Collect. Math. **44** (1993), 301–306

On the weak star uniformly rotund points of Orlicz spaces

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Abstract

We obtain the necessary and sufficient condition of weak star uniformly rotund point in Orlicz spaces.

In this paper, X denotes a Banach space, $B(X), S(X)$ denote respectively the unit ball and the unit sphere of X. $x \in S(X)$ is said to be a UR (WUR, W*UR) point provided that $x_n \in B(X), \|x_n + x\| \to 2 \Rightarrow \|x_n - x\| \to 0 \, (x_n - x \xrightarrow{\text{w}} 0, x_n - x \xrightarrow{\text{w}^*} 0)$. Obviously, URP ⇒ WURP ⇒ W*URP. If all points of $S(X)$ are W*UR points, X is locally weak star uniformly rotund.

$M(\cdot), N(\cdot)$ denote a pair of complemented N-functions (see [3]); "$M \in \Delta_2$" ("$M \in \nabla_2$") means that $M(\cdot)$ satisfies the ∇_2-condition (N satisfies the ∇_2-condition). (G, Σ, μ) stands for a non-atomic finite measure space; $L_M(G, \Sigma, \mu)$ expresses the Orlicz space generated by $M(\cdot)$:

$$L_M(G, \Sigma, \mu) = \left\{ x(t) \colon \exists a > 0, R_M\Big(\frac{x}{a}\Big) = \int_G M\Big(\frac{x(t)}{a}\Big) d\mu < \infty \right\}$$

with the norm

$$\|x\| = \inf \left\{ a > 0 \colon R_M\Big(\frac{x}{a}\Big) \le 1 \right\}.$$

For an Orlicz function space L_M and a sequence space l_M, a criterion of UR (WUR) point was obtained in recent years [1,2]. Here we give the criterion of W*UR point, but the statement and method is different to WURP.

Theorem 1

*Let $x \in S(L_M)$. x is a W*URP if and only if*
(1) $\exists \tau > 0, R_M(\frac{x}{1-\tau}) < \infty$
(2) $\mu\{t \in G\colon |x(t)| \in \mathbb{R} \setminus S_M\} = 0$
(3) $\mu\Big\{t \in G\colon |x(t)| \in \bigcup\limits_{i=1}^{\infty}\{b_i\}\Big\} = 0$ or $\mu\Big\{t \in G : |x(t)| \in \bigcup\limits_{i=1}^{\infty}\{a_i\}\Big\} = 0$ *and* $M \in \nabla_2$,
where S_M is the set of all strictly convex points of $M(\cdot)$, (a_i, b_i) are affine segments of $M(\cdot)$ for $i = 1, 2, \dots$

Proof. Without loss of generality, we assume $x(t) \geq 0$.

Necessity. Suppose that $R_M(\frac{x}{1-\varepsilon}) = \infty$ for all $\varepsilon > 0$. Take c large enough such that $\mu G_c = \mu\{t \in G\colon x(t) \leq c\} > 0$. Define $x' = -x\chi_{G_c} + x\chi_{G\setminus G_c}$; then $\|x'\| = \|x\| = 1$, but for all $\varepsilon > 0$

$$R_M\Big(\frac{x + x'}{2(1-\varepsilon)}\Big) = \int_{G\setminus G_c} M\Big(\frac{x(t)}{1-\varepsilon}\Big) d\mu = \infty$$

so $\|x + x'\| = 2$. From $x \neq x'$, we get a contradiction with the fact that x is a W*UR point, which shows that (1) is true.

Since a W*UR point is an extreme point, from [3], we get that (2) is true.

Suppose that there exist affine segments $[a, b], [c, d]$ of $M(\cdot)$ such that $A = \{t\colon x(t) = a\}, B = \{t\colon x(t) = d\}$ are sets with positive measure. Take $E \subset A, F \subset B$ satisfying $(M(b) - M(a))\mu E = (M(d) - M(c))\mu F$. Define $x' = x\chi_{G\setminus(E\cup F)} + b\chi_E + c\chi_F$, then $x' \neq x, R_M(x) = R_M(x') = 1$, and $R_M(\frac{x+x'}{2}) = \frac{R_M(x)+R_M(x')}{2} = 1$, which contradicts the fact that x is a W*UR point.

