Bounded variation functions of order \( k \) on sequence spaces

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Abstract

In this paper, we generalize some results concerning bounded variation functions on sequence spaces.

Some properties of bounded variation functions on sequence spaces were investigated by Wu Congxin [1], [2], [3]. Later on, he and Zhao Linsheng in [4], [5], [6] introduced and discussed bounded variation functions of order 2 on sequence spaces. In this paper, we generalize these results to bounded variation functions of order \( k \) on sequence spaces.

Let \( \lambda \) be a real linear sequence space. The Köthe dual \( \lambda^* \) of \( \lambda \) is the real linear sequence space consisting of all real sequences \( U = (u_1, u_2, \ldots) \) satisfying
\[
\sum_{k=1}^{\infty} |u_k x_k| < \infty
\]
for all \( X = (x_1, x_2, \ldots) \in \lambda \). When \( \lambda = \lambda^{**} \), we say that \( \lambda \) is a perfect space.

For a real function \( x(t) \) defined on \([a, b]\) and \( k+1 \) different points \( t_0, t_1, \ldots, t_k \in [a, b] \), we denote
\[
Q_k(x; t_0, t_1, \ldots, t_k) = \sum_{i=0}^{k} \frac{x(t_i)}{\prod_{j=0, i\neq j}^{k} (t_i - t_j)}
\]

Definition 1 [7, p. 87]. The variation of order \( k \) of a function \( x(t) \) defined on \([a, b]\) is
\[
V_{a, k}^b (x) \triangleq \sup_{\pi} \sum_{i=0}^{n-k} |Q_{k-1}(x; t_{i+1}, \ldots, t_{i+k-1}) - Q_{k-1}(x; t_{i+1}, \ldots, t_{i+k})|.
\]
Where the “sup” is taken over all partitions \( \pi: a = t_0 < t_1 < \cdots < t_n = b \) of \([a, b]\). When \( V^b_{a,k}(x) < \infty \), we say that \( x(t) \) is a bounded variation function of order \( k \) and denote by \( x(t) \in V^k_{a,k}(a, b) \).

**Lemma 1** [7, p. 88].
For any \( x \in V^k_{a,k}(a, b) \) and \( k \) different points \( a_0, a_1, \ldots, a_{k-1} \) in \([a, b]\), we have

\[
|Q_{k-1}(x; a_0, a_1, \ldots, a_{k-1})| \leq |Q_{k-1}(x; a_0, \ldots, a_{k-1})| + 2 V^b_{a,k}(x).
\]

**Lemma 2** [7, p. 79].
\[
Q_{r-1}(x; t_1, t_2, \ldots, t_r) - Q_{r-1}(x; t_0, t_1, \ldots, t_{r-1}) = (t_r - t_0) Q_r(x; t_0, \ldots, t_r).
\]

**Lemma 3**
For any \( k \geq 3 \), we have \((k - 1)! V^b_{a,k}(x) = b_{a,2} x^{(k-2)}(x)\).

**Lemma 4** [11, p. 179].
Let \( x: [a, b] \to \mathbb{R}, y: [a, b] \to \mathbb{R} \), then
\[
Q_k(xy; t_0, t_1, \ldots, t_k) = y(x_0) Q_k(x; t_0, t_1, \ldots, t_k) + Q_1(y; t_0, t_1) Q_{k-1}(x; t_1, \ldots, t_k) + Q_2(y; t_0, t_1, t_2) Q_{k-2}(x; t_2, t_3, \ldots, t_k) + \cdots + Q_k(y; t_0, t_1, \ldots, t_k) x(t_k).
\]

**Lemma 5**
Suppose \( x: [a, b] \to \mathbb{R} \) and \( a_0, a_1, \ldots, a_r \in [a, b] \) (\( a_i \neq a_j \) when \( i \neq j \)), then

\[
|Q_r(x; a_0, a_1, \ldots, a_r)| \leq \frac{1}{\min |a_i - a_j|} \sum_{i=0}^{r} |x(a_i)| \quad r = 1, 2, \ldots.
\]

