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On complemented subspaces of rearrangement invariant function spaces

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Abstract

A necessary and sufficient condition is given for a r.i. function space to contain a complemented isomorphic copy of $\ell_1(\ell_2)$.

1. Introduction

In the paper [3] was investigated the existence of complemented copies of the space ℓ_2 in rearrangement invariant ("r.i") function spaces. We showed in particular that if the r.i. space X does not contain a subspace isomorphic to c_0, then it contains a complemented copy of ℓ_2 iff either it contains a complemented sublattice isomorphic to ℓ_2 or X and its Köthe dual X' both contain a Gaussian variable. In the same paper was also investigated the existence of an isomorphism between X and its Hilbert-valued extension $X(\ell_2)$ (which is in fact equivalent to the existence of a complemented copy of $X(\ell_2)$ in X), in the case where X is a q-concave ($q < 2$) r.i. function space over $I = [0, 1]$. In this case, a necessary and sufficient condition is that the multiplication operator $M_G: L_0(I) \to L_0(I \times I), f \mapsto f \otimes G$ operates from $X'(I)$ into $X'(I \times I)$ (where G is a normal Gaussian variable and $f \otimes G(s,t) = f(s)G(t)$).

Here we are interested in the existence of a complemented copy of the space $\ell_1(\ell_2)$ in X. When X contains itself ℓ_1 as complemented sublattice (which is the case in particular if simple integrable functions are not dense in X'), it is clear that this question is intermediate between the two preceding; thus the criterion we find is naturally intermediate between the two criterions given above. In the case where X has finite upper Boyd index, and $\ell_1(\ell_2)$ does not embed in X as complemented sublattice, the criterion reduces to the fact that the domain of M_G in X' is not included in the closure of simple integrable functions.

We state now our main results.

Proposition 1

Let X be a rearrangement invariant function space over $(\Omega, \mathcal{A}, \mu)$, not containing c_0. Suppose that X does not contain $\ell_1(\ell_2)$ as complemented sublattice. Then X contains $\ell_1(\ell_2)$ as complemented subspace iff there exist disjoint functions $A_i,\ i \geq 1$ in $X_+(\Omega)$ and an element $B \geq 0$ in the Köthe dual $X'(\Omega)$, such that, denoting by G a normal Gaussian variable defined on the auxiliary probability space (S, Σ, σ),
i) $\forall \geq 1, \|A_i \otimes G\|_X = 1 = \langle A_i, B\rangle$ *and*
ii) $B \otimes G$ *belongs to* $X'(\Omega \times S)$.

Corollary 2

Suppose that X satisfies the hypotheses of Proposition 1 and, moreover, has finite upper Boyd index. Then a necessary and sufficient condition for X to contain $\ell_1(\ell_2)$ as complemented subspace is the existence of an element B of X' which is not in the closure of the space of simple integrable functions but such that $B \otimes G$ still belongs to $X'(\Omega \times S)$.

We give now some definitions.
If X is a r.i. space over $I = [0, m]$ and $(\Omega, \mathcal{A}, \mu)$ is a measure space with $\mu(\Omega) = m$ (possibly infinite), we denote by $X(\Omega, \mathcal{A}, \mu)$ the space of measurable functions over $(\Omega, \mathcal{A}, \mu)$ whose non-increasing rearrangement is in $X = X(I)$.

We say that a bounded sequence $(x_n)_n$ in the r.i. space X is *X-equiintegrable* if the following conditions are satisfied:

$$i) \quad \lim_{\mu(A)\to 0} \sup_n \|\mathbf{1}_A x_n\|_X = 0 \qquad ii) \quad \inf_{\mu(A)<\infty} \sup_n \|\mathbf{1}_{A^c} x_n\|_X = 0$$

where A^c denotes the complementary set of A.

We say that a sequence $(x_n)_n$ converges *weakly conditionally in distribution* (in short "wcd") if there exists a measurable function $Y \in L_0(\Omega \times S)$, defined on a superspace of measure $(\Omega \times S, \mathcal{A} \otimes \Sigma, \mu \otimes \sigma)$ (where σ is a probability measure) such that for every μ-integrable subset U of Ω, and every bounded continuous function φ on $\mathbb{R}$, $\int_U \varphi(x_n) d\mu \xrightarrow[n\to\infty]{} \int_{U\times S} \varphi(Y) d\mu d\sigma$.

We say that Y is *conditionally Gaussian* (r.r. to the first variable) iff for μ-a.e. $\omega \in \Omega$, the partial function $Y_\omega = Y(\omega, \cdot)$ has Gaussian probability distribution (hence is equimeasurable with $A(\omega)G(\cdot)$, where G is a normal gaussian variable).

The main tool used here (as in [3]) is the following: for every ℓ_2-basic sequence $(x_n)_n$ in $L_1(\Omega)$, there exists a sequence of successive normalized blocks (y_n) build on the x_n which converges wcd to a conditionally Gaussian variable. This is for instance a consequence of [1] and [4], as noticed in [5].

In section 2 below, we prepare the proof of Proposition 1 by several technical lemmas. The proof of Proposition 1 itself and of its corollary are given in section 3.

Unexplained notions or facts about r.i. spaces can be found in [2], which we follow in particular for the precise definition of r.i. spaces ([2], 2a1).

