Collect. Math. **48**, 3 (1997), 347–354

On the Gelfand-Phillips property in Banach spaces with PRI

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Received May 14, 1996

Abstract

It is proved that every Banach space belonging to a certain class called the class $\mathcal{P}$ possesses the Gelfand-Phillips property. Consequently, so does every weakly countably determined Banach space, every Banach space with an M-basis whose dual unit ball is weak* angelic and $C(K)$ spaces for Valdivia compact K.

1. Introduction and Preliminaries

A subset K of Banach space E is said to be limited if, for every weak*-null sequence (f_n) in the dual space E^*, we have $\lim_{n\to\infty} \sup_{x\in K} |f_n(x)| = 0$, i.e., (f_n) converges uniformly on K. Every limited subset is bounded and every relatively norm compact subset is limited. A Banach space E is said to have the Gelfand-Phillips property if every limited subset of E is relatively compact. Every separable Banach space, in general, all Banach spaces with weak* sequentially compact dual unit balls have the

* The second author is supported by the CSIR (India).

Gelfand-Phillips property [3, p. 238]. On the other hand the set $\{e_n : 1 \leq n < \infty\}$ of unit coordinate vectors is a limited set in l_∞ which is not relatively compact. Some other Banach spaces with the Gelfand-Phillips property are those with Schur property and the $C(K)$ spaces, where K is both a compact and sequentially compact Hausdorff space.

Many new results have poured in since mid eighties in a bid to find new classes of Banach spaces with the Gelfand-Phillips property and to construct new Banach spaces with this property from those already known to have it. The inheritance of the property in injective and projective tensor products, in the space of bounded linear operators, and in Banach spaces with Schauder decompositions from their constituent spaces was studied in [4, 5, 6]. Furthermore, Gelfand-Phillips property in $C(K)$ spaces, in vector valued Köthe function spaces and in interpolation spaces, was studied in [5, 6, 11, 12]. In the present note we show that every Banach space belonging to the class $\mathcal{P}$, a special class of Banach spaces with PRI to be defined later, has the Gelfand-Phillips property. As a consequence, it is proved that several large classes of Banach spaces; for instance, all weakly countably determined Banach spaces, all Banach spaces with extended M-bases such that their dual unit balls are weak* angelic, all Banach spaces such that the dual spaces with the weak*-topology belong to a certain class $\sum$ and $C(K)$ spaces for Valdivia compact K, have the Gelfand-Phillips property.

Let E denote a Banach space, E^* its dual and B_E and B_{E^*} the closed unit balls of E and E^*, respectively. By dens E we shall denote the cardinality of the smallest dense subset in E. Let ω denote the first infinite ordinal with cardinality $\mathcal{N}_0$ and ω_1 the first ordinal with the cardinality of the continuum. Other symbols used for ordinals are α, β, λ and μ. We shall call an ordinal number α to be of type C if α is a limit ordinal such that there is a sequence of ordinals (α_n) with $\alpha_n < \alpha\,(n = 1, 2, \dots)$ and (α_n) is cofinal in α, i.e., for each ordinal $\beta < \alpha$, there is an integer $k > 0$ such that $\beta \leq \alpha_k$. Note that besides the denumerable ordinals many non-denumerable ordinal numbers are also of type C, while ω_1 is not of type C.

Let μ be the first ordinal of cardinality dens E. Then, a long sequence $\{p_\alpha\}_{\omega \leq \alpha \leq \mu}$ of linear projections on E is said to be a projectional resolution of the identity (in short, PRI) of E if it satisfies the following

$$\|p_\alpha\| = 1 \qquad (\omega \leq \alpha \leq \mu)\,, \tag{1.1}$$

$$p_\alpha p_\beta = p_\beta p_\alpha = p_\beta \qquad (\omega \leq \beta \leq \alpha \leq \mu)\,, \tag{1.2}$$

$$\text{dens } p_\alpha(E) \leq \text{card } \alpha \qquad (\omega \leq \alpha \leq \mu)\,, \tag{1.3}$$

$$p_\alpha(E) = \overline{\bigcup \{p_{\beta+1}(E) : \omega \leq \beta < \alpha\}} \qquad (\omega \leq \alpha \leq \mu)\,, \tag{1.4}$$

$$p_\mu \text{ is the identity operator on } E\,. \tag{1.5}$$

If a Banach space E has a $PRI(p_\alpha)_{\omega\le\alpha\le\mu}$, then for each $x \in E$ the map $\alpha \to p_\alpha(x)$ from $[\omega,\mu]$ into E is continuous and if $\beta \le \mu$ is a limit ordinal then $p_\beta(x) = \lim_{\alpha<\beta} p_\alpha(x)$ for each $x \in E$. Amir and Lindenstrauss [1] constructed a PRI in every weakly compactly generated Banach space and also showed that its dual unit ball is weak* sequentially compact, whence it has the Gelfand-Phillips property. Some classes of Banach spaces which are known to have PRI can be found in [1, 9, 13, 14, 15].