**Definition:** Let \( X(t) = (x_1(t), x_2(t), \ldots) \) be an abstract function from \([a, b]\) to a sequence space \( \lambda \). If for each \( U = (u_1, u_2, \ldots) \in \lambda^* \), we have

\[
V^b_{a,k}(X, U) \Delta \sup_{\pi} \sum_{i=0}^{n-k} \sum_{m=1}^{\infty} |u_m [Q_{k-1}(x; t_i, \ldots, t_{i+k-1}) - Q_{k-1}(x; t_{i+1}, \ldots, t_{i+k})]| < \infty
\]

then \( X(t) \) is called a bounded variation function of order \( k \) and denoted by \( X(t) \in V^k_{a,k}(a, b, \lambda) \).
Theorem 1

\[ X(t) \in V_k([a, b], \lambda) \text{ iff} \]

1° \( x_m(t) \in V_k[a, b], \ m = 1, 2, \ldots \) and

2° \( \sum_{m=1}^{\infty} \left\{ V_{\frac{b}{a}}(x_m) \right\} \in \lambda^{**}. \)

Proof. Necessity. 1° Pick \( U = (0, 0, \ldots, 0, 1, 0, \ldots) \in \lambda^{*}, \) then from

\[
\sup_{\pi} \sum_{i=0}^{n-1} \sum_{m=1}^{\infty} |u_m| \left| t_{i+k} - t_i \right| \left| Q_k(x_m; t_i, \ldots, t_{i+k}) \right| = \sup_{\pi} \sum_{i=0}^{n-k} \left| t_{i+k} - t_i \right| \left| Q_k(x_m; t_i, \ldots, t_{i+k}) \right| < \infty
\]

we see \( x_s \in V_k[a, b], s = 1, 2, \ldots \)

Next we turn to 2°. If 2° is not true, then there exist \( U^{(0)} = (u_1^{(0)}, u_2^{(0)}, \ldots) \in \lambda^*, u_m^{(0)} \neq 0, \ m = 1, 2, \ldots \) and \( N_n \geq 1, \varepsilon_n > 0 \) such that

\[
\sum_{m=1}^{N_n} |u_m^{(0)}| V_{\frac{b}{a}}(x_m) = n + \varepsilon_n.
\]

Since \( x_m \in V_k[a, b], m = 1, 2, \ldots, N_n, \) there exists a partition \( \pi_m: a = t_i^{(m)} < t_i^{(m)} < \ldots < t_i^{(m)} = b \) such that

\[
V_{\frac{b}{a}}(x_m) \leq \sum_{i=0}^{n-m-k} \left| t_{i+k} - t_i \right| \left| Q_k(x_m; t_i^{(m)}, \ldots, t_{i+k}) \right| + \varepsilon_n \frac{2m+1}{u_m^{(0)}}.
\]

Let \( \pi \) be the partition consisting of all points \( \{ t_i^{(m)} \mid i \leq n_m, m \leq N_n \}: \pi: a = s_1^{(N_n)} < s_2^{(N_n)} < \cdots < s_{l(N_n)}^{(N_n)} = b \) then by Theorem 3 in [8], we have

\[
V_{\frac{b}{a}}(x_m) \leq \sum_{i=1}^{l(N_n)-k} \left| s_i^{(N_n)} - s_{i+k}^{(N_n)} \right| \left| Q_k(x_m; s_i^{(N_n)}, \ldots, s_{i+k}^{(N_n)}) \right| + \varepsilon_n \frac{2m+1}{u_m^{(0)}}.
\]
Hence
\[
\sum_{i=0}^{\lfloor(N_n)^{-k}\rfloor} \sum_{m=1}^{\infty} \sum_{i=0}^{\lfloor(N_n)^{-k}\rfloor} \lvert u_m^{(0)} \rvert \lvert t_{i+k} - t_i \rvert \lvert Q_k(x_m; t_i, \ldots, t_{i+k}) \rvert
\geq \sum_{m=1}^{N_n} \sum_{i=0}^{\lfloor(N_n)^{-k}\rfloor} \lvert u_m^{(0)} \rvert \lvert s_{i+k}^{(N_n)} - s_i^{(N_n)} \rvert \lvert Q_k(x_m; s_i^{(N_n)}), \ldots, s_{i+k}^{(N_n)}) \rvert
\]

contradicting that \( X(t) \in V_k([a, b], \lambda) \).