Suppose now that there is an affine segment $[a, b]$ with positive measure set $D = \{t\colon x(t) = b\}$ and $M \notin \nabla_2$, i.e. there are $u_n \nearrow \infty$ such that $u_1 > b$ and $M(\frac{u_n}{2}) > (1 - \frac{1}{n})\frac{M(u_n)}{2}$ $(n = 1, 2, \dots)$. Take subsets $E_n, D \supset E_1 \supset E_2 \supset \dots$ satisfying

$$M(u_n - b)\mu E_n + M(a)\mu(D \setminus E_n) = M(b)\mu D$$

then $\mu E_n \to 0$. Define $x_n = x\chi_{G\setminus D} + a\chi_{D\setminus E_n} + (u_n - b)\chi_{E_n}$ so $R_M(x_n) = R_M(x) = 1$ $(n = 1, 2, \dots)$ and

$$\begin{aligned}R_M\Big(\frac{x_n+x}{2}\Big) &= R_M(x\chi_{G\setminus D}) + M\Big(\frac{a+b}{2}\Big)\mu(D\setminus E_n) + M\Big(\frac{u_n}{2}\Big)\mu E_n\\ &\ge R_M(x\chi_{G\setminus D}) + \frac{M(a)+M(b)}{2}\mu(D\setminus E_n) + \frac{1}{2}\Big(1-\frac{1}{n}\Big)M(u_n)\mu E_n\\ &\ge \frac{R_M(x\chi_{G\setminus D}) + M(b)\mu(D\setminus E_n)}{2}\\ &\quad + \frac{R_M(x\chi_{G\setminus D}) + M(a)\mu(D\setminus E_n) + (1-\frac{1}{n})M(u_n-b)\mu E_n}{2}\\ &\longrightarrow R_M(x) = 1.\end{aligned}$$

From $\langle x - x_n, \chi_{D\setminus E_1}\rangle = (b-a)\mu(D\setminus E_1) > 0$, we get a contradiction since x is a W*UR point.

Sufficiency. Suppose that $x_n \in B(L_M)$, $\|x_n + x\| \to 2$. We will first show that $R_M(x_n) \to 1$ and $R_M(\frac{x_n+x}{2}) \to 1$.

Suppose that $R_M(x_n) \le 1-\delta$ for some $\delta > 0$ $(n = 1, 2, \ldots)$. Take ε small enough such that $\frac{1+\varepsilon}{1-\varepsilon} < \frac{1}{1-\delta}$. For such ε, while n is large enough we have $\|(1+\varepsilon)\frac{x_n+x}{2}\| > 1$, and so $R_M((1+\varepsilon)\frac{x_n+x}{2}) > 1$. Hence applying assumption (1) we get

$$\begin{aligned}1 < R_M\Big((1+\varepsilon)\frac{x_n+x}{2}\Big) &= R_M\Big(\frac{1+\varepsilon}{2}x_n + \frac{1-\varepsilon}{2}\frac{1+\varepsilon}{1-\varepsilon}x\Big)\\ \le \frac{1+\varepsilon}{2}R_M(x_n) + \frac{1-\varepsilon}{2}R_M\Big(\frac{1+\varepsilon}{1-\varepsilon}x\Big) &\le \frac{1+\varepsilon}{2}(1-\delta)\\ + \frac{1-\varepsilon}{2}\big(1+o(\varepsilon)\big)R_M(x).\end{aligned}$$

If $\varepsilon \to 0$, $1 \le 1-\delta/2$, so we get $R_M(x_n) \to 1$.

Clearly $\|\frac{x_n+x}{2} + x\| \to 2$, so we get similarly $R_M(\frac{x_n+x}{2}) \to 1$.

In the following, we show $x_n - x \xrightarrow{w^*} 0$, and so it is enough to show $x_n - x \xrightarrow{\mu} 0$. Denote $E = \{t \in G \colon x(t) \in S_M \setminus (\{a_i\} \cup \{b_i\})\}$. We first show $x_n - x \xrightarrow{\mu} 0$ on E. Suppose for a contrary that there exist $\varepsilon, \sigma > 0$ with $\mu\{t \in E \colon |x_n(t) - x(t)| \ge \varepsilon\} \ge \sigma$. Since

$$1 \ge R_M(x_n) \ge \int\limits_{\{t:|x_n(t)|\ge d\}} M\big(x_n(t)\big)d\mu \ge M(d)\mu\{t \colon |x_n(t)| > d\},$$

we have that for d large enough $\mu\{t \colon |x_n(t)| > d\} < \sigma/3$ $(n = 1, 2, \ldots)$ and $\mu\{t \colon |x(t)| > d\} < \sigma/3$. Thus $\mu E_n \ge \sigma/3$, where

$$E_n = \big\{t \in E \colon |x_n(t) - x(t)| \ge \varepsilon, \quad |x_n(t)| \le d, \quad |x(t)| \le d\big\}.$$