2. Main result

We start with the following lemma about the projections of limited sets in Banach spaces.

Lemma 2.1

The linear projection of a limited set in a Banach space into a complemented subspace is a limited set therein.

Proof. Let p be a bounded linear projection on a Banach space E and $(g_n) \subset p^*(E^*)$ (note that $(p(E))^*$ is isomorphic to $p^*(E^*)$) be a $\sigma(p^*(E^*)),\ p(E))$-null sequence. Then, $\lim_{n\to\infty} g_n(p(x)) = 0\,(x \in E)$. Now, there is a sequence $(f_n) \subset E^*$ such that $g_n = p^*(f_n)(n = 1, 2, \ldots)$. Then

$$\lim_{n\to\infty} \big(p^* \cdot f_n\big)(x) = \lim_{n\to\infty} g_n\big(p(x)\big) = 0 \qquad (x \in E)\,.$$

Thus, $(p^* \cdot f_n)$ is a $\sigma(E^*, E)$-null sequence. Since K is limited, given $\varepsilon > 0$, there is an integer $m_0 > 0$ such that

$$\sup_{x\in K} \big|g_n(p(x))\big| = \sup_{x\in K} \big|(p^* \cdot f_n)(x)\big| < \varepsilon \qquad (n \ge m_0)\,.$$

It follows that (g_n) is $\sigma(p^*(E^*),\ p(E))$-null uniformly on $p(K)$. $\square$

Before stating our main result we shall define a particular class of Banach spaces.

Definition 2.2. Let $\mathcal{P}$ be the class of Banach spaces such that every E in $\mathcal{P}$ admits a PRI $(p_\alpha)_{\omega\le\alpha\le\mu}$ such that $p_\alpha(E)$ (respectively, $(p_{\alpha+1} - p_\alpha)(E)$) belongs to $\mathcal{P}$ for every $\alpha \in [\omega, \mu)$.

This class of Banach spaces was defined in [2, p. 286] as the class $\mathcal{P}$ where it has also been shown that these Banach spaces admit equivalent locally uniformly rotund norms. Note that Vašák [15] has shown that every weakly countably determined Banach space has a PRI and the property of being weakly countably determined is hereditary. Thus all weakly countably determined Banach spaces belong to the class $\mathcal{P}$.

Now we prove the main result of this note.

Theorem 2.3

Every Banach space belonging to the class $\mathcal{P}$ has the Gelfand-Phillips property.

Proof. Let E be a Banach space belonging to the class $\mathcal{P}$ and K be a limited set in E. We shall proceed by transfinite induction on dens E. If dens $E = \mathcal{N}_0$ then E is separable and so, it has the Gelfand-Phillips property, whence K is relatively compact.

Now let dens $E > \mathcal{N}_0$. Let μ be the first ordinal of cardinality dens E. We set up an induction hypothesis that every Banach space F belonging to the class $\mathcal{P}$ and with dens $F < \operatorname{card}\mu$ has the Gelfand-Phillips property.

Since E belongs to the class $\mathcal{P}$, there is a long sequence $\{p_\alpha\}_{\omega\leq\alpha\leq\mu}$ of linear projections on E satisfying conditions (1.1) to (1.5) such that $p_\alpha(E)$ (respectively, $(p_{\alpha+1} - p_\alpha)(E)$) is also in the class $\mathcal{P}$ for each $\omega \leq \alpha < \mu$. To begin with we note that by the induction hypothesis and by Lemma 2.1, $p_\alpha(K)$ and $(p_{\alpha+1} - p_\alpha)(K)$ are relatively compact for each $\omega \leq \alpha < \mu$.

Let (x_n) be a sequence in K. If μ is a limit ordinal then we shall discuss the construction of a cauchy subsequence of (x_n) separately in the following two cases.