2. Some technical lemmas

The first lemma is a refinement of Lemma 10 of [3]:

Lemma 3

Let X be a r.i. function space not containing c_0 and $(x_{j,n})_{j,n\in\mathbb{N}}$ be a system of elements of X such that for each $j \in \mathbb{N}$, the sequence $(x_{j,n})_n$ is X-equiintegrable and converges wcd to a conditionally Gaussian variable. Then for each j there is a subsequence $(x_{j,n_\ell^{(j)}})_\ell$ such that for every finite system $(\lambda_{j,\ell})$ of reals:

$$\Big\| \sum_{j,\ell} \lambda_{j,\ell} x_{j,n_\ell^{(j)}} \Big\| \underset{1+\varepsilon}{\sim} \Big\| \sum_j \Big(\sum_\ell |\lambda_{j,\ell}|^2 \Big)^{1/2} x_{j,n_\ell^{(j)}} \Big\|. \tag{1}$$

Moreover we can choose these subsequences such that each $F_j = \overline{\operatorname{span}}[x_{j,n_\ell^{(j)}}]_\ell$ has X-equiintegrable unit ball, and every weakly null subsequence of F_j converges wcd to a conditionally Gaussian variable.

Proof. We have $x_{j,n} \xrightarrow[n\to\infty]{\text{wcd}} A_j \otimes G_j \in X(\Omega \times S)$, where we may suppose the G_j to be independent. We suppose that $L_0(S)$ contains a sequence (G'_j) of Gaussian variables which are independent and independent of the G_j. We fix a sequence of positive reals ε_j with $\varepsilon = \sum_j \varepsilon_j$. Suppose we have chosen the $n_\ell^{(j)}$, with $j, \ell \geq 1$ and $j + \ell \leq m$, verifying:

$$H_m \left\{ \begin{array}{l} \text{For every system } (\lambda_j, \ell) \text{ with } j \geq 1, \ell \geq 1 \text{ and } j+\ell \leq m \\ \text{and every sequence } (\rho_j), j \leq m\text{:} \\ \Big\| \sum\limits_{\substack{j,\ell\geq 1 \\ j+\ell \geq m}} \lambda_{j,\ell} x_{j,n_\ell^{(j)}} + \sum\limits_{j=1}^m \rho_j A_j \otimes G_j \Big\| \\ \qquad - \Big\| \sum\limits_{j=1}^m \Big(\sum\limits_{\ell=1}^{m-j} |\lambda_{j,\ell}|^2 + |\rho_j|^2 \Big)^{1/2} A_j \otimes G_j \Big\| \leq \sum\limits_{j=1}^m \varepsilon_j \ . \end{array} \right.$$

Then we have for every systems $(\lambda_{j,\ell})_{j,\ell\geq 1, j+\ell\leq m+1}$ and $(\rho_j)_{j\leq m+1}$:

$$\begin{aligned}
u_{n_1,n_2,\dots,n_m}\big((\lambda_{j,\ell}),(\rho_j)\big) &:= \sum_{\substack{j,\ell\geq 1\\ j+\ell\leq m}} \lambda_{j,\ell} x_{j,n_\ell^{(j)}} + \sum_{j=1}^{m} \lambda_{j,m+1-j} x_{j,n_j} \\
&\qquad + \sum_{j=1}^{m+1} \rho_j A_j \otimes G_j \\
\underset{n_m\to\infty; n_{m-1}\to\infty;\dots n_1\to\infty}{\overset{\text{wcd}}{\longrightarrow}} & \sum_{\substack{j,\ell\geq 1\\ j+\ell\leq m}} \lambda_{j,\ell} x_{j,n_\ell^{(j)}} + \sum_{j=1}^{m} \lambda_{j,m+1-j} A_j \otimes G'_j \\
&\qquad + \sum_{j=1}^{m+1} \rho_j A_j \otimes G_j \\
\overset{\text{dist}}{\sim} \sum_{\substack{j,\ell\geq 1\\ j+\ell\leq m}} \lambda_{j,\ell} x_{j,n_\ell^{(j)}} &+ \sum_{j=1}^{m+1} (|\lambda_{j,m+1-j}|^2 + |\rho_j|^2)^{1/2} A_j \otimes G_j \\
&=: u_\infty\big((\lambda_{j,\ell}),(\rho_j)\big)\,.
\end{aligned}$$

Hence we deduce the convergence a.e. of the rearrangements:

$$u_{n_1,n_2,\dots,n_m}\big((\lambda_{j,\ell}),(\rho_j)\big)^* \to u_\infty\big((\lambda_{j,\ell}),(\rho_j)\big)^*\,. \tag{2}$$

As in the proof of Lemma 10 in [3], using the order-continuity of X, we deduce the convergence of:

$$\begin{aligned}
F_{n_1,n_2,\dots,n_m}\big((\lambda_{j,\ell}),(\rho_j)\big) := \Bigg\| &\sum_{\substack{j,\ell\geq 1\\ j+\ell\leq m}} \lambda_{j,\ell} x_{j,n_\ell^{(j)}} + \sum_{j=1}^{m} \lambda_{j,m+1-j} x_{j,n_j} \\
&+ \sum_{j=1}^{m+1} \rho_j A_j \otimes G_j \Bigg\|_X
\end{aligned}$$

to

$$\begin{aligned}
F_\infty\big((\lambda_{j,\ell}),(\rho_j)\big) := \Bigg\| &\sum_{\substack{j,\ell\geq 1\\ j+\ell\leq m}} \lambda_{j,\ell} x_{j,n_\ell^{(j)}} + \sum_{j=1}^{m+1} (|\lambda_{j,m+1-j}|^2 \\
&+ |\rho_j|^2)^{1/2} A_j \otimes G_j \Bigg\|_X
\end{aligned}$$

and by Ascoli's theorem, this convergence is uniform on each set

$$\Big\{\bigvee_{j=1}^{m+1}\Big(\sum_{\ell=1}^{m+1-j}|\lambda_{j,\ell}|^2+|\rho_j|^2\Big)^{1/2}\le K\Big\}.$$

