) and applying Lemma 1, we now have

\[
I^n_x(D(y)) + I^{n-1}_x([D(x),y]) + \cdots + I_x([D(x),x]_{n-2},y) + [D(x),x]_{n-1},y = 0. \tag{1}
\]

Since \( I^n_x(D(y)) = xI^n_x(D(y)) + I^n_x(D(x),y) \),

\[
I^{n-1}_x([D(x),xy]) = xI^{n-1}_x([D(x),y]) + I^{n-1}_x([D(x),x]y); \\
I^k_x([D(x),x]_{n-k-1},xy] = xI^k_x([D(x),x]_{n-k-1},y) + I^k_x([D(x),x]_{n-k},y),
\]

for \( k = 1, 2, \cdots, n-2 \).
Replacing $y$ by $xy$ in (1) yields
\[ I^n_x(D(x)y) + I^{n-1}_x([D(x), x]y) + \cdots + I_x([D(x), x]_{n-1}y) = 0. \tag{2} \]
Taking $y = [D(x), x]_{n-2}x$ in (2), and noting $I^j_x([D(x), x]_{n-2}) = 0$ for $j \geq 2$ and $I^k_x(ab) = \sum_{j=0}^k \binom{k}{j} I^{k-j}_x(a) I^j_x(b)$, we then gain
\[ n([D(x), x]_{n-1})^2 x + (n-1)([D(x), x]_{n-1})^2 x + \cdots + ([D(x), x]_{n-1})^2 x = 0, \]
thus $\frac{n(n+1)}{2}([D(x), x]_{n-1})^2 x = 0$ and $([D(x), x]_{n-1})^2 x = 0$. □

**Lemma C**

$[D(x), x]_{n-1} = 0$ for all $x \in U$.

**Proof.** Take a family $\Omega = \{p_\alpha | \alpha \in \Lambda \}$ of prime ideals of $R$ such that $\cap P_\alpha = \{0\}$, and let $\Omega_1 = \{p_\alpha \in \Omega | D(U) \subseteq P_\alpha \}$.

For each $P \in \Omega_1$, and for each $P \in \Omega \setminus \Omega_1$ such that $0 < \text{char}(R/P) \leq n + 1$, we have $(n+1)! [D(x), x]_{n-1} \in P$ for all $x \in U$.

Suppose that there is a $P \in \Omega \setminus \Omega_1$ such that $\text{char}(R/P) = 0$ or $\text{char}(R/P) > n + 1$, we shall show that $[D(x), x]_{n-1} \in P$. Firstly we show that $a^2 \in P$ implies $a \in P$ for $a \in U$. From $[D(x), x]_n = 0$, we arrive at
\[ x^n D(x) + \frac{n}{1} x^{n-1} D(x) x + \cdots + (-1)^n D(x) x^n = 0. \tag{3} \]
For any $a \in U$ with $a^2 \in P$, replace $x$ by $ra$ in (3) and then right-multiply by $a$, we have $(ra)^n r D(a) a \in P$ for all $r \in R$, and apply Lemma 2 to conclude $D(a) a \in P$.

If we replace $x$ by $ra + a$ in (3), and apply the condition that $D(ra + a) a \in P$, we get
\[
((ra)^n + a(ra)^{n-1}) D(ra + a) \\
+ \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} ((ra)^{n-k} + a(ra)^{n-k-1}) D(ra + a)(ra)^k \\
+ (-1)^n D(ra + a)(ra)^n \in P,
\]
and by [2, Lemma 2], we conclude that
\[
(ra)^n D(a) + a(ra)^{n-1} D(ra) \\
+ \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} ((ra)^{n-k} D(a)(ra)^k + a(ra)^{n-k-1} D(ra)(ra)^k \\
+ (-1)^n D(a)(ra)^n \in P, \tag{4}
\]
and

\[ a(ra)^{n-1}D(a) + \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} a(ra)^{n-k-1} D(a)(ra)^k \in P. \]  

(5)

Left-multiplying (5) by \( r \), and in conjunction with (4), shows that

\[ a(ra)^{n-1}D(ra) + \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} a(ra)^{n-k-1} D(ra)(ra)^k + (-1)^n D(a)(ra)^n \in P. \]

Left-multiplying this last condition by \( a \), we have \((-1)^n aD(a)(ra)^n \in P\); and by Lemma 2, \( aD(a) \in P \).