Case 1. Let μ be not of type C. Since $\lim_{\alpha<\mu} p_\alpha(x_n) = x_n (n = 1, 2, \ldots)$, there is a sequence (α_n) of ordinals with $\omega < \alpha_n < \mu$ and $\alpha_{n-1} < \alpha_n (n = 1, 2, \ldots)$ such that

$$\big\|p_\alpha(x_n) - x_n\big\| < \frac{1}{n} \qquad \big(\alpha \geq \alpha_n, \quad n = 1, 2, \ldots\big)\,. \tag{2.2.1}$$

Since μ is not of type C, (α_n) can not be cofinal in μ. Thus, there is an ordinal $\alpha_0 < \mu$ such that $\alpha_n \leq \alpha_0 (n = 1, 2, \ldots)$, whence by (2.2.1), we have

$$p_\alpha(x_n) = x_n \qquad \big(\alpha \geq \alpha_0, \quad n = 1, 2, \ldots\big)\,. \tag{2.2.2}$$

Now $p_{\alpha_0}(K)$ being relatively compact in $p_{\alpha_0}(E)$ and since $p_{\alpha_0}(x_n) \in p_{\alpha_0}(K)(n = 1, 2, \ldots)$, given an $\varepsilon > 0$, there exist a subsequence (x_{n_k}) of (x_n) and an integer $m_0 > 0$ such that

$$\big\|x_{n_k} - x_{n_j}\big\| = \big\|p_{\alpha_0}(x_{n_k}) - p_{\alpha_0}(x_{n_j})\big\| < \varepsilon \qquad \big(k, j \geq m_0\big)\,.$$

Thus, (x_{n_k}) is a Cauchy subsequence of (x_n).

Case 2. Let μ be of type C. Choose a sequence (α_n) of ordinals with $\omega < \alpha_n < \mu$ and $\alpha_{n-1} < \alpha_n$ $(n = 1, 2, \ldots)$ such that

$$\|p_\alpha(x_n) - x_n\| < \frac{1}{n} \qquad \big(\alpha \geq \alpha_n, \quad n = 1, 2, \ldots\big).$$

If (α_n) is not a cofinal sequence in μ, then proceeding as in case 1 we can construct a Cauchy subsequence (x_{n_k}) of (x_n). Finally, let us assume that (α_n) is cofinal in μ. In this case

$$\lim_{n\to\infty} p_{\alpha_n}(x) = x \qquad (x \in E).$$

We claim that this convergence is uniform over K. For otherwise, there exists an $\varepsilon_0 > 0$, an integer $k_n \geq n$ and a $z_n \in K$ for each n, such that

$$\big\|p_{\alpha_{k_n}}(z_n) - z_n\big\| \geq \varepsilon_0 \qquad (n = 1, 2, \ldots). \tag{2.2.3}$$

By Hahn-Banach theorem there are $f_n \in E^*$ with $\|f_n\| = 1$ $(n = 1, 2, \ldots)$ such that

$$\big|f_n((p_{\alpha_{k_n}} - I)\,(z_n))\big| = \big\|(p_{\alpha_{k_n}} - I)(z_n)\big\| \qquad (n = 1, 2, \ldots),$$

whence by (2.2.3) it follows that

$$\big|((p_{\alpha_{k_n}} - I)^* \cdot f_n)(z_n)\big| \geq \varepsilon_0 \qquad (n = 1, 2, \ldots). \tag{2.2.4}$$

Since (α_{k_n}) is a subsequence of (α_n) and $(f_n) \subset B_{E^*}$, it is easy to see that $((p_{\alpha_{k_n}} - I)^* \cdot f_n)$ is weak*-null and hence converges uniformly on the limited set K. But (z_n) being a sequence in K we arrive at a contradiction to (2.2.4). Thus, given $\varepsilon > 0$ there is an integer $m_1 > 0$ such that

$$\big\|p_{\alpha_n}(x) - x\big\| < \frac{\varepsilon}{3} \qquad (n \geq m_1,\ x \in K). \tag{2.2.5}$$

Now by hypothesis $p_{\alpha_k}(K)$ is a relatively compact subset of $p_{\alpha_k}(E)$ for each k. Since $(p_{\alpha_1}(x_n))$ is a sequence in $p_{\alpha_1}(K)$ there is a subsequence $(x_{1,n})$ of (x_n) such that $p_{\alpha_1}(x_{1,n})$ is Cauchy. Again, $p_{\alpha_2}(x_{1,n})$ being a sequence in $p_{\alpha_2}(K)$ there is a subsequence $(x_{2,n})$ of $(x_{1,n})$ such that $p_{\alpha_2}(x_{2,n})$ is Cauchy. Continuing in this way, for each integer $k > 0$, the sequence $(x_{k-1,n})$ has a subsequence $(x_{k,n})$ and an integer $n_k > 0$ with $n_{k-1} < n_k$, $n_0 = 0$ and $x_{0,n} = x_n (n = 1, 2, \ldots)$ satisfying

$$\big\|p_{\alpha_k}(x_{k,m}) - p_{\alpha_k}(x_{k,n})\big\| < \frac{\varepsilon}{3} \qquad \big(m, n \geq n_k\,, \quad k = 1, 2, \ldots\big).$$