Hence we can choose $n_m^{(1)}, n_{m-1}^{(2)}, \dots, n_1^{(m)}$ such that:

$$\Big|F_{n_m^{(1)},n_{m-1}^{(2)},\dots,n_1^{(m)}}\big((\lambda_{j,\ell}),(\rho_j)\big)-F_\infty\big((\lambda_{j,\ell}),(\rho_j)\big)\Big|\le\varepsilon_{m+1}$$

uniformly on the set

$$\Big\{(\lambda_{j,\ell}),(\rho_j):\Big\|\sum_{j=1}^{m+1}\sum_{\ell=1}^{m+1-j}(|\lambda_{j,m+1-j}|^2+|\rho_j|^2)^{1/2}A_j\otimes G_j\Big\|_X\le 1\Big\}.$$

Together with (H_m) we obtain (H_{m+1}). The subsequences $(x_{j,n_\ell^{(j)}})_\ell$ we obtain satisfy then the equivalence (1) (take m sufficiently large and $\rho_j=0$ in (H_m)).

Finally the assertions about equiintegrability and wcd convergence of blocks are also a consequence of the convergence of rearrangements (2) (see [3] for more details). □

Lemma 4

Let X be a r.i. space over $(\Omega,\mathcal{A},\mu)$, not containing c_0. If X contains $\ell_1(\ell_2)$ as complemented subspace, but not as complemented sublattice, then $X(\Omega\times[0,1])$ contains a complemented subspace with a $\ell_1(\ell_2)$-basis of the form $(A_j\otimes G_j^n)_{j,n\ge1}$, where $A_j\in L_0^+(\Omega)$ and the $G_j^n\in L_0([0,1])$ are independent Gaussian variables.

Proof. A) Let E be a complemented subspace of X, isomorphic to $\ell_1(\ell_2)$; write $E=\oplus_j E_j$, where the "fibers" E_j are isomorphic to ℓ_2, and the direct sum is a ℓ_1 sum: for every finite sequence $(x_j)_j, x_j\in E_j$, we have $\|\sum_j x_j\|\sim\sum_j\|x_j\|$. We remark first that for all but a finite numbers of indices j, there exists a subset U_j of Ω, of finite μ-measure, such that the X-norm and the $L_1(U_j)$-norm are equivalent on E_j (these equivalence need not be uniform with respect to j).

For, if not we have an infinite subset J of $\mathbb{N}$, such that for each $j\in J$, and every μ-finite subset U of Ω, the $L_1(U)$-norm and the X-norm are not equivalent on E_j. It is then easy to find, for each j, a normalized sequence $(f_{j,n})_n$ in E_j which is weakly null and converges to 0 locally in measure. It follows that this sequence is quasidisjoint for both the lattice structures of X and of E_j (this last one being given by the isomorphism with ℓ_2), i.e. $f_{j,n} - g_{j,n} \xrightarrow[n\to\infty]{} 0$ and $f_{j,n} - h_{j,n} \xrightarrow[n\to\infty]{} 0$, where $(g_{j,n})_n$ is disjoint in X and $(h_{j,n})_n$ is disjoint in E_j w.r. to the ℓ_2-basis. Then $(h_{j,n})_{j\in J, n\in\mathbb{N}}$ is a $\ell_1(\ell_2)$-basic sequence, spanning a complemented subspace of X. If we suppose, as we may, that: $\|h_{j,n} - g_{j,n}\|_X \le \varepsilon 2^{-(j+n)}$, then by Bessaga-Pełczyński perturbation principle the same holds for the doubly indexed sequence $(g_{j,n})_{j\in J, n\in\mathbb{N}}$, providing a complemented sublattice of X isomorphic to $\ell_1(\ell_2)$, a contradiction.

B) From now on we suppose that $J = \mathbb{N}$. There exists for each j a normalized sequence $(x_{j,n})_n$ in E_j which converges wcd on U_j to a conditionally Gaussian variable. This can be done in fact on every $U \supset U_j$ (since the $L_1(U)$ and the X-norm are still equivalent on E_j), hence by a diagonal argument we can obtain this wcd convergence on the whole of Ω. Now the subspace $F_j = \overline{\text{span}}[x_{j,n}]$ is C-complemented in the hilbertian space E_j (with C independent from j), hence $F = \oplus F_j$ is complemented in E. Thus we suppose from now on that E has a $\ell_1(\ell_2)$-basis $(x_{j,n})_{j,n}$ such that for every j, the sequence $(x_{j,n})_n$ converges wcd to a conditionally gaussian variable.

Now using a "subsequence splitting lemma" (see [6] for instance), after extraction, we may decompose: $x_{j,n} = x'_{j,n} + x''_{j,n}$, where $x'_{j,n} \perp x''_{j,n}$, the sequence $(x'_{j,n})_n$ is X-equiintegrable and the sequence $(x''_{j,n})_n$ is disjoint. We have for all fixed j two operators S'_j and $S''_j\colon E_j \to X$, such that $S' x_{j,n} = x'_{j,n}$ and $S'' x_{j,n} = x''_{j,n}$ and which are uniformly bounded (w.r. to j). Since $E = \oplus E_j$ is a ℓ_1-direct sum (up to isomorphism), we deduce the existence of two bounded operators $S', S''\colon E \to X$, whose restriction to each subspace E_j are respectively S'_j and S''_j.

C) Let P be a projection from X onto E. For each j, n, we denote by $E_{j,n}$ the closed span of $(x_{j,m})_{m>n}$.

