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Measures of non-compactness in Orlicz modular spaces

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Abstract

In this paper we show that the ball measure of non-compactness of a norm bounded subset of an Orlicz modular space L^ψ is equal to the limit of its n-widths. We also obtain several inequalities between the measures of non-compactness and the limit of the n-widths for modular bounded subsets of L^ψ which do not have Δ_2-condition. Minimum conditions on ψ to have such results are specified and an example of such a function ψ is provided.

Introduction and Preliminaries

We start by recalling the usual definitions of Orlicz and modular spaces.

Definition A. Let X be a vector space. A functional $\rho: X \to [0,\infty]$ is called a modular; if for $f, g \in X$ the following is true:
(i) $\rho(f) = 0$ if and only if $f = 0$
(ii) $\rho(af) = \rho(f)$ if $|a| = 1$

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(iii) $\rho(af+bg) \le \rho(f)+\rho(g)$ for $a+b=1$ and $a,b \ge 0$.

If (iii) is replaced by

$$\text{(iii)}' \qquad \rho(af+bg) \le a^s\rho(f)+b^s\rho(g) \quad \text{for} \quad f,g \in X, 0 \le a,b, a^s+b^s=1$$

with $0 < s \le 1$ fixed, ρ is called *s-convex modular* (convex if $s=1$).

For example, every monotone F-norm ρ is a modular, every norm is a convex modular, and $\rho(x) = \sqrt{|x|}$ for $x \in R$ is a $\frac{1}{2}$-convex modular. To a modular we associate a modular space. Let X be a real vector space and ρ be a modular on X. We define the *modular space* X_ρ by

$$X_\rho = \left\{ f \in X \colon \lim_{\alpha \to 0} \rho(\alpha f) = 0 \right\} .$$

Obviously X_ρ is a vector subspace of X.

DEFINITION B. An *Orlicz function* $\psi \colon R \to R^+$ is a continuous nondecreasing function with $\psi(0)=0, \psi(t) \to \infty$ as $t \to \infty$ and $\psi(-x)=\psi(x)$, i.e. ψ behaves similarly to power function $\psi(t)=t^p$.

Let ψ be an Orlicz function and let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Then for every measurable real valued function f on X, we define the *Orlicz modular* by

$$\rho(f) = \int_X \psi(|f(x)|) d\mu.$$

ρ is convex if ψ is convex.

The *Orlicz space* is the space of all (equivalence classes of) measurable real valued functions f on X so that $\lim_{\lambda \to 0} \rho(\lambda f) = 0$. Obviously, an Orlicz space L^ψ is a generalization of the classical L^p-spaces. Although ψ behaves similarly to the power function $\psi(t)=t^p$, the convexity of the Orlicz function can be omitted; two examples of such functions are:

$$\psi(t) = e^t - 1 \quad \text{and} \quad \psi(t) = \ln(1+t).$$

The vector space L^ψ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf \left\{ \lambda > 0 \colon \rho\Big(\frac{f}{\lambda}\Big) \le \lambda \right\} .$$

If ρ is convex, then

$$\|f\|_\rho = \inf \left\{ \lambda > 0 \colon \rho\Big(\frac{f}{\lambda}\Big) \le 1 \right\}$$

will define a norm on L^ψ, in either case the norm is called a Luxemburg norm.

With this norm $(L^{\psi}, \| \ \|_{\rho})$ is a Banach space in case ρ is convex [9]. One has two structures on L^{ψ}; one is that of Banach space induced by the norm $\| \ \|_{\rho}$, and the other is the structure of a modular space induced by the Orlicz modular ρ.

Although the study of structure of L^{ψ} spaces is interesting in itself, many applications to differential and integral equations with kernels of nonpower types are the basic reason for the development of Orlicz spaces. Also, it should be noted that the most commonly used rearrangement invariant functions spaces, beside L^{p}-space are the Orlicz function spaces. (See e.g. J. Lindenstrauss and L. Tzafriri [11].)

Let ρ be an Orlicz modular on L^{ψ}, a sequence (f_k) of elements L^{ψ} is called modular convergent (or ρ-convergent) to $f \in L^{\psi}$ if

$$\rho(f_k - f) \to 0 \ \text{ as } \ h \to \infty.$$

Norm-convergence in L^{ψ} implies ρ-convergence, but ρ-convergence does not imply norm-convergence. In case the measure space is σ-finite, the following theorem gives the equivalence. We say ψ satisfies Δ_2-condition if

(i) $\limsup_{u\to\infty} \psi(2u)/\psi(u) < \infty$ and $\limsup_{u\to 0} \psi(2u)/\psi(u) < \infty$ in case the measure μ is atomless and infinite.