Since \( aD(a) \) and \( D(a)a \) are in \( P \), we obtain

\[
(ra + ara)^n D(ra + ara) - ((ra)^n + a(ra)^n) D(ra) \in P; \\
(ra + ara)^n D(ra + ara)(ra + ara)^k - ((ra)^n + a(ra)^n) D(ra)(ra)^k \in P
\]

for \( k = 1, 2, \cdots, n - 1 \);

\[
D(ra + ara)(ra + ara)^n - (D(ra) + D(arar))(ra)^n \in P.
\]

Substituting \( ra + ara \) for \( x \) in (3), we have

\[
((ra)^n + a(ra)^n) D(ra) + \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} ((ra)^{n-k} + a(ra)^{n-k}) D(ra)(ra)^k \\
+ (-1)^n (D(ra) + D(arar))(ra)^n \in P,
\]

and using the condition \([D(ra), ra]_n = 0\), we obtain \( D(a)(ra)^{n+1} \in P \), so that Lemma 2 yields either \( a \in P \) or \( D(a) \in P \). For all \( x \in R \), \( axa \in U \) and \((xax)^2 \in P\), that is, \( axa \) satisfies our original hypotheses on \( a \), therefore for each \( x \in R \), either \( axa \in P \) or \( D(axa) \in P \). Since the sets \( \{ x \in R | axa \in P \} \) and \( \{ x \in R | D(axa) \in P \} \) are additive subgroups of \( R \), we conclude that either \( aRa \subseteq P \) or \( D(aDa) \subseteq P \). The former implies that \( a \in P \) and in this event we are done. We assume henceforth that

\[ a \notin P, \quad D(a) \in P \quad \text{and} \quad D(aRa) \subseteq P. \]

It follows immediately that \( aD(ya) \in P \) for all \( y \in R \). Substituting \( ya \) for \( x \) in (3), \((-1)^n D(ya)(ya)^n \in P \) and \( D(ya)(ya)^n \in P \). Now, right-multiplying the equation \( D(axya) = D(ax)(ya) + axD(ya) \) by \( (ya)^n \), we see that

\[
D(axya)(ya)^n = D(ax)(ya)^{n+1} + axD(ya)(ya)^n.
\]
Since $D(axya) \in P$ and $D(ya)(ya)^n \in P$, we have $D(ax)(ya)^{n+1} \in P$ for all $x, y \in R$. By Lemma 2, we obtain $D(ax) \in P$ and $aD(x) \in P$ for all $x \in R$. Therefore $aD(xy) = aD(x)y + axD(y) \in P$, $axD(y) \in P$ and $D(y) \in P$ for all $y \in R$, contradicting the hypothesis that $P \notin \Omega_1$. Hence $a \in P$. Henceforth $(P + U)/P$ contains no left zero divisors of $R$ in view of [2, Lemma 3], but by Lemma A, we get $[D(x), x]_{n-1} x = x[D(x), x]_{n-1}$, and Lemma B shows that $(x[D(x), x]_{n-1})^2 = ([D(x), x]_{n-1})^2 = 0$, so that $x[D(x), x]_{n-1} \in P$ and $[D(x), x]_{n-1} \in P$.

Now, we establish that $(n+1)! [D(x), x]_{n-1} \in P$ for all $P \in \Omega$ and for all $x \in U$, thus $(n+1)! [D(x), x]_{n-1} \in \cap P_{\alpha} = \{0\}$; hence $[D(x), x]_{n-1} = 0$. □

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References