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Remarks on independent sequences and dimension in topological linear spaces

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Abstract

We show that for a metrizable locally convex space X the following conditions are equivalent: (i) every linearly independent sequence in X has an ω-independent subsequence; (ii) X contains no subspace isomorphic to φ; (iii) X admits a continuous norm. We also show that a dual Banach space equipped with the weak* topology satisfies (i). Moreover, we are concerned with the algebraic dimension of closed convex subsets of F-spaces.

1. Introduction

The main body of the paper consists of Sections 4, 5 and 6, which are, as far as the proofs are concerned, mutually independent. Section 3 contains two lemmas, which are then applied in Section 4, while Section 2 is entirely devoted to notation and definitions.

The material of Sections 3–5 has origin in the theorem that a linearly independent sequence in a Banach space has an ω-independent subsequence, which is due essentially to Erdős and Straus [4]. This theorem, clearly, extends to topological linear spaces which admit a continuous norm. In the class of locally convex F-spaces a converse holds, as shown by Kadets [7]. We generalize this equivalence to the class of metrizable locally convex spaces (Theorem 4 of Section 4). In this connection, we establish in Sections 3 and 4 some properties of the space φ of scalar sequences with finite support equipped with the product topology. In particular, we prove some results on the containment of φ in topological linear spaces (Theorems 1 and 3 of Section 4), generalizing the corresponding results on the containment of ω in F-spaces due to Bessaga, Pełczyński and Rolewicz [1], [2].

Section 5 is concerned with a strengthening of the Erdős–Straus theorem to the effect that a linearly independent sequence in a dual Banach space has a subsequence which is ω-independent with respect to the weak* topology (Theorem 5).

The final Section 6 presents two different proofs of the theorem that the algebraic dimension of a closed convex subset of an F-space is either finite or at least $2^{\aleph_0}$ (Theorem 6). This result will be applied in a forthcoming paper [11].

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2. Notation and definitions

Throughout the paper X denotes a real topological linear space. (All the results remain valid in the case of complex scalars, with occasional obvious changes in the proofs.) The topological dual of X is denoted by X^*.

If X is assumed metrizable, $|\cdot|$ stands for an F-norm generating the topology of X such that for all $\lambda, \mu \in \mathbb{R}$ with $|\lambda| \le |\mu|$ and $x \in X$ we have $|\lambda x| \le |\mu x|$ (cf. [15], p. 4 and Theorems 1.1.1 and 1.2.2). If X is a normed space, $\|\cdot\|$ stands for the norm in X.

By ω we denote the linear space $\mathbb{R}^{\mathbb{N}}$ equipped with the product topology and by φ its subspace consisting of sequences with finite support. (In the older literature the former space was often denoted by (s).)

Let e_n stand for an element of φ with 1 as the nth term and 0 otherwise. We call (e_n) the *standard basis of* φ (and of other sequence spaces containing φ).

We set

$$\delta = \{(\lambda_n) \in \omega : \quad |\lambda_n|^{1/n} \to 0\} .$$

This is a subspace of ω, familiar from the theory of power series.

We say that a sequence (x_n) in X is *ω-independent* if for each $(\lambda_n) \in \omega$ with $\Sigma_{n=1}^{\infty} \lambda_n x_n = 0$ we have $(\lambda_n) = 0$. (In [16], Chapter I, § 6, and [9], [10], the names *ω-linearly independent* and *topologically linearly independent*, respectively, are used instead.) The notions of c_0-, l_1- and *δ-independence*, which we also use below, are defined analogously.

3. Two properties of φ

We shall establish two auxiliary results to be used in Section 4. Namely, Lemma 1 is an essential ingredient of the proof of Theorem 1(b) while Lemma 2 yields the implication (ii) $\Rightarrow$ (iii) of Theorem 4.

Lemma 1

Let σ be a locally convex topology on φ strictly stronger than the product topology. Then there exists a σ-continuous linear functional F on φ with $F(e_n) \neq 0$ for infinitely many $n \in \mathbb{N}$.