(ii) $\limsup_{u\to\infty} \psi(2u)/\psi(u) < \infty$ in case the measure μ is atomless and finite.

(iii) $\limsup_{u\to 0} \psi(2u)/\psi(u) < \infty$ in case the measure μ in case the measure μ is purely atomic.

All of them imply that there exists $K, c > 0$ such that for all $f \in L^{\psi}$ we have $\rho(2f) \le K\rho(f) + c$.

Theorem ([9])

Norm convergence and ρ-convergence are equivalent in L^{ψ} if and only if ψ satisfies the Δ_2-condition.

It should be remarked that Orlicz spaces L^{ψ} with the Δ_2-condition are not far from L^{p}-spaces in the sense that there are analogous theorems about separability. However, in the spaces which lack a Δ_2-condition, the fact that ρ-convergence is not reducible to norm convergence makes modular convergence interesting. For further theory of Orlicz modular spaces, we refer to [8], [9], [12] and [14].

As for the measures of non-compactness [2], they are of interest in many spaces. They are used in fixed point theory (see Darbo [3], Sadovskii [17], Reich [15], [16]), and also in the study of the essential spectrum (see Nussbaum [13], Lebow-Schechter [10], Aksoy [1]). Measures of non-compactness of embeddings in the context of

Sobolev spaces are given by D.E. Edmunds and W.D. Evans [6]. The (ball) measure of non-compactness $\alpha(T)$ of T is defined to be:

$$\alpha(T) = \inf \Big\{ \varepsilon > 0 \colon T(B_x) \ \text{ can be covered by finitely many balls of radius } \varepsilon \Big\}.$$

The estimates of α for embedding maps can be found in [4]. Two types of measures of non-compactness, namely entropy and approximation numbers of embeddings in Orlicz spaces, are also studied in [5]. In [7], one can find fixed point theorems in Orlicz modular spaces.

The purpose of this paper is to study measures of non-compactness in the context of Orlicz spaces, where the Orlicz space under consideration is either equipped with the norm or just an Orlicz modular. We will investigate equality of certain measures of non-compactness in L^{ψ} even if ψ does not satisfy the Δ_2-condition. From this point on, ψ is assumed to be convex.

DEFINITION 1. Let $\xi > 0$ be a fixed real number and let $f \in L^{\psi}$. We define $\|f\|$, the norm of f, as:

$$\|f\| = \frac{\xi}{s(f)} \quad \text{where} \quad s(f) = \sup\{s \colon \rho(sf) \leq \xi\} > 0\,.$$

Proposition 1

$\|f\| = \frac{\xi}{s(f)}$ satisfies the properties of a norm.

Proof. Suppose $f = 0$, then using the fact that $\psi(0) = 0$, we obtain $\rho(sf) = 0 \leq \xi$ which implies that $\|f\| = 0$. On the other hand, if $\|f\| = 0$, from the definition there is $s_n \to \infty$ such that $\rho(s_n f) \leq \xi$ or equivalently $\xi \geq \int \psi(|s_n f(x)|)d\mu$. Since ψ is lower semi-continuous, we have

$$\begin{aligned}
\xi &\geq \underline{\lim} \int \psi(|s_n f(x)|)d\mu \geq \int \underline{\lim} \psi(|s_n f(x)|)d\mu \\
&\geq \int \psi(\underline{\lim}|s_n f(x)|)d\mu = \int_{\{x : f(x)=0\}} + \int_{\{x : f(x)\neq 0\}} \\
&= \int_{\{x : f(x)=0\}} \psi(0)d\mu + \int_{\{x : f(x)\neq 0\}} \psi(\infty)d\mu \\
&= \mu(\{x \colon f(x) = 0\}) \cdot \psi(0) + \mu(\{x \colon f(x) \neq 0\}) \cdot \psi(\infty)
\end{aligned}$$

Again using the facts that $\psi(0) = 0$ and $\psi(\infty) = \infty$, we obtain:

$$\geq 0 + \infty \cdot \mu(\{x \colon f(x) \neq 0\})$$

which implies $\mu(\{x \colon f(x) \neq 0\}) = 0$ or $f = 0$ a.e. μ.