We claim that for all but a finite number of indices j, there exist a positive real σ_j and an integer N_j such that:

$$\forall y \in E_{j,N_j},\ \|PS'y\| \ge \sigma_j \|y\|.$$

For, if not, there exist an infinite subset J of $\mathbb{N}$, and for all $j \in J$ a sequence $y_{j,n}$ in E_j, such that:

$$\|u_{j,n}\| = 1, \quad y_{j,n} \xrightarrow[n\to\infty]{w} 0 \ \text{ and } \ \|PS'y_{j,n}\| \xrightarrow[n\to\infty]{} 0\,.$$

We can suppose that $\|PS'y_{j,n}\| \le \varepsilon 2^{-(j+n)}$. After extraction, the doubly indexed sequence $(y_{j,n})_{j,n}$ is equivalent to the $\ell_1(\ell_2)$ basis and spans a 2-complemented subspace F of E. Let Q be a projection from E onto F with $\|Q\| \le 2$. Since

$y_{j,n} - QPS''y_{j,n} = QPS'_{j,n}$, we see that $J = QPS''$ is an isomorphism of F; $S''F$ is isomorphic to F and complemented in X (by $S''J^{-1}QP$). Thus $(S''y_{j,n})_{j,n}$ spans a complemented $\ell_1(\ell_2)$ subspace of X. For each $j, (y''_{j,n})_n$ is a disjoint sequence. Using the order continuity of X and the Bessaga-Pełczyński perturbation principle, we deduce that $\ell_1(\ell_2)$ embeds as complemented sublattice in X, a contradiction.

D) We want now to extract subsequences $(x_{j,n_\ell^{(j)}})_{\ell=1}^\infty$ such that for some $\delta > 0$ and every element y of the closed span F of $(x_{j,n_\ell^{(j)}})_{j,\ell}$, one has $\|S'y\| \geq \delta\|y\|$. (It will be also useful for the sequel that the unit ball of each closed subspace F'_j generated by the sequence $(x'_{j,n_\ell^{(j)}})_\ell$ is X-equiintegrable, and that every weakly null sequence in F'_j converges wcd to a conditionally Gaussian variable).

Using Lemma 3, we can extract subsequences $(x_{j,n_\ell^{(j)}})_{\ell=1}^\infty$ such that:

$$\Big\| \sum_{j,\ell} x'_{j,n_\ell^{(j)}} \Big\|_X \underset{1+\varepsilon}{\sim} \Big\| \sum_j \Big(\sum_\ell |\lambda_{j,l}|^2 \Big)^{1/2} x'_{j,n_\ell^{(j)}} \Big\|_X \ . \tag{3}$$

Relabeling, in order to simplify notations, we can suppose $n_\ell^{(j)} = \ell (\forall j, \ell)$. If $(x'_{j,1})_{j=1}^\infty$ is not equivalent to the ℓ_1-basis, there exist ℓ_1-normalized blocks $y_p = \sum_{j \in J_p} \alpha_j x'_{j,1}$, $(\sum_{j \in J_p} |\alpha_j| = 1)$ such that $\|y_p\| \xrightarrow[p\to\infty]{} 0$. Supposing $\forall p, \|y_p\| \leq \varepsilon \ \| \sum_{j \in J_p} \alpha_j x_{j,1} \|$, and setting: $z_{p,\ell} = \sum_{j \in J_p} \alpha_j x_{j,\ell}$, we obtain for every finite system of scalars $(\lambda_{p,\ell})$:

$$\Big\| \sum_{p,\ell} \lambda_{p,\ell} PS' z_{p,\ell} \Big\| \leq (1+\varepsilon)\|P\| \ \Big\| \sum_p \Big(\sum_\ell |\lambda_{p,\ell}|^2 \Big)^{1/2} y_p \Big\|$$

$$\leq \varepsilon(1+\varepsilon)\|P\| \sum_p \Big(\sum_\ell |\lambda_{p,\ell}|^2 \Big)^{1/2} \Big\| \sum_{j \in J_p} \alpha_j x_{j,1} \Big\| \leq C\varepsilon(1+\varepsilon)\|P\| \ \Big\| \sum_{p,\ell} \lambda_{p,\ell} z_{p,\ell} \Big\|$$

where C is the equivalence constant of $(x_{j,\ell})$ with the basis of $\ell_1(\ell_2)$. Hence for small ε, the operator $(I - PS') = PS''$ is an isomorphism from $Z = \overline{\text{span}}[z_{p,\ell}]$ onto its image. In particular, $S''Z$ and $PS''Z$ are isomorphic to $\ell_1(\ell_2)$. The subspace $PS''Z$, being a copy of $\ell_1(\ell_2)$ in the space E, which is itself isomorphic to $\ell_1(\ell_2)$, contains a further subspace G, which is isomorphic to $\ell_1(\ell_2)$, build on a subset of the $\ell_1(\ell_2)$-basis of $PS''Z$, and complemented in E. Let $Z_1 = (PS'')^{-1}(G)$, and Q be a projection of E onto G. Let $J = P_{|_{S''Z_1}}$. Then $J^{-1}QP$ is a projection from X onto $S''Z_1$, which proves that X contains a complemented $\ell_1(\ell_2)$ sublattice, a contradiction.

Hence in fact the sequence $(x'_{j,1})_j$ is equivalent to the ℓ_1-basis, which implies by (3) that $(x'_{j,\ell})$ is equivalent to the $\ell_1(\ell_2)$-basis (and S' is an isomorphism on its range).