Sufficiency. Notice that
\[
\sup_{\pi} \sum_{m=1}^{n-k} \sum_{i=0}^{\infty} \lvert u_m \rvert \lvert t_{i+k} - t_i \rvert \lvert Q_k(x_m; t_i, \ldots, t_{i+k}) \rvert
\leq \sum_{m=1}^{\infty} \sup_{\pi} \sum_{i=0}^{n-k} \lvert u_m \rvert \lvert t_{i+k} - t_i \rvert \lvert Q_k(x_m; t_i, \ldots, t_{i+k}) \rvert
\]
\[
= \sum_{m=1}^{\infty} \lvert u_m \rvert \frac{b}{a_k} V_k(x_m) < \infty
\]

we find that \( X(t) \) is bounded variation of order \( k \). □

**Theorem 2**

\( V_k([a, b], \lambda) \subset V_r([a, b], \lambda) \) for all \( 1 \leq r < k \).

**Proof.** It is sufficient to consider the case \( r = k-1 \). By Theorem 1, \( X(t) = \{x_m(t)\}_{m=1}^{\infty} \in V_k([a, b], \lambda) \) implies

\[
x_m(t) \in V_k[a, b] \quad \text{and} \quad \{\frac{b}{a_k} V_k(x_m)\} \in \lambda^{**}.
\]

For \( k \) different points \( a_0 < a_1 < \cdots < a_{k-1} \) in \([a, b]\), by Lemma 1, we have

\[
\lvert Q_{k-1}(x_m; t_i, \ldots, t_{i+k-1}) \rvert \leq \lvert Q_{k-1}(x_m; a_0, \ldots, a_{k-1}) \rvert + 2 \frac{b}{a_k} V_k(x_m).
\]
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Hence

$$\sup_{\pi} \sum_{i=0}^{n-k+1} \sum_{m=1}^{\infty} |u_m| |t_{i+k-1} - t_i| |Q_{k-1}(x_m; t_i, \ldots, t_{i+k-1})|$$

$$\leq \sup_{\pi} \sum_{i=0}^{n-k+1} \sum_{m=1}^{\infty} |u_m| |t_{i+k-1} - t_i| (|Q_{k-1}(x_m; a_0, a_1, \ldots, a_{k-1})|$$

$$+ 2 b_{a_k}(x_m))$$

$$\leq \sup_{\pi} \sum_{i=0}^{n-k+1} |t_{i+k-1} - t_i| \sum_{m=1}^{\infty} |u_m| (|Q_{k-1}(x_m; a_0, a_1, \ldots, a_{k-1})|$$

$$+ 2 b_{a_k}(x_m))$$

$$\leq k(b-a) \left( \frac{1}{\min_{i \neq j, i, j=0,1, r, \ldots, k-1} |a_i - a_j|^k} \sum_{m=1}^{\infty} \sum_{i=0}^{k-1} |x_m(a_i)|$$

$$+ 2 \sum_{m=1}^{\infty} |u_m b_{a_k}(x_m)| \right) < \infty.$$ 

It follows that $X(t) \in V_{k-1}([a, b], \lambda)$. □

Corollary 1

If $\{b_{a_k}(x_m)\} \in \lambda^{**}$, then $\{b_{a_r}(x_m)\} \in \lambda^{**}$ for all $1 \leq r < k$.