Since $(x_{j,n})$ is a subsequence of $(x_{k,n})$ for $j \geq k$, we have

$$\text{(2.2.6)} \qquad \left\| p_{\alpha_k}(x_{i,m}) - p_{\alpha_k}(x_{j,n}) \right\| < \frac{\varepsilon}{3} \qquad (i \geq j \geq k\,,\ m,n \geq n_k\,,\ k = 1,2,\ldots)\,.$$

Let us construct a subsequence (y_k) of (x_k) by setting $y_k = x_{k,n_k}(k = 1,2,\ldots)$. Thus, by (2.2.5) and (2.2.6), it follows that

$$\begin{aligned} \left\| y_m - y_j \right\| &\leq \left\| p_{\alpha_{m_1}}(y_m) - y_m \right\| + \left\| p_{\alpha_{m_1}}(y_j) - y_j \right\| \\ &\quad + \left\| p_{\alpha_{m_1}}(y_m) - p_{\alpha_{m_1}}(y_j) \right\| < \varepsilon \qquad (m \geq j \geq m_1)\,. \end{aligned}$$

Hence, (y_n) is a Cauchy subsequence of (x_n).

Thus, we find that every sequence in K has a Cauchy subsequence. Hence, K is relatively compact. This completes the proof. □

Now, we give a few notations in order to enable us to state the following corollary neatly. Given a set I, let us denote by $\sum^{(I)}$ the topological subspace of $\mathbb{R}^I$ formed by points $(x_i : i \in I)$ such that $\{i \in I : x_i \neq 0\}$ is countable. A topological space X is said to belong to the class $\sum$ if there is some set I together with a continuous one-to-one mapping from X into $\sum^{(I)}$. It has been shown by Valdivia [13] that every Banach space whose dual with the weak* topology belongs to the class $\sum$, has a PRI and the dual of every complemented subspace with the weak* topology also belongs to the class $\sum$. Thus Banach spaces whose duals with the weak* topology belong to the class $\sum$ belong to the class $\mathcal{P}$. It has also been shown in [13] that for a Banach space E with an extended M-basis, E^* with the weak* topology belongs to the class $\sum$ if B_{E^*} is weak* angelic. Further, it has been shown in [2, p. 286] that $C(K)$ spaces for Valdivia compact K also belong to the class $\mathcal{P}$.

The fruits of the above theorem can be seen in the form of the following corollary which adds some new classes of Banach spaces to the list of those which are already known to have the Gelfand-Phillips property.

Corollary 2.4

The following Banach spaces have the Gelfand-Phillips property

a) Weakly countably determined Banach spaces.

b) Banach spaces whose dual spaces with the weak*-topology belong to the class $\sum$.

c) Banach spaces with extended M-bases whose dual unit balls are weak*-angelic.

d) $C(K)$ spaces for Valdivia compact K.

However, it is not necessary for a Banach space to belong to the class $\mathcal{P}$, or for that matter to have even a PRI, in order to possess the Gelfand-Phillips property. Indeed, consider the separable Banach E given by Hagler [10] which does not contain any copy of l_1 such that the dual E^* is not separable. By an argument given in [9, Remark 4], assuming the continuum hypothesis, E^* does not have a PRI. But E^* being a Schur space has the Gelfand-Phillips property. On the other hand note that if Γ is a set with card $\Gamma >$ dens l_∞ then the Banach space $E = l_2(\Gamma) \oplus_2 l_\infty$ has a PRI; but E does not belong to the class $\mathcal{P}$ (see [2, p. 286]). The set $K = \{0 \oplus e_n : 1 \leq n < \infty\}$, where e_n is the n-th unit coordinate vector in l_∞, is a limited set in E which is not relatively compact. Thus, Theorem 2.3 can not be proved for all Banach spaces with the PRI.

Finally, the authors are thankful to the referee for his valuable suggestions towards the improvement of the paper.

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