Proof. By assumption, there is a σ-continuous (homogeneous) seminorm p on σ with $p(e_n) > 0$ for infinitely many n. Consider the quotient mapping $Q : \varphi \to \varphi/p^{-1}(0)$. Clearly, $Q(e_n) \neq 0$ for infinitely many n. Therefore, we can find a continuous linear functional G on the normed space $\varphi/p^{-1}(0)$ with $G(Q(e_n)) \neq 0$ whenever $Q(e_n) \neq 0$ (see, e.g., [13], Theorem 1). Defining $F = G \circ Q$, we get a functional with the desired properties. $\square$

The local convexity assumption is essential for the validity of Lemma 1. Indeed, define an F-norm on φ by $|(\lambda_n)|_0 = \Sigma_{n=1}^{\infty} n^{-1}|\lambda_n|(1+|\lambda_n|)^{-1}$, and let σ be the linear topology on φ generated by $|\cdot|_0$ (cf. [2], Example; see Example 1 of Section 4 for a more sophisticated linear topology on φ with the same property).

Lemma 2 improves, in some sense, the result that the linearly independent sequence $(2^n, 3^n, ...)$, $n \in \mathbb{N}$, in ω has no ω-independent subsequence. The latter was announced in [17], p. 858, and proved in [7]. It is also a direct consequence of a result of Kalton (see [8], Corollary to Theorem 6, or [12], Theorem (19.6)). The proof below is a modification of Kadets' proof [7].

Lemma 2

The linearly independent sequence $x_n = (2^n, ..., (n+1)^n, 0, 0, ...)$, $n \in \mathbb{N}$, in φ has no δ-independent subsequence.

Proof. Denote by $\mathcal{H}$ the locally convex F-space of entire functions on $\mathbb{C}$ equipped with the topology of uniform convergence on compact sets (see, e.g., [12], § 3). Fix an infinite subset M of $\mathbb{N}$, and set

$$\mathcal{M} = \{f \in \mathcal{H} : \quad f^{(n)}(0) = 0 \quad \text{for} \quad n \in (\mathbb{N} \setminus M) \cup \{0\}\} .$$

Then $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ ([12], Proposition (3.13)). Define a sequence (F_k) in $\mathcal{M}^*$ as follows:

$$F_k(f) = f(k) \quad \text{for} \quad k = 1, 2,$$

$$F_k(f) = f(k) - \sum_{n=1}^{k-2} \frac{f^{(n)}(0)}{n!}\, k^n \quad \text{for} \quad k = 3, 4, \ldots$$

We also define a sequence (p_k) of continuous seminorms on $\mathcal{M}$ by

$$p_k(f) = \sup\left\{ |f(z)| \;:\; |z| \le k - \frac{1}{2} \right\} \quad \text{for} \quad k = 1, 2, \ldots$$

Clearly, F_k is continuous with respect to p_{k+1} but not with respect to p_k. By Eidelheit's theorem ([3], Satz 2 and Bemerkung 3; see also [15], Theorem 4.4.7, and [7], Lemma 1), the mapping $\mathcal{M} \ni f \to (F_k(f)) \in \mathbb{C}^{\mathbb{N}}$ is surjective. In particular, there exists $g \in \mathcal{M}$ with $F_1(g) \neq 0$ and $F_k(g) = 0$ for $k = 2, 3, \ldots$. Let (a_n) be the coefficients in the power series representation of g. We have $(a_n) \neq 0$ and $a_n = 0$ for $n \in (\mathbb{N} \setminus M) \cup \{0\}$. Moreover,

$$\sum_{n=k-1}^{\infty} a_n\, k^n = 0 \quad \text{for} \quad k = 2, 3, \ldots$$

It follows that the sequence x_n, $n \in M$, is not δ-independent. □

4. Containment of φ and ω-independent subsequences

The following result is basic for the rest of this section. In our proof we apply the main idea of the proof of Theorem 9 in [2].

Theorem 1

Let X be a topological linear space. If $x_n \in X$, $x_n \neq 0$, and $\mu_n x_n \to 0$ for all $(\mu_n) \in \omega$, then
(a) (x_n) has an ω-independent subsequence.
(b) (x_n) has a subsequence equivalent to the standard basis of φ provided that X is metrizable or locally convex.

The assertion of (b) means, by definition, that there exists (x_{n_k}) such that the mapping which assigns x_{n_k} to e_k extends (uniquely) to an isomorphism of φ into X.