Theorem 3

Suppose $k \geq 3$, then $X(t) \in V_k([a, b], \lambda)$ iff $X'(t) \equiv \{x'_m(t)\} \in V_{k-1}([a, b], \lambda)$.

Proof. Necessity. By Theorem 1, $x'_m(t) \in V_{k-1}[a, b]$. Moreover, by Lemma 3, when $k = 3$, from $\{b_{a_3}(x_m)\} \in \lambda^{**}$, we have $\{b_{a_2}(x'_m)\} \in \lambda^{**}$, and when $k > 3$, we have

$$(k-2)! b_{a_{k-1}}(x'_m) = b_{a_2}(x^{(k-2)}).$$

Hence, $\{b_{a_{k-1}}(x'_m)\} \in \lambda^{**}$ and the conclusion follows from Theorem 1.
Sufficiency. By Theorem 1 in [9], we have \( x_m(t) \in V_k[a, b] \). Observing that 
\( k = 3 \) implies \( 2! \left[ \frac{b}{a} \right]_3 (x_m') = \left[ \frac{b}{a} \right]_2 (x_m^{(2)'}(t)) = (k - 1)! \left[ \frac{b}{a} \right]_k (x_m) \)

we find that \( \left\{ \frac{b}{a} (x_m) \right\} \in \lambda^*(k \geq 3) \) and that Theorem 1 implies \( X(t) \in V_k[a, b, \lambda] \). □

Theorem 1 and Theorem 3 imply

Corollary 2

Let \( k \geq 3 \) then the following are equivalent

1° \quad X(t) \in V_k([a, b], \lambda);

2° \quad \forall 2 \leq r < k, \quad X^{(k-r)}(t) - X^{(k-r)}(a) \in V_r([a, b], \lambda);

3° \quad \exists 2 \leq r < k, \quad \text{such that} \quad X^{(k-r)}(t) - X^{(k-r)}(a) \in V_r([a, b], \lambda);

4° \quad \forall 2 \leq r < k, \quad \text{we have}

(i) \quad x_m^{(k-r)}(t) \in V_r[a, b], m = 1, 2, \ldots

(ii) \quad \left\{ \frac{b}{a} x_m^{(k-r)} \right\} \in \lambda^{**}

5° \quad \exists 2 \leq r < k \quad \text{such that}

(i) \quad x_m^{(k-r)}(t) \in V_r[a, b], m = 1, 2, \ldots

(ii) \quad \left\{ \frac{b}{a} x_m^{(k-r)} \right\} \in \lambda^{**}.

Theorem 4

Assume that \( \lambda \) is a perfect space, then \( X(t) \in V_k([a, b], \lambda) \) iff there exist convex functions \( X^{(i)}(t) \in V_k([a, b], \lambda) \) \( (i = 1, 2) \) of order \( k \) such that

\[ X(t) = X^{(1)}(t) - X^{(2)}(t) \quad (t \in [a, b]) \]

\( (X(t) \) is called a convex function of order \( k \), if for each natural number \( m, x_m(t) \) is a usual convex function of order \( k \) and \( x(t) \) is called a usual convex function of order \( k \), if for any partition \( \pi: a = t_0 < t_1 < \cdots < t_k = b \) of \([a, b]\), we have \( Q_k(x; t_0, t_1, \ldots, t_k) \geq 0 \).
Proof. The sufficiency is obvious. Now we prove the necessity. The necessity is already known for $k = 1$ and $k = 2$. Suppose that the condition is necessary for $k = m - 1$, we investigate the case $k = m$. Since by Theorem 3, $X'(t) \in V_{m-1}([a, b], \lambda)$ by the assumption, there exist convex functions $Y^{(i)}(t) \in V_{m-1}([a, b], \lambda)$ of order $m - 1$ ($i = 1, 2$) such that $X'(t) = Y^{(1)}(t) - Y^{(2)}(t)$. For any $c \in (a, b)$, we have

$$X(t) = \int_0^t X'(s) ds - X(c) = \int_0^t Y^{(1)}(s) ds - \int_0^t Y^{(2)}(s) ds - X(c).$$