E) From now on we suppose that $\|S'y\| \geq \delta\|y\|$ for $y \in E$, and we prove now the existence of a closed subspace F of E, generated by a system of block sequences $(y_{j,\ell})_\ell$ of the $(x_{j,\ell})_\ell(j \in \mathbb{N})$, on which PS' is an isomorphism, i.e. $\forall y \in E, \|PS'y\| \geq \rho\|y\|$. Since the subspace $PS'E_j$ is hilbertian by the point C) above, so is the subspace $S'PS'E_j$ (S' being an isomorphism), and we can find appropriate normalized successive blocks $(y_{j,\ell})_\ell$ on each sequence $(x_{j,\ell})_\ell$ such that the sequence $(S'PS'y_{j,\ell})_\ell$ converges wcd to a (nonzero) conditionally Gaussian variable. By Lemma 3, we can suppose that:

$$\Big\| \sum_{j,\ell} \lambda_{j,\ell} S'PS'y_{j,\ell} \Big\| \underset{1+\varepsilon}{\sim} \Big\| \sum_j \Big(\sum_\ell |\lambda_{j,\ell}|^2 \Big)^{1/2} S'PS'y_{j,1} \Big\|$$

hence, using again the fact that S' is an isomorphism, we have:

$$\Big\| \sum_{j,\ell} \lambda_{j,\ell} PS'y_{j,\ell} \Big\| \underset{1+\varepsilon}{\sim} \Big\| \sum_j \Big(\sum_\ell |\lambda_{j,\ell}|^2 \Big)^{1/2} PS'y_{j,1} \Big\|$$

and, reasoning as in the point D above (using $PS' = I - PS''$ on E), this implies that $(PS'y_{j,1})_j$ is equivalent to the ℓ_1-basis, hence:

$$\Big\| \sum_{j,\ell} \lambda_{j,\ell} S'PS'y_{j,\ell} \Big\| \sim \sum_j \Big(\sum_\ell |\lambda_{j,\ell}|^2 \Big)^{1/2}.$$

F) Let $Y = \overline{\mathrm{span}}[y_{j,\ell}]_{j,\ell}$. The subspace $PS'(Y)$, being a $\ell_1(\ell_2)$-subspace of the $\ell_1(\ell_2)$-subspace E, contains a subspace G which is isomorphic to $\ell_1(\ell_2)$, complemented in E and spanned by a subset of the $PS'y_{j,\ell}$. Set $Z = (PS')^{-1}(G)$: then $S'Z$ is a subspace of X isomorphic to $\ell_1(\ell_2)$ and complemented in X. Note that the basis $z_{i,m}$ of $S'Z$ is a subset of the basis of $S'Y$: $z_{i,m} = S'y_{j(i),\ell(i,m)}$, hence each sequence $(z_{i,m})_m$ is X-equiintegrable and converges wcd to a conditionally Gaussian variable.

G) We have thus reduced the situation to the case where the elements $(x_{j,n})_{j,n}$ of a $\ell_1(\ell_2)$ basis (spanning a complemented closed subspace of X) are, for each fixed j, X-equiintegrable and converging wcd to a conditionally Gaussian variable.

Now we apply the ultrapower procedure of the §3 in [3]. We have for every doubly indexed finite set of natural numbers $S = (n(j,\ell))_{1 \leq j,l \leq k}$ a projection $\pi_S: X \to$

span $[x_{j,n(j,\ell)}]_{1\le j,\ell\le k}$ with norm bounded by a constant K (independent of S). Let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. Passing to the ultrapower $\widetilde{X}$ (of X) relative to the iterated limit:

$$\lim_{k,\mathcal{U}}\ \lim_{n(k,k),\mathcal{U}}\dots\lim_{n(k,1),\mathcal{U}}\ \lim_{n(k-1,k),\mathcal{U}}\dots\lim_{n(1,1),\mathcal{U}}$$

we obtain a doubly indexed sequence $(\xi_{j,n})_{j,n\ge1}$ in $\widetilde{X}$, which is equivalent to the $\ell_1(\ell_2)$-basis, and a bounded projection $\pi\colon\widetilde{X}\to\overline{\mathrm{span}}[\xi_{j,n}]_{j,n}$. Moreover the $\xi_{j,n}$ lie in fact in $\widetilde{X}^{eq}$, the subspace of $\widetilde{X}$ whose elements can be defined by X-equiintegrable families of elements of X, since for each j, the sequence $(x_{j,n})_n$ is in fact X-equiintegrable. It is known that $\widetilde{X}^{eq}$ identifies to a space $X(\widetilde{\Omega},\widetilde{\mathcal{A}},\widetilde{\mu})$, the measure space $(\Omega,\mathcal{A},\mu)$ identifying to that generated by a sub-σ-algebra in $(\widetilde{\Omega},\widetilde{\mathcal{A}},\widetilde{\mu})$ ([6]). The conditional distribution of the elements $\xi_{j,n}$, w.r. to the initial σ-field $\mathcal{A}$ is the same as the limit conditional distribution of the $x_{j,n}$, i.e. that of a sequence $A_j\otimes G_j^n$ in $X(\Omega\times[0,1])$, where the A_j are nonnegative, $\mathcal{A}$-measurable, and the G_j^n are independent normal Gaussian variables in $[0,1]$.