To show $\|\lambda f\| = |\lambda|\,\|f\|$, consider

$$s(\lambda f) = \sup\{s\colon \rho(s\lambda f) \le \xi\} = \sup\left\{\frac{s}{|\lambda|}\colon \rho\Big(\frac{s}{\lambda}\cdot \lambda f\Big) \le \xi\right\}$$
$$= \frac{1}{|\lambda|}\sup\{s\colon \rho(sf) \le \xi\} = \frac{1}{|\lambda|}s(f)$$

so

$$\|\lambda f\| = \frac{\xi}{s(\lambda f)} = \frac{|\lambda|\xi}{s(f)} = |\lambda|\,\|f\|\,.$$

To show the triangle inequality $\|f+g\| \le \|f\| + \|g\|$, let s_f and s_g denote the $s(f)$ and $s(g)$, respectively. Then

$$\frac{s_f\cdot s_g}{s_f+s_g}(f+g) = \frac{s_g}{s_f+s_g}(s_f\cdot f) + \frac{s_f}{s_f+s_g}(s_g\cdot g) \le \xi\,.$$

Since ψ is convex

$$\rho\left(\frac{s_f\cdot s_g}{s_f+s_g}(f+g)\right) = \frac{s_g}{s_f+s_g}\rho(s_f\cdot f) + \frac{s_f}{s_f+s_g}\rho(s_g\cdot g) \le \xi\,.$$

Thus $s(f+g) \ge \frac{s_f\cdot s_g}{s_f+s_g}$ and hence $s(f+g) \ge \frac{s(f)\cdot s(g)}{s(f)+s(g)}$. Now

$$\|f+g\| = \frac{\xi}{s(f+g)} \le \xi\cdot\frac{s(f)+s(g)}{s(f)\cdot s(g)} = \|f\| + \|g\|. \ \square$$

Remark 1. Proof of Proposition 1 can be shortened if one makes the following observations: Let $v > 0$ and consider the new modular $\rho_v = v\rho$, then

$$\|f\|_{\rho_v} = \inf\left\{t>0\colon \rho\Big(\frac{f}{t}\Big) \le \frac{1}{v}\right\}.$$

Let $i(f) = \inf\{t\colon \rho(\frac{f}{t}) \le \xi\}$, then $i(f) = \|f\|_{\frac{1}{\xi}\rho}$ and since $s(f) = \frac{1}{i(f)}$, $\frac{\xi}{s(f)} = \xi\|f\|_{\frac{1}{\xi}\rho}$ holds and clearly defines a norm.

For any modular, it is known that $\rho(f) \le 1$ if and only if $\|f\|_\rho \le 1$. Therefore, using the above Remark 1, we can conclude that:

$$\|f\|_{\frac{1}{\xi}\rho} \le 1 \ \text{ iff } \ \frac{1}{\xi}\rho(f) \le 1 \ \text{ iff } \ \rho(f) \le \xi.$$

Therefore, $\xi\|f\|_{\frac{1}{\xi}\rho} \le \xi$ iff $\rho(f) \le \xi$ and hence in our notation:

$$\|f\| \le \xi \ \text{ iff } \ \rho(f) \le \xi.$$

Notations

$B_{\|\ \|}(r) = \{f\colon \|f\| \le r\}, B_\rho(r) = \{f\colon \rho(f) \le r\}$ denotes the norm-ball and ρ-ball centered at 0 and radius r, respectively, where $\|\cdot\|$ is the norm defined in Proposition 1 and ρ is the Orlicz modular on L^ψ.

Furthermore, we will use $r^\pm$ for any number of the form $r \pm \varepsilon$ for any $\varepsilon > 0$ small enough, if there is no ambiguity.

Proposition 2

$f \in B_{\|\cdot\|}(r)$ if and only if $\rho(\frac{\xi f}{r}) \leq \xi$.

Proof. If $\rho(\frac{\xi f}{r}) \leq \xi$, then $s(f) \geq \frac{\xi}{r}$, thus $\|f\| = \frac{\xi}{s(f)} \leq r$. Conversely let $f \in B_{\|\cdot\|}(r)$, then $\frac{\xi}{r} \leq s(f)$. First suppose $\frac{\xi}{r} = s(f)$, then since ψ is increasing there is $s_f = s(f)^-$ such that $\rho(s_f^- f) \leq \xi$. Now using the Fatou property and the fact that ψ is lower semi-continuous, we have

$$\underline{\lim}\rho(s_f^- f) \geq \int \underline{\lim}\psi(s_f^- f) \geq \int \psi(\underline{\lim} s_f^- f) = \rho(s(f)\cdot f)\,.$$

There, $\rho(\frac{\xi}{r} f) \leq \xi$.