Set

$$X^{(1)}(t) = \int_0^t Y^{(1)}(s) ds, \quad X^{(2)}(t) = \int_0^t Y^{(2)}(s) ds + X(c)$$

then by Theorem 13 in [8], $X^{(i)}(t)$ is convex of order $m, i = 1, 2$. But $(X^{(i)}(t))' \in V_{m-1}([a, b], \lambda)$, by Theorem 3, $X^{(i)}(t) \in V_{m}([a, b], \lambda)$ $i = 1, 2$. Clearly, $X(t) = X^{(1)}(t) - X^{(2)}(t)$, and $X^{(1)}(t) \in \lambda$ (and thus $X^{(2)}(t) \in \lambda$) can be deduced as follows

$$|x^{(1)}_m(t)| \leq \int_a^b |y^{(1)}_m(s)| ds \leq \int_a^b |y^{(1)}_m(a)| + \frac{b}{a} (y^{(1)}_m) ds$$

$$+ (b-a) (|y^{(1)}_m(a)| + \frac{b}{a} (y^{(1)}_m))$$

therefore, $\{X^{(1)}_m(t)\} \in \lambda^{**} = \lambda$. □

Theorem 5

Let $\lambda$ be perfect, then $X(t), Y(t) \in V_k([a, b], \lambda)$ implies $X(t)Y(t) \in V_k([a, b], \lambda)$ iff for any $Z(t) \in V_k([a, b], \lambda), U \in \lambda^*$ and $c \in [a, b]$, we have

$$\left\{ |u_m| (|Z_m(c)| + \frac{b}{a} V (Z_m) + \frac{b}{a_2} V (Z_m) + \cdots + \frac{b}{a_k} (Z_m)) \right\} \in \lambda^*.$$  

Proof. Sufficiency. Let $X(t) \in V_k([a, b], \lambda), Y(t) \in V_k([a, b], \lambda)$, by Lemma 4,
\[
\sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| |Q_k(x_m; t_i, t_{i+1}, \ldots, t_{i+k})| \\
= \sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| |y_m(t_i)Q_k(x_m; t_i, \ldots, t_{i+k}) + Q_1(y_m; t_i, t_{i+1})Q_{i-1}(x_m; t_{i+1}, \ldots, t_{i+k}) + \cdots + Q_k(y_m; t_i, \ldots, t_{i+k})x_m(t_{i+k})| \\
\leq \sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| |y_m(t_i)Q_k(x_m; t_i, \ldots, t_{i+k}) + Q_1(y_m; t_i, t_{i+1})Q_{i-1}(x_m; t_{i+1}, \ldots, t_{i+k}) + Q_k(y_m; t_i, \ldots, t_{i+k})x_m(t_{i+k})| \\
\leq \sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| (|y_m(a_0)| + 2 V_a(y_m)) + \sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| (|Q_k(y_m; t_i, \ldots, t_{i+k})|(|x_m(a_0)| + 2 V_a(x_m)),
\]

where \(\{a_i\}_{i=0}^{k-1}\) are different points in \((a, b)\). For the first term, we have

\[
\sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| (|y_m(a_0)| + 2 V_a(y_m))|Q_k(x_m; t_i, \ldots, t_{i+k})| \\
\leq \sum_{m=1}^{\infty} |u_m| |y_m(a_0)| + 2 V_a(y_m) \sup_{\pi} \sum_{i=0}^{n-k} |t_{i+k} - t_i| |Q_k(x_m; t_1, \ldots, t_{i+k})| \\
= \sum_{m=1}^{\infty} |u_m| (|y_m(a_0)| + 2 V_a(y_m)) \sqrt{b_a(x_m)} < \infty
\]

(note that \(|u_m|(|y_m(a_0)| + 2 V_a(y_m)) \in \lambda^*, \{V_{a,k}^b(x_m)\} \in \lambda^{**}\).
Similarly, for the last term, we have
\[
\sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| |Q_k(y_m; t_i, \ldots, t_{i+k})| (|x_m(a_0)| + 2 V^b(a) (x_m)) < \infty.
\]