The last point is to use a transformation of the measure algebra $(\widetilde{\mathcal{A}},\widetilde{\mu})$ conserving the measure, leaving the elements of $\mathcal{A}$ invariant and carrying each $\xi_{j,n}$ on $A_j\otimes G_j^n$, where A_j is $\mathcal{A}$-measurable, and G_j^n are normal Gaussian variables independent of $\mathcal{A}$. That such a transformation exists, at least after enlarging $(\widetilde{\mathcal{A}},\widetilde{\mu})$ is an easy measure-theoretic exercise. □

Lemma 5

Let X be a r.i. space not containing c_0 and $(x_n)_n$ a basic sequence in X, equivalent to the ℓ_1-basis. There exists a sequence of ℓ_1-normalized successive blocks f_i build on the sequence (x_n), which is quasidisjoint in X (i.e. $\forall i, \||f_i|\wedge|f_j|\|_X \xrightarrow[j\to\infty]{} 0$).

Proof. Passing to a subsequence, we may suppose (by the subsequence splitting lemma) that we have a decomposition: $x_n = x'_n + x''_n$, into a X-equiintegrable part $(x'_n)_n$ and a disjoint part $(x''_n)_n$, with $\forall n, x'_n \perp x''_n$. Let $F=\overline{\mathrm{span}}[x_n], F'=\overline{\mathrm{span}}[x'_n], F''=\overline{\mathrm{span}}[x''_n]$ and $S'\colon F\to F'$, (resp. $S''\colon F\to F''$) be the bounded linear operator such that $S'x_n=x'_n$, (resp. $S''x_n=x''_n$). If there are no $n_0\in\mathbb{N}$ and $\delta>0$ such that $\|S'x\|\ge\delta\|x\|$ for every $x\in F_{n_0}=\overline{\mathrm{span}}[x_n]_{n\ge n_o}$, then there is a sequence $(y_n)_n$ of successive normalized blocks on the basis of F such that $\|y_n-S''y_n\|\to0$, hence (y_n) is quasidisjoint. If at the contrary $\|S'x\|\ge\delta\|x\|$ for every $x\in F_{n_0}$, then $(x'_n)_{n\ge n_0}$ is equivalent to the ℓ_1-basis. This implies that the X-norm and the $L_1(U)$-norm are equivalent on F'_{n_0} for no integrable subset U of Ω (since a ℓ_1-basic sequence in $L_1(U)$ cannot be equiintegrable for the norm of $L_1(U)$, and a fortiori for that of X). Hence there exists a sequence of normalized successive blocks on the x'_n which converges to 0 in measure, hence is quasidisjoint (by order continuity). The homologous blocks build on the x_n are also quasidisjoint. □

Lemma 6

Let X be a r.i. space not containing c_0 nor $\ell_1(\ell_2)$ as a complemented sublattice. If X contains $\ell_1(\ell_2)$ as a complemented subspace, then there exists in the extended r.i. space $X(\Omega \times S, \mathcal{A}\overline{\otimes}\Sigma, \mu \otimes \sigma)$ a $\ell_1(\ell_2)$-basis (spanning a complemented closed subspace too) having the form $(A_j \otimes G_j^n)_{j,n}$, where the $G_j^n \in L_0(S)$ are independent normal Gaussian variables, and the functions $A_j \in L_0^+(\Omega)$ have disjoint supports.

Proof. We start from the elements $(A_j \otimes G_j^n)$ given by Lemma 4. By applying Lemma 5 to the ℓ_1-basic sequence $(A_j \otimes G_j^1)_j$, we obtain a quasidisjoint sequence of successive ℓ_1-normalized blocks $y_\ell = \sum_{j \in J_\ell} \alpha_j A_j \otimes G_j^1$; due to the symmetry of the variables G_j^1, we may suppose in fact that the α_j are nonnegative. We set now: $y_\ell^n = \sum_{j \in J_\ell} \alpha_j A_j \otimes G_j^n$. The doubly indexed sequences $(y_\ell^n)_{\ell,n}$ and $(B_\ell \otimes G_\ell^n)_{\ell,n}$, where $B_\ell = (\sum_{j \in J_\ell} (\alpha_j A_j)^2)^{1/2}$, are equivalent in distribution (in fact, conditionally w.r. to the first coordinate).

It follows that $F = \overline{\text{span}}[B_\ell \otimes G_\ell^n]$ is complemented in $X(\Omega \times S)$. For, we may suppose that the variables (G_ℓ^n) generate the σ-algebra Σ. There exists a measure presenting set transformation T defined on the $\mathcal{A} \otimes \Sigma$, with $\mu \otimes \sigma$-measurable values, whose associated isometry $\widetilde{T}: X(\Omega \otimes S) \to X(\Omega \times S)$ maps $\phi \otimes G_\ell^n$ onto $\frac{\phi}{B_\ell} \sum_{j \in J_\ell} \alpha_\ell A_\ell \otimes G_\ell^n$, for all $\phi \in L_\infty(\Omega)$: then $\widetilde{T}(B_\ell \otimes G_\ell^n) = y_\ell^n$. The subspace $\overline{\text{span}}[y_\ell^n] = \widetilde{T}F$ is complemented in $E = \overline{\text{span}}[A_j \otimes G_\ell^n]$ by the projection $Q: \sum_{jn} \lambda_{j,n} A_j \otimes G_{j,n} \mapsto \sum_{\ell,n} (\sum_{j \in J_\ell} \lambda_{j,n}) y_{\ell,n}$. Since E is itself complemented in $X(\Omega \times S)$ by a projection P, F is also complemented by the projection $\widetilde{T}^{-1} QP\widetilde{T}$.