Secondly, suppose $\frac{\xi}{r} < s(f)$, then there is $s_f = s(f)^-$ such that $\rho(s_f^- f) \leq \xi$, but $\frac{\xi}{r} < s_f^- \leq s(f)$. Again using the fact that ψ is increasing we obtain

$$\rho\left(\frac{\xi}{r} f\right) \leq \rho(s_f^- f) \leq \xi. \ \square$$

The following result uses Proposition 2 to illustrate the relationship between ρ-balls and norm-balls of L^ψ.

Proposition 3

(i) When $r \leq \xi$, we have $B_{\|\cdot\|}(r) \subseteq B_\rho(r)$.
(ii) When $\xi \leq r$, we have $B_\rho(r) \subseteq B_{\|\cdot\|}(r)$.

Proof. (i) If $f \in B_{\|\cdot\|}(r)$, then by Proposition 2, $\rho(\frac{\xi f}{r}) \leq \xi$. Since $\psi(0) = 0$ and ψ is convex, we have

$$\begin{aligned}\rho(f) = \rho\left(\frac{r}{\xi}\cdot\frac{\xi f}{r}\right) &\leq \frac{r}{\xi}\rho\left(\frac{\xi f}{r}\right) + \left(1 - \frac{r}{\xi}\right)\rho(0)\\ &\leq \frac{r}{\xi}\cdot\xi = r\end{aligned}$$

which shows that $f \in B_\rho(r)$.
(ii) If $f \in B_\rho(r)$, then $\rho(f) \leq r$. Again using Proposition 2 together with the fact $\psi(0) = 0$ and ψ convex will yield:

$$\rho\left(\frac{\xi}{r} f\right) \leq \frac{\xi}{r}\rho(f) + \left(1 - \frac{\xi}{r}\right)\rho(0) \leq \xi$$

thus $f \in B_{\|\cdot\|}(r)$. $\square$

Remark 2. Notice that, although in the above proof of Proposition 3 we are using the facts $\psi(0) = 0$ and ψ is convex, in fact what we need is ψ satisfying

$$\psi(ax) \leq a\psi(x) \ \text{ for } \ 0 \leq a \leq 1\,.$$

Corollary

Let D be a subset of L^ψ. Then D is ψ-bounded implies D is $\|\cdot\|$-bounded.

Proof. Since ψ is increasing if $r_1 \leq r_2$, then $D \subset B_\rho(r_1)$ implies $D \subset B_\rho(r_2)$. Now if $D \subseteq B_\rho(r)$ by (ii) of the previous proposition, we have

$$D \subset B_\rho\big(\max(r,\xi)\big) \subseteq B_{\|\cdot\|}\big(\max(r,\xi)\big)\,. \ \square$$

DEFINITION 2. Let D be a norm-bounded subset of L^ψ. The *norm n-th width of D* in the sense of Kolmogorov is denoted by $d^n_{\|\,\cdot\,\|}$ and defined as

$$d^n_{\|\cdot\|}(D) = \inf\{r > 0\colon D \subseteq B_{\|\cdot\|}(r) + A_n \quad \text{where } A_n \text{ is a vector space with dim of } A_n \leq n\}$$

and *norm-ball measure of non-compactness* $\alpha_{\|\,\cdot\,\|}(D)$ is defined as

$$\alpha_{\|\cdot\|}(D) = \inf\left\{ r > 0\colon D \subseteq \bigcup_{i=1}^{k} B_{\|\cdot\|}(x_i; r) \right\}.$$

Here k is arbitrary but finite; notice that

$$\bigcup_{i=1}^{k} b_{\|\cdot\|}(x_i; r) = B_{\|\cdot\|}(r) + \bigcup_{i=1}^{k} \{x_i\}\,.$$

Theorem 1

Let D be a $\|\cdot\|$-bounded subset of L^ψ. Then

$$\alpha_{\|\cdot\|}(D) = \lim_n d^n_{\|\cdot\|}(D).$$

Proof. We obviously have $\alpha_{\|\cdot\|}(D) \geq \lim_n d^n_{\|\cdot\|}(D)$. To show the reverse inequality, suppose we choose an admissible r and A_n such that $D \subset B_{\|\cdot\|}(r) + A_n$, then we can write $D \subseteq D_1 + D_2$ where $D_1 \subset B_{\|\cdot\|}(r)$ and $D_2 \subset A_n$. Observe that D_2 is $\|\cdot\|$-bounded, because for every $f \in D$ one has $f = f_1 + f_2$ where $f_2 \in D_2$ and

$$\|f_2\| = \|f - f_1\| \leq \|f\| + \|f_1\|\,.$$

Now if we use the seminorm property in A_n which is finite, we obtain: for every $\varepsilon > 0$, there exists a finite covering for D_2 by balls of radius ε, i.e.