Now, we show that the other terms are also bounded. Without loss of generality, we only consider the term
\[
\sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| (|Q_1(y_m; a_0, a_1)| + 2 V^b(a) (y_m)) (|Q_{k-1}(x_m; a_0, \ldots, a_{k-1})| + 2 V^b(a) (x_m)).
\]

By Lemma 5,
\[
\sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m| |t_{i+k} - t_i| (|Q_1(y_m; a_0, a_1)| + 2 V^b(a) (y_m)) (|Q_{k-1}(x_m; a_0, \ldots, a_{k-1})| + 2 V^b(a) (x_m)) \leq k(b-a) \sum_{m=1}^{\infty} |u_m| (|Q_1(y_m; a_0, a_1)| + 2 V^b(a) (y_m)) (|Q_{k-1}(x_m; a_0, \ldots, a_{k-1})| + 2 V^b(a) (x_m)) < \infty.
\]

Thus, \(X(t)Y(t) \in V_k([a, b], \lambda)\).

Necessity. By Theorem 2.6 in [7], the condition is necessary for \(k = 1\). Now, suppose \(k \geq 2\). Define
\[
x_m(t) = (|Z_m(c)| + V^b(a) (Z_m) + \ldots + V^b(a_k) (Z_m)) t^{k-1} \quad (a \leq t \leq b)
\]
\[
y_m(t) = |x_m^{(0)}| t
\]
then from
\[
x_m^{(k-1)}(t) = (k-1)! (|Z_m(c)| + V^b(a) (Z_m) + \ldots + V^b(a_k) (Z_m))
\]
\[
y_m^{(k-1)}(t) = \begin{cases} |x_m^{(0)}|, & k = 2 \\ 0, & k > 2 \end{cases}
\]
we have
\[ x^{(k-2)}_m \in V_2[a, b], \quad y^{(k-2)}_m \in V_2[a, b]. \]

Hence, Theorem 1 in [9] implies \( x_m, y_m \in V_k[a, b] \), and Proposition 3.4 in [7] claims
\[ V^b_{a,k}(x_m) = V^b_{a,k}(y_m) = 0. \]
Thus, \( \{ V^b_{a,k}(x_m) \} \in \lambda^{**} \) and \( \{ V^b_{a,k}(y_m) \} \in \lambda^{**} \), and so, by Theorem 1,
\[ X(t) = \{ x_m(t) \} \in V_k([a, b], \lambda) \]
\[ y(t) = \{ y_m(t) \} \in V_k([a, b], \lambda). \]

Therefore \( X(t)y(t) \in V_k([a, b], \lambda) \).

For any \( t_0 \neq t_1 \neq \cdots \neq t_k \) in \((a, b)\), by Proposition 3.5 in [7] p. 82,
\[ Q_k(x_m y_m; t_0, \ldots, t_k) = |x_0^{(0)}||(Z_m(c)) + V^b_{a,k}(Z_m) + \cdots + V^b_{a,k}(Z_k)| \]
and from
\[ \infty \geq \frac{h}{a} V(XY; U) \geq \sum_{m=1}^{\infty} |u_m| |t_k - t_0| |Q_k(x_m y_m; t_0, \ldots, t_k)| \]
\[ = |t_k - t_0| \sum_{m=1}^{\infty} |u_m| |(Z_m(c)) + V^b_{a,k}(Z_m) + \cdots + V^b_{a,k}(Z_k)| |x_0^{(0)}| \]
we find
\[ \{ |u_m| |(Z_m(c)) + V^b_{a,k}(Z_m) + \cdots + V^b_{a,k}(Z_k)| \} \in \lambda^{*}. \] □

References

1. W. Congxin, Bounded variation functions on sequence spaces (I), J. Harbin Inst. of Tech. no. 2 (1959), 93–100.