On the other hand, the sequence $(z_\ell)_\ell := (B_\ell \otimes G_\ell^1)_\ell$ is quasidisjoint in X (since it is equimeasurable with $(y_\ell^1)_\ell$). This implies that (z_ℓ) converges to zero locally in measure. Hence (B_ℓ) itself goes to zero locally in measure; hence $\|(B_\ell \wedge B_p) \otimes G\| \xrightarrow[p \to \infty]{} 0$ by ordercontinuity, and we can find a disjoint sequence (C_ℓ) with $\|(B_\ell - C_\ell) \otimes G\| \xrightarrow[l \to \infty]{} 0$; equivalently, $\|B_\ell \otimes G_\ell^1 - C_\ell \otimes G_\ell^1\| \to 0$, and by a standard reasoning, we deduce that for some subsequence $(D_\ell)_\ell$ of $(C_\ell)_\ell$, the double sequence

$(D_\ell \otimes G_\ell^n)_{\ell,n}$ is equivalent to the $\ell_1(\ell_2)$-basis and spans a closed subspace which is complemented. □

3. Proof of the main results

Proof of Proposition 1. A) We prove first the necessity of the given conditions.

Suppose that X contains $\ell_1(\ell_2)$ as subspace, and let $(A_j \otimes G_j^n)_{j,n}$ be a special $(\ell_1(\ell_2))$-basis given by Lemma 6. If P is a projection from $X(\Omega \times S)$ onto $E = \overline{\text{span}}[A_j \otimes G_j^n]_{j,n}$, we can replace it by another projection Q conserving the support U_j of each function A_j; we set simply:

$$Qf = \sum_j \mathbf{1}_{U_j} P \mathbf{1}_{U_j} f\,.$$

This series is norm-convergent in X, and $\|Qf\| \leq \|P\|\|f\|$. For, denote by S_j the isometry of X defined by: $S_j f = \mathbf{1}_{U_j} f - \mathbf{1}_{U_j^c} f$, and define recursively a sequence (P_n) of projections by: $P_0 = P, P_{j+1} = \frac{1}{2}(P_j + S_j P_j S_j)$. Then $\|P_j\| \leq \|P\|$ and $P_n f = \sum_{j=1}^{n} \mathbf{1}_{U_j} P \mathbf{1}_{U_j} f + R_n f$, with $R_n f = \mathbf{1}_{\bigcap_{j=1}^n U_j^c} P \mathbf{1}_{\bigcap_{j=1}^n U_j^c} f$. Then $\|R_n f\| \leq \|\mathbf{1}_{\bigcap_{j=1}^n U_j^c} f\|$ which goes to zero as $n \to \infty$ when f is supported by the union of the U_j, by order-continuity of X, and the sequence $(P_n f)_n$ is stationary when f is supported in a finite union of the U_j. Denoting by π the band projection in X defined by the union of the U_j, this shows that $P_n \pi \xrightarrow[n\to\infty]{} Q$ (in strong operator topology).

Now let $Q_j = \mathbf{1}_{U_j} Q \mathbf{1}_{U_j}$, which acts as a projection from $X_j = X(U_j \times S)$ onto $E_j = \overline{\text{span}}[A_j \times G_j^n]_{n\geq 1}$. By [3] §2, there is another projection $R_j : X_j \to E_j$, having the form:

$$R_j f = \sum_n \langle f, B_j, \otimes G_j^n \rangle A_j \otimes G_j^n$$

with $B_j \otimes G \in X', \langle A_j, B_j \rangle = 1$ and $\|R_j\| \leq \|Q_j\|$. More precisely, the proof in [3] shows that:

$$\forall f \in X, \quad \| \, R_j \mathbf{1}_{U_j} f + \mathbf{1}_{U_j^c} f \, \| \leq \| \, Q_j \mathbf{1}_{U_j} f + \mathbf{1}_{U_j^c} f \, \|$$

hence:

$$\Big\| \sum_{j=1}^{n} R_j \mathbf{1}_{U_j} f + \mathbf{1}_{\bigcap_{j=1}^n U_j} f \, \Big\| \leq \Big\| \sum_{j=1}^{n} Q_j \mathbf{1}_{U_j} f + \mathbf{1}_{\bigcap_{j=1}^n U_j} f \, \Big\| \,.$$

Note that we may suppose that B_j is supported by U_j, and that $B_j \geq 0$ (replacing if necessary B_j by $\frac{|B_j|}{\langle A_j, |B_j|\rangle}$). Thus $Rf := \sum_{j=1}^{\infty} R_j \mathbf{1}_{U_j} f$ converges in norm and $\|Rf\| \leq \|Qf\|$. We have then:

$$Rf = \sum_{j,n} \langle f, B_j \otimes G_j^n \rangle A_j \otimes G_j^n.$$

Since $(B_j \otimes G_j^n)_{j,n}$ is biorthogonal with $(A_j \otimes G_j^n)_{j,n}$, which spans a complemented $\ell_1(\ell_2)$ subspace, it is a $c_0(\ell_2)$-basis, and in particular:

$$\Big\| \sum_{j=1}^{n} B_j \otimes G_j^1 \Big\|_{X'} \leq \|P\| \Big\| \sum_{j=1}^{n} B_j \otimes G_j^1 \Big\|_{E^*} \leq C\|P\|$$

but since the B_j are disjoint, $\Big\| \sum\limits_{j=1}^{n} B_j \otimes G_j^1 \Big\|_{X'} = \Big\| \sum\limits_{j=1}^{n} B_j \otimes G \Big\|_{X'}$ (same distribution), and consequently $B \otimes G \in X'$, where $B = \sum\limits_{j=1}^{\infty} B_j$.

B) Now we prove the sufficiency of the conditions.