Clearly there is $\delta > 0$ such that for $t \in E_n$,

$$M\Big(\frac{x_n(t)+x(t)}{2}\Big) \le (1-\delta)\frac{M(x_n(t))+M(x(t))}{2}.$$

Hence

$$\begin{aligned} 0 \leftarrow & \frac{R_M(x_n)+R_M(x)}{2} - R_M\Big(\frac{x_n+x}{2}\Big) \\ &= \int_G \Big(\frac{M(x_n(t))+M(x(t))}{2} - M\Big(\frac{x_n(t)+x(t)}{2}\Big)\Big)d\mu \\ &\ge \int_{E_n} \Big(\frac{M(x_n(t))+M(x(t))}{2} - M\Big(\frac{x_n(t)+x(t)}{2}\Big)\Big)d\mu \\ &\ge \frac{\delta}{2}\int_{E_n} \big(M(x_n(t)\big) + M\big(x(t)\big)d\mu \ge \frac{\delta}{2}M\Big(\frac{\varepsilon}{2}\Big)\frac{\sigma}{3}. \end{aligned}$$

This contradiction shows that $x_n - x \xrightarrow{\mu} 0$ on E. Denote

$$\begin{aligned} F_a^+(n) &= \{t \in G : x(t) \in \{a_i\}, x_n(t) \ge x(t)\} \\ F_a^-(n) &= \{t \in G : x(t) \in \{a_i\}, x_n(t) < x(t)\} \\ F_b^+(n) &= \{t \in G : x(t) \in \{b_i\}, x_n(t) \ge x(t) \quad \text{or} \quad x_n(t) < 0\} \\ F_b^-(n) &= \{t \in G : x(t) \in \{b_i\}, x_n(t) < x(t) \quad \text{and} \quad x_n(t) \ge 0\}\,. \end{aligned}$$

Analogously as above we have that for any $\varepsilon > 0$

$$\begin{aligned} &\mu\{t \in F_a^-(n) : x_n(t) \le x(t) - \varepsilon\} \to 0 \quad (n \to \infty) \\ &\mu\{t \in F_b^+(n) : x_n(t) \ge x(t) + \varepsilon\} \to 0 \quad (n \to \infty), \end{aligned}$$

whence

$$\limsup_{n\to\infty} \int_{E\cup F_a^-(n)\cup F_b^+(n)} M\big(x_n(t)\big)d\mu \ge \limsup_{n\to\infty} \int_{E\cup F_a^-(n)\cup F_b^+(n)} M\big(x(t)\big)d\mu.$$

Consider the case $\mu(F_b^+(n) \cup F_b^-(n)) = 0 \quad (n = 1, 2, \ldots)$. From

$$\limsup_{n\to\infty} \int_{E\cup F_a^-(n)} M\big(x_n(t)\big)d\mu \ge \limsup_{n\to\infty} \int_{E\cup F_a^-(n)} M\big(x(t)\big)d\mu$$

and $R_M(x_n) \to 1 = R_M(x)$, it follows that

$$\liminf_{n\to\infty} \int_{F_a^+(n)} M\big(x_n(t)\big)d\mu \le \liminf_{n\to\infty} \int_{F_a^+(n)} M\big(x(t)\big)d\mu$$

thus

$$\liminf_{n\to\infty}\int_{F_a^+(n)}\big(M(x_n(t))-M(x(t))\big)d\mu=0$$

so for any $\varepsilon>0$

$$\mu\{t\in F_a^+(n)\colon x_n(t)\ge x(t)+\varepsilon\}\to 0\quad (n\to\infty).$$

Combining the above, we have $x_n-x\xrightarrow{\mu}0$ on $E\cup F_a^-(n)\cup F_a^+(n)=G$.

Consider the case of $\mu(F_a^+(n)\cup F_a^-(n))=0\quad (n=1,2,\ldots)$ and $M\in\nabla_2$. We first prove that $\lim\limits_{\mu e\to 0}\sup\limits_n\int_e M(x_n(t))d\mu=0$. Suppose that there exist $\varepsilon>0$ and $e_n\subset G$ with $\mu e_n\searrow 0$ such that $\int_{e_n} M(x_n(t))d\mu\ge\varepsilon$.