$$D_2 \subseteq \bigcup_{\text{finite}} B_{\|\cdot\|}(x_i; \varepsilon)\,.$$

So $B_{\|\cdot\|}(r) + B_{\|\cdot\|}(x_i;\varepsilon) \subseteq B_{\|\cdot\|}(x_i; r+\varepsilon)$ which implies $\alpha_{\|\cdot\|}(D) \leq r + \varepsilon$. $\square$

Remark 3. In the above proof to show

$$B_{\|\cdot\|}(r) + B_{\|\cdot\|}(x_i; \varepsilon) \subseteq B_{\|\cdot\|}(x_i; r + \varepsilon)$$

we used the triangle property of our norm. But all we need is

$$\|f + g\| \leq \|f\| + C\|g\| \quad \text{for fixed} \quad C > 0.$$

This inequality holds if the Orlicz function ψ satisfies the condition:

$$\psi(ax + by) \leq a\psi(x) + bC\psi(y)$$

with $a + bC = 1,\ a, b \geq 0\ \ C > 0$ fixed.

Notice that by replacing norm-balls by ρ-balls in Definition 2, one can similarly define modular n-width of $D, d_\rho^n(D)$ and modular-ball measure of non-compactness $\alpha_\rho(D)$ for a ρ-bounded subset D as follows:

$$\begin{aligned} d_\rho^n(D) = \inf\{r > 0\colon D \subseteq B_\rho(r) + A_n \ \text{ where } \ A_n \\ \text{is a vector space with } \dim \text{ of } \ A_n \leq n\} \end{aligned}$$

$$\alpha_\rho(D) = \inf\left\{ r > 0\colon D \subset \bigcup_{i=1}^{k} B_\rho(x_i; r) \right\} .$$

Obviously we have $\lim_n d_\rho^n(D) \leq \alpha_\rho(D)$. Therefore, one can ask whether Theorem 1 type of equality holds with respect to modular, too. Following Theorem 2 gives an affirmative answer to this question in case ψ satisfies Δ_2-condition. Later by Theorem 3 we give partial answers to the same questions in case ψ does not satisfy Δ_2-condition.

Theorem 2

Suppose that ψ satisfies the Δ_2-condition, then for a ρ-bounded subset D of L^ψ we have:

$$\lim_{n\to\infty} d_\rho^n(D) = \alpha_\rho(D).$$

Proof. Since D is ρ-bounded, there exists M with $\rho(d) \leq M$ for all $d \in D$. On the other hand, since ψ satisfies Δ_2-condition, there are $K, C > 0$ such that $\rho(2f) \leq K_\rho(f) + C$. Now choose an admissible r and A_n such that $D \subset B_\rho(r) + A_n$. Next we claim that there is K_1 such that $\rho(x) \leq K_1$ for each $x \in A_n \cap D$, because if $x \in A_n \cap D$, then $x = x_1 - x_2$ with $x_1 \in D$ and $x_2 \in D \cap B_\rho(r)$ and hence

$$\rho(x) = \rho\left(\frac{2x_1 - 2x_2}{2}\right) \leq \frac{1}{2}\rho(2x_1) + \frac{1}{2}\rho(2x_2)$$
$$\frac{1}{2}K\rho(x_1) + \frac{1}{2}K\rho(x_2) + 2C \leq \frac{1}{2}KM + \frac{1}{2}KM + 2C.$$

Now choose K_1 so that $\frac{k_1}{\xi} \geq 1$, then using convexity we have $\rho(\frac{\xi}{K_1}x) \leq \frac{\xi}{K_1}\rho(x) \leq \xi$ which implies that $\|x\| \leq K_1$. Since $A_n \cap D$ is norm-bounded and finite dimensional, for any $\varepsilon > 0$, there exists $\{y_i\}_{i=1}^n$ such that

$$A_n \cap D \subset \bigcup_{\text{finite}} B_{\|\ \|}(y_i;\varepsilon), \quad 0 < \varepsilon < 1.$$

Using Proposition 3 (i), (take $\xi = 1$) we obtain

$$D \subset B_\rho(r) + \bigcup_{\text{finite}} B_\rho(y_i;\varepsilon) \subseteq \bigcup_{\text{finite}} B(y_i; r+\varepsilon)$$

which implies $\alpha_\rho(D) \leq \lim\limits_{n\to\infty} d_\rho^n(D)$. □

Lemma 1

Suppose D is a ρ-bounded subset of L^ψ, then we have one of the following:

(i) $\alpha_\rho(D) \geq \alpha_{\|\cdot\|}(D) \geq \xi$
(ii) $\alpha_\rho(D) \leq \alpha_{\|\cdot\|}(D) < \xi$
(iii) $\alpha_{\|\cdot\|}(D) = \xi$.