Let U_i be the support of A_i and $B_i = \mathbf{1}_{U_i} B$. Then in X', the sequence $(B_i \otimes G)_i$ is equivalent to the c_0 basis, since $\|B_i \otimes G\| \geq \langle B_i \otimes G, A_i \otimes G\rangle = \langle B_i, A_i \rangle = 1$, and $\forall n, \| \sum\limits_{i=1}^{n} B_i \otimes G\| \leq \|B \otimes G\|$; the biorthogonal sequence $(A_i \otimes G)_i$ in X is thus equivalent to the ℓ_1-basis; then $F = \overline{\text{span}}[A_i \otimes G]_i$ is complemented by the projection $\overline{R}$ defined by $\overline{R}f = \sum\limits_i \langle f, B_i \otimes G \rangle A_i \otimes G$, since:

$$\begin{aligned} \|\overline{R}f\| &\leq \sum_i |\langle f, B_i \otimes G \rangle| \leq \sum_i \langle f, B_i \otimes |G| \rangle \leq f, \Big\langle \Big(\sum_i B_i \Big) \otimes |G| \Big\rangle \\ &= \langle f, B \otimes |G| \rangle \leq \|f\|_X \|B \otimes G\|_{X'}. \end{aligned}$$

We remark now that the same happens when we replace the $A_i \otimes G$ by elements $A_i \otimes G_i$, where $(G_i)_i$ is a sequence of independent normal Gaussian variables. For, we may suppose that $G_i = Z_i G$, where Z_i is an onto isometry of $X(S)$ induced by a measure preserving set transformation. Let $\widetilde{Z}_i = \text{id} \otimes Z_i$ be the natural extensions of these isometries to $X(\Omega \times S)$. If $f \in X(\Omega \times S)$, set $f_i = \mathbf{1}_{U_i} f$ and $\widetilde{Z} f = \sum\limits_i Z_i f_i$. We define $Rf = \sum\limits_i \langle f, B_i \otimes G_i \rangle A_i \otimes G_i$. We have then:

$$\begin{aligned} Rf &= \sum_i \langle f_i, B_i \otimes G_i \rangle A_i \otimes G_i = \sum_i \langle \widetilde{Z}_i^{-1} f_i, B_i \otimes G \rangle \\ &= \sum_i \langle \widetilde{Z}^{-1} f, B_i \otimes G \rangle A_i \otimes G_i \end{aligned}$$

which is equimeasurable with $\sum_i \langle \widetilde{Z}^{-1} f, B_i \otimes G\rangle A_i \otimes G = \overline{R}(\widetilde{Z}^{-1} f)$; hence $\|R\| \leq \|\overline{R}\widetilde{Z}^{-1}\| = \|\overline{R}\|$.

Finally let $(G_{i,n})_{i,n}$ be a doubly indexed sequence of independent normal Gaussian variables. Then $(A_i \otimes G_{i,n})$ is equivalent to the $\ell_1(\ell_2)$-basis. Let us show that $E = \overline{\operatorname{span}}[A_i \otimes G_{i,n}]_{i,n}$ is complemented. We set:

$$Rf = \sum_{i,n} \langle f, B_i \otimes G_{i,n}\rangle A_i \otimes G_{i,n} \,.$$

We have:

$$\begin{aligned} Rf &= \sum_i \Big(\sum_n |\langle f, B_i \otimes G_{i,n}\rangle|^2\Big)^{1/2} A_i \otimes G_i \\ &= \sum_i \sum_n \lambda_{i,n} \langle f, B_i \otimes G_{i,n}\rangle A_i \otimes G_i \end{aligned}$$

(for some sequence $(\lambda_{i,n})$ with $\sum_n |\lambda_{i,n}|^2 = 1$)

$$\begin{aligned} &= \sum_i \Big\langle f, B_i \otimes \Big(\sum_n \lambda_{i,n} G_{i,n}\Big)\Big\rangle A_i \otimes G_i \\ &= \sum_i \langle f, B_i \otimes \Gamma_i\rangle A_i \otimes G_i \end{aligned}$$

where the $\Gamma_i = \sum_n \lambda_{i,n} G_{i,n}$ are independent normal Gaussian variables. By the preceding, we deduce:

$$\|Rf\| \leq \|f\|_X \|B \otimes G\|_{X'}$$

R is the desired projection. □

Proof of Corollary 2. If X contains $\ell_1(\ell_2)$ as a complemented subspace, let A_i, B be given by Proposition 1, U_i be the support of A_i, and $B_i = \mathbf{1}_{U_i} B$. Then $B \in X'(\Omega)$ and $\|A_i\|_X \leq (E|G|)^{-1} \|A_i \otimes G\| = C$, hence $\|B_i\| \geq \frac{1}{C}\langle B_i, A_i\rangle \geq \frac{1}{C}$. The order ideal generated by B in X' is thus not order continuous, while the closure $S_{X'}$, of the space of simple integrable functions in X' is order continuous (since $X' \neq L_\infty$). Hence $B \notin S_{X'}$.

Conversely if $B \geq 0$ verifies: $B \otimes G \in X', B \notin S_{X'}$, there exists a disjoint sequence (B_i) in X'_+, which is bounded from below (say $\|B_i\| \geq 1$), and such that $B = \bigvee_i B_i$. Let $A_i \in X_+$ verify $1 \leq \|A_i\| \leq (1+\varepsilon)\langle A_i, B_i\rangle$, with support included in that of B_i. Since X has finite upper Boyd index, we have $\|f \otimes G\|_X \leq C\|f\|_X$ for every $f \in X$; in particular:

$$\|A_i \otimes G\| \leq C' \langle A_i, B_i \rangle = C' \langle A_i, B \rangle \,.$$

Since $\|A_i \otimes G\| \geq (E|G|)\|A_i\| = E|G|$, we see that up to a constant factor the conditions i) and ii) of Proposition 1 are fulfilled. □

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