Take $c>0$ small enough, $M(c)\mu G<\varepsilon/2$, denote $e_n'=\{t\in e_n\colon|x_n(t)|\ge c\}$. Then

$$\begin{aligned}\int_{e_n'}M\big(x_n(t)\big)d\mu&=\int_{e_n}M\big(x_n(t)\big)d\mu-\int_{e_n\setminus e_n'}M\big(x_n(t)\big)d\mu\\&\ge\varepsilon-M(c)\mu G\ge\frac{\varepsilon}{2}.\end{aligned}$$

Combining $M\in\nabla_2$, for $\tau,\delta>0$, while $u\ge c, M(\frac{u}{1+\tau})\le(1-\delta)\frac{M(u)}{1+\tau}$ whence

$$\begin{aligned}1\leftarrow R_M\Big(\frac{x_n+x}{2}\Big)&=R_M\Big(\frac{x_n+x}{2}\chi_{G\setminus e_n'}\Big)+R_M\Big(\frac{x_n+x}{2}\chi_{e_n'}\Big)\\
&\le\frac{R_M(x_n\chi_{G\setminus e_n'})+R_M(x\chi_{G\setminus e_n'})}{2}+\int_{e_n'}M\Big(\frac{1+\tau}{2}\frac{x_n(t)}{1+\tau}+\frac{1-\tau}{2}\frac{x(t)}{1-\tau}\Big)d\mu\\
&\le\frac{R_M(x_n\chi_{G\setminus e_n'})+R_M(x\chi_{G\setminus e_n'})}{2}+\frac{1+\tau}{2}\int_{e_n'}M\Big(\frac{x_n(t)}{1+\tau}\Big)d\mu\\
&\quad+\frac{1-\tau}{2}\int_{e_n'}M\Big(\frac{x(t)}{1-\tau}\Big)d\mu\\
&\le\frac{R_M(x_n\chi_{G\setminus e_n'})+R_M(x\chi_{G\setminus e_n'})}{2}+\frac{1+\tau}{2}\frac{1-\delta}{1+\tau}\int_{e_n'}M\big(x_n(t)\big)d\mu\\
&\quad+\frac{1-\tau}{2}R_M\Big(\frac{x}{1-\tau}\chi_{e_n'}\Big)\\
&\le\frac{R_M(x_n\chi_{G\setminus e_n'})+R_M(x\chi_{G\setminus e_n'})}{2}+\frac{1-\tau}{2}o(\mu e_n')+\frac{R_M(x_n\chi_{e_n'})}{2}-\frac{\delta\varepsilon}{4}\\
&\le\frac{R_M(x_n)+R_M(x)}{2}-\frac{\delta\varepsilon}{4}+o(\mu e').\end{aligned}$$

If $n\to\infty$, it gives a contradiction: $1\le 1-\delta\varepsilon/4$. So we get

$$\limsup_{n\to\infty}\int_{E\cup F_b^+(n)}M\big(x_n(t)\big)d\mu=\limsup_{n\to\infty}\int_{E\cup F_b^+(n)}M\big(x(t)\big)d\mu$$

hence

$$\liminf_{n\to\infty} \int_{F_b^-(n)} M\big(x_n(t)\big)d\mu = \liminf_{n\to\infty} \int_{F_b^-(n)} M\big(x(t)\big)d\mu.$$

From $\liminf\limits_{n\to\infty} \int\limits_{F_b^-(n)} (M(x(t)) - M(x_n(t)))d\mu = 0$, it follows that $x_n - x \xrightarrow{\mu} 0$ on $F_b^-(n)$, and consequently $x_n - x \xrightarrow{\mu} 0$ on G. □

By an analogous argumentation, we get the same result for Orlicz sequence spaces.

Theorem 2

*Let $x \in S(l_M)$. Then x is a W*URP point if and only if*

(1) there is $\tau > 0, R(\frac{x}{1-\tau}) < \infty$

(2) if there is "i", $|x(i)| \in (a, b]$, then $M \in \nabla_2$ and there is no "j", $j \neq i$ with $|x(j)| \in [c, d)$, where $[a, b]$ and $[c, d]$ are affine segments of $M(\cdot)$.

References

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