Proof. We start with establishing the first part of (b). Set

$$d_n = \sup\{\, |\lambda x_n| : \ \lambda \in \mathbb{R}\} \quad \text{for} \quad n = 1, 2, ...$$

By assumption, $d_n \to 0$. Choose $n_1 < n_2 < ...$ so that $d_{n_1} < 1$ and $d_{n_{k+1}} < \frac{1}{3} d_{n_k}$. It follows that $\Sigma_{l=k+1}^{\infty} d_{n_l} < \frac{1}{2} d_{n_k}$ for all $k \in \mathbb{N}$. In particular, $\Sigma_{k=1}^{\infty} d_{n_k} < \infty$. Define $T : \varphi \to X$ by $T\big((\lambda_k)\big) = \Sigma_{k=1}^{\infty} \lambda_k \, x_{n_k}$. We claim that T has all the desired properties.

To show the continuity of T at 0, fix $\varepsilon > 0$, and choose k_0 with $\Sigma_{k=k_0+1}^{\infty} d_{n_k} < \frac{\varepsilon}{2}$. There exists $\eta > 0$ such that $|\lambda| < \eta$ implies $|\lambda x_{n_k}| < \varepsilon/(2k_0)$ for $k = 1, ..., k_0$. It follows that, given $(\lambda_k) \in \varphi$ with $|\lambda_k| < \eta$ for $k = 1, ..., k_0$, we have

$$|T\big((\lambda_k)\big)| \leq \sum_{k=1}^{k_0} |\lambda_k x_{n_k}| + \sum_{k=k_0+1}^{\infty} |\lambda_k \, x_{n_k}| < \varepsilon \ .$$

To show the remaining properties of T, observe that for every $k \in \mathbb{N}$ we can find $\alpha_k > 0$ with

$$\Big| \sum_{l=k}^{\infty} \lambda_l \, x_{n_l} \Big| > \frac{1}{4} \, d_k \quad \text{whenever} \quad (\lambda_n) \in \varphi \quad \text{and} \quad |\lambda_k| > \alpha_k \ .$$

Indeed, choose $\alpha_k > 0$ that $|\lambda \, x_{n_k}| > \frac{3}{4} \, d_{n_k}$ whenever $|\lambda| > \alpha_k$. Then

$$\Big| \sum_{l=k}^{\infty} \lambda_l \, x_{n_l} \Big| \geq |\lambda_k \, x_{n_k}| - \sum_{l=k+1}^{\infty} |\lambda_l \, x_{n_l}| > \frac{1}{4} \, d_{n_k}$$

whenever $|\lambda_k| > \alpha_k$. This observation yields that $T^{-1}(0) = \{0\}$. Moreover, it implies, by induction on k, that if $T((\lambda_k^i)) \to 0$ when $i \to \infty$, where $(\lambda_k^i) \in \varphi$, then $\lambda_k^i \to 0$ for all k.

To establish (a) and the second part of (b), assume that X is of countable dimension, and fix a weaker metrizable linear topology τ on X (see [9], Lemma 3). Applying the first part of (b) to (X, τ), we get (x_{n_k}) which is ω-independent with respect to τ, and so with respect to the original topology of X. Thus, (a) holds. The corresponding isomorphism T of φ into (X, τ) yields, in the locally convex situation, a locally convex topology σ on φ stronger than the product topology. Moreover, $\mu_k \, e_k \to 0$ with respect to σ for all $(\mu_k) \in \omega$. Hence, in view of Lemma 1, σ coincides with the product topology of φ, and so T is an isomorphism of φ into X. □

We shall construct an example to the effect that Theorem 1(b) fails in general topological linear spaces.

EXAMPLE 1: Denote by $\mathcal{N}$ the set of all strictly increasing functions from $\mathbb{N}$ into $\mathbb{N}$. For $f \in \mathcal{N}$ and $(\lambda_n) \in \varphi$ set

$$|(\lambda_n)|_f = \sum_{n=1}^{\infty} \frac{1}{n} \frac{|\lambda_{f(n)}|}{1 + |\lambda_{f(n)}|} .$$

Clearly, $|\cdot|_f$ is an F-seminorm on φ. Moreover, $|\mu_n e_n|_f \to 0$ for every $(\mu_n) \in \omega$. Let σ be the linear topology on φ generated by the family $|\cdot|_f$, $f \in \mathcal{N}$. It follows that (φ, σ) and (e_n) satisfy the assumption of Theorem 1. On the other hand, we can find $1 = k_1 < k_2 < ...$ so that

$$\Big| \sum_{k=k_i}^{k_{i+1}} e_{f(k)} \Big|_f \geq 1 \quad \text{for all} \quad f \in \mathcal{N} \quad \text{and} \quad i = 1, 2, ...$$

Thus, the assertion of Theorem 1(b) fails in (φ, σ).

Theorem 2

For an arbitrary topological linear space X the following two conditions are equivalent:
(i) every linearly independent sequence in X has an ω-independent subsequence;
(ii) every linearly independent sequence in X has a c_0-independent subsequence.

Proof. Suppose (ii) holds and let (x_n) be a linearly independent sequence in X. In view of Theorem 1(a), it is enough to consider the case where there exists a neighborhood V of 0 in X and $(\mu_n) \in \omega$ with $\mu_n x_n \notin V$ for all n. In view of (ii), we can find $(\mu_{n_k} x_{n_k})$ which is c_0-independent, and so ω-independent. Hence (x_{n_k}) is also ω-independent. □

In the case where X is an F-space the next result is due to Bessaga, Pełczyński and Rolewicz ([2], Theorem 9; see also [15], Proposition 4.2.7). In that case X contains an isomorphic copy of φ if and only if it contains an isomorphic copy of ω.

Theorem 3

For a metrizable topological linear space X the following two conditions are equivalent:
(i) X contains a subspace isomorphic to φ;
(ii) for every neighborhood V of 0 in X there exists an $x \in X$ with $x \neq 0$ and $\{\lambda x : \lambda \in \mathbb{R}\} \subset V$.

In the terminology of [15], p. 196, X satisfying (ii) is said to contain arbitrarily short lines.

Proof. We only need to show that (ii) implies (i). To this end, let (V_n) be a base of neighborhoods of 0 in X with $V_1 \supset V_2 \supset \ldots$. In view of (ii), there exist $x_n \in X$ with $x_n \neq 0$ and $\lambda x_n \in V_n$ for $\lambda \in \mathbb{R}$. Hence $\mu_n x_n \to 0$ for every $(\mu_n) \in \omega$. This implies (i), by Theorem 1(b). □

We note that the metrizability assumption is essential for the validity of the implication (ii) ⇒ (i) of Theorem 3. Both examples below are, in fact, locally convex spaces.

Example 2: Let X be an infinite–dimensional linear space, and denote by X' its algebraic dual. Then $\big(X, \sigma(X, X')\big)$, clearly, satisfies (ii) but it does not contain an infinite–dimensional metrizable subspace (see [9], Lemma 2).

Example 3: Let X be an infinite–dimensional normed space equipped with its weak topology. Then (ii), clearly, holds. On the other hand, for every sequence (x_n) in X with $x_n \neq 0$ we can find $(\mu_n) \in \omega$ so that $(\mu_n x_n)$ does not converge weakly to 0. Therefore, (i) does not hold. The latter also follows by Lemma 2 and [10], Theorem′.

In a special case, Theorem 2 can be improved and completed as follows:

Theorem 4

For a metrizable locally convex space X the following four conditions are equivalent:
(i) every linearly independent sequence in X has an ω-independent subsequence;
(ii) every linearly independent sequence in X has a δ-independent subsequence;
(iii) X contains no subspace isomorphic to φ;
(iv) X admits a continuous norm.

Proof. Clearly, (i) implies (ii). In view of Lemma 2, (ii) implies (iii).

Let (p_n) be an increasing sequence of seminorms generating the topology of X. If (iii) holds, then $p_n^{-1}(0) = \{0\}$ for some n (see Theorem 3, (ii) ⇒ (i)). Thus, (iv) follows.

Finally, it is a known result that (i) holds for every normed space X (see the bibliographical comments below), and so (iv) implies (i). □

Under the additional assumption that X be complete, in which case X is called a Fréchet space or a B_0-space, the implication (i) $\Rightarrow$ (iv) is due to Kadets ([7], Theorem; see also [19], Theorem III). In fact, the stronger implication (ii) $\Rightarrow$ (iv) is implicit in his proof. Under the same assumption, the equivalence of (iii) (with "ω" instead of "φ") and (iv) is due to Bessaga and Pełczyński ([1], Corollary 1 and Lemma 1).

That a normed space X satisfies (i) is due essentially to Erdős and Straus ([4]; see also [17], Theorem III.6.1 and pp. 756 and 857–858). Other proofs have been given in [5], [10] and [18]. The proof in [5] is, however, distorted by misprints (cf. [10], Remark 2). In particular, formula (1) thereof should read:

$$\rho_n = \inf_{t\in\mathbb{R}} \inf_{\max\limits_k |\alpha_k|\geq 1} \left\| t\, x_0 - \sum_1^n \alpha_k\, x_k \right\| .$$

This correction has been recently communicated to the author by V. M. Kadets. We also note that the Erdős–Straus result is strengthened in [10], Theorem′; for a generalization of the latter see Theorem 5 below.

The metrizability assumption is essential for the validity of the implication (i) $\Rightarrow$ (iv) of Theorem 4. Indeed, the spaces considered in Examples 2 and 3 both satisfy (i). This is clear in the first case and is the content of [10], Theorem′, in the second case. However, neither of these spaces satisfies (iv).

5. ω-independent subsequences with respect to the weak* topology

The following lemma is a special case of a result of Johnson and Rosenthal ([6], Theorem III.1 and Remark III.1; cf. also [17], Theorem III.1.5). For the reader's convenience we present a proof which is much simpler than the proof of that result given in [6] or [17].

Lemma 3

Let X be a Banach space and let (x_n^*) be a sequence in X^* with the following two properties:

(1) $x_n^* \to 0$ *with respect to* $\sigma(X^*, X)$,

(2) $\inf_{n\in\mathbb{N}} \|x_n^*\| > 0$.

Then there exists a subsequence (y_n^*) of (x_n^*) and a sequence (y_n) in X such that $y_n^*(y_m) = \delta_{n\,m}$ for all n, $m \in \mathbb{N}$ with $n \leq m$. In particular, (y_n^*) is ω-independent with respect to $\sigma(X^*, X)$.

The sequences in question are easily constructed, by induction, with the help of the following sublemma.

Sublemma

Under the assumptions of Lemma 3, there exist $(x^*_{n_k})$ *and* $x \in X$ *with* $x^*_{n_1}(x) = 1$ *and* $x^*_{n_k}(x) = 0$ *for* $k = 2, 3, ...$

Proof. We begin with establishing the following property of (x^*_n): for every closed subspace X_1 of X of finite codimension there exists $m \in \mathbb{N}$ with $\inf_{n \geq m} \|x^*_n|X_1\| > 0$. To this end, let P be a projection of X onto X_1. Then, by (1), we have $\|x^*_n(I-P)\| \to 0$. Moreover,

$$\|x^*_n\| - \|x^*_n(I-P)\| \leq \|x^*_n P\| \leq \|P\| \, \|x^*_n|X_1\| \ .$$

Therefore, (2) yields the property in question.

We shall now choose $n_1 < n_2 < ...$ and $x_k \in X$ with $x^*_{n_1}(x_1) = 1$ and, for $k = 2, 3, ...,$

$$x^*_{n_i}(x_k) = 0 \quad \text{whenever} \quad i < k, \ \|x_k\| \leq \frac{1}{(k+1)^2} \quad \text{and} \quad x^*_{n_k}\Big(\sum_{i=1}^{k} x_i\Big) = 0 \ .$$

Let $n_1 = 1$, and take $x_1 \in X$ with $x^*_{n_1}(x_1) = 1$. Suppose now that $n_1 < ... < n_k$ and $x_1, ..., x_k \in X$ with the desired properties have been already defined. Set

$$X_1 = \{x \in X : \ x^*_{n_i}(x) = 0 \quad \text{for} \quad i = 1, ..., k\} \ .$$

By what we have established in the first part of the proof and (1), there exists $n_{k+1} > n_k$ with

$$\Big| x^*_{n_{k+1}}\Big(\sum_{i=1}^{k} x_i\Big)\Big| < \frac{1}{(k+1)^2} \, \|x^*_{n_{k+1}} \mid X_1\| \ .$$

Hence we can find $x_{k+1} \in X_1$ so that

$$\|x_{k+1}\| \leq \frac{1}{(k+1)^2} \quad \text{and} \quad x^*_{n_{k+1}}(x_{k+1}) = -x^*_{n_{k+1}}\Big(\sum_{i=1}^{k} x_i\Big) \ .$$

It follows that $(x^*_{n_k})$ and $x = \Sigma_{k=1}^{\infty} x_k$ satisfy the assertion. □

Since, for every normed space X, we can identify $\big(X, \sigma(X, X^*)\big)$ with a subspace of $\big(X^{**}, \sigma(X^{**}, X^*)\big)$, the next result generalizes [10], Theorem$'$.

Theorem 5

If X *is a Banach space, then every linearly independent sequence* (x^*_n) *in* X^* *has a subsequence which is* ω*-independent with respect to* $\sigma(X^*, X)$.

Proof. We first assume additionally that X is separable. Then, as well known, the unit ball of X^* is weak* compact and metrizable. We may also confine ourselves to the case where $\|x_n^*\| = 1$ for all n, and so assume that (x_n^*) is weak* convergent. According as the limit is 0 or not, the assertion now follows from Lemma 3 above or [10], Theorem (a).

In the general case, we can find a separable closed subspace X_0 of X such that $(x_n^*|X_0)$ is linearly independent. Indeed, choose $x_n \in X$ so that, for all n,

$$x_n^*(x_n) = 1 \quad \text{and} \quad x_i^*(x_n) = 0 \quad \text{for} \quad i = 1, ..., n-1$$

(cf. [15], Lemma 5.2.2), and let X_0 be the closed linear span of (x_n) in X. Applying the already established special case of the theorem to X_0 and $(x_n^*|X_0)$, we easily get the assertion. □

The following simple example shows that the completeness assumption on X in Theorem 5 in essential.

EXAMPLE 4: Let X be the subspace of c_0 algebraically identical with φ. Then $(X^*, \sigma(X^*, X))$ is isomorphic to l_1 equipped with the product topology. Therefore, in view of Lemma 2, it does not satisfy the assertion of Theorem 5.

We also note that there exist dual Banach spaces in which ω-independence with respect to the weak* topology is strictly stronger than that with respect to the weak topology.

EXAMPLE 5: (cf. [16], Example I.13.3, and [10], p. 96): Let $X = c_0$, and set $x_n^* = e_n - e_{n-1}$, where $e_0 = 0$ and (e_n) is the standard basis of l_1. Then $\Sigma_{n=1}^{\infty} x_n^* = 0$ with respect to $\sigma(l_1, c_0)$. On the other hand, (x_n^*) is ω-independent with respect to $\sigma(l_1, l_\infty)$. Indeed, setting

$$x_1^{**} = (1, 1, ...) \quad \text{and} \quad x_{n+1}^{**} = x_1^{**} - \sum_{k=1}^{n} e_k \quad \text{for} \quad n = 1, 2, ... ,$$

we have $x_n^{**}(x_m^*) = \delta_{nm}$ for all $n, m \in \mathbb{N}$. (In fact, (x_n^*) is a Schauder basis of l_1.)

6. Dimension of closed convex sets in F-spaces

By the dimension of a subset W of a linear space Y we mean below the algebraic dimension of lin W, the linear span of W in Y.

The second part of the next result extends a classical theorem that the dimension of an F-space is either finite or at least $2^{\aleph_0}$ (see [9], Corollary 2, for a proof and more references).

Theorem 6

Let X be an F-space and let W be a closed convex subset of X with $\dim W \geq \aleph_0$. *Then there exists a continuous injective linear operator $T : l_1 \to \operatorname{lin} W$. In particular,* $\dim W \geq 2^{\aleph_0}$.

Proof. By taking a translate of W, we may assume that $0 \in W$. Then $tW \subset W$ for $0 \leq t \leq 1$, so that we can find a linearly independent sequence (x_n) in W with $\Sigma_{n=1}^{\infty}|x_n| < \infty$. In view of [9], Proposition 3, we may also assume that (x_n) is l_1-independent. Define $T\big((\lambda_n)\big) = \Sigma_{n=1}^{\infty}\lambda_n x_n$ for $(\lambda_n) \in l_1$. Clearly, T is an injective linear operator and

$$T\Big(\Big\{(\lambda_n) \in l_1 : \sum_{n=1}^{\infty} |\lambda_n| \leq 1\Big\}\Big) \subset W - W \ .$$

Moreover, T is easily seen to be continuous.

Since $\dim l_1 = 2^{\aleph_0}$ (see, e.g., [9], Corollary 2), the second assertion follows from the first one.

We shall give another proof