Proof. **Case 1.** Suppose $\alpha_\rho(D) \geq \xi$. Then using Proposition 3(ii), we have $r^+ \geq \alpha - \rho(D) \geq \xi$ such that

$$D \subset \bigcup_{\text{finite}} B_\rho(x_i;r^+) \subseteq \bigcup_{\text{finite}} B_{\|\cdot\|}(x_i;r^+)$$

which implies $\alpha_{\|\cdot\|}(D) \leq \alpha_\rho(D)$.

Case 2. Suppose $\alpha_{\|\cdot\|}(D) < \xi$. Then there is r^+ such that $\alpha_{\|\cdot\|}(D) \leq r^+ < \xi$. But by Proposition 3 (i) we have

$$D \subset \bigcup_{\text{finite}} B_{\|\cdot\|}(x_i, r^+) \subseteq \bigcup_{\text{finite}} B_\rho(x_i, r^+)$$

and therefore $\alpha_\rho(D) \leq \alpha_{\|\cdot\|}(D) < \xi$. □

Lemma 2

Suppose D is a ρ-bounded subset of L^ψ, then we have only one of the following:
(i) $\delta_\rho(D) \geq \delta_{\|\cdot\|}(D) \geq \xi$
(ii) $\delta_\rho(D) \leq \delta_{\|\cdot\|}(D) < \xi$
(iii) $\delta_{\|\cdot\|}(D) = \xi$
where $\delta_{\|\cdot\|}(D) = \lim_n d^n_{\|\cdot\|}(D)$ *and* $\delta_\rho(D) = \lim_n d^n_\rho(D)$.

Proof. **Case 1.** Suppose $\delta_\rho(D) \geq \xi$, then there is $r^+ \geq \delta_\rho(D) \geq \xi$ and A_n such that

$$D \subset B_\rho(r^+) + A_n \subseteq B_{\|\cdot\|}(r^+) + A_n \,.$$

In the last inclusion we used Proposition 3(ii) again. Thus we have

$$\begin{cases} \delta_{\|\cdot\|}(D) \leq \delta_\rho(D) \\ \delta_\rho(D) \geq \xi \end{cases}$$

Case 2 of this Lemma is similar to Case 2 of Lemma 1. □

Combining the results in Lemma 1 and Lemma 2 and Theorem 1 we obtain:

Theorem 3

Let D be a ρ-bounded subset of L^ψ, then we have one of the following:
(i) $\alpha_\rho(D) \geq \delta_\rho(D) \geq \alpha_{\|\cdot\|}(D) = \delta_{\|\cdot\|}(D) \geq \xi$
(ii) $\delta_\rho(D) \leq \alpha_\rho(D) \leq \alpha_{\|\cdot\|}(D) = \delta_{\|\cdot\|}(D) < \xi$
(iii) $\delta_{\|\cdot\|}(D) = \alpha_{\|\cdot\|}(D) = \xi$.

Remark 4. Combining Remark 2 after Proposition 3 and Remark 3 after Theorem 1 we deduce that the conditions we need to put on ψ in order for the above theorem to hold are:
1. $\psi(0) = 0$
2. $\psi(ax + by) \leq a\psi(x) + b\psi(y)$ with $a + bC = 1$ $a, b \geq 0, C > 0$ fixed.

These two conditions together imply that $\psi(ax) \leq a\psi(x)$, which implies that ψ is increasing. Also condition 2 above implies ψ is lower semicontinuous.

It is clear that condition 2 is satisfied by convex functions but does not imply convexity for ψ. Therefore Theorem 1 and 3 are valid for larger classes of functions than convex functions. For example, the function

$$\psi(x) = \min\left(x, \max\left(\frac{1}{2}x, \frac{3}{2}x - \frac{1}{2}\right)\right)$$

satisfies the conditions given in Remark 4 for $C = 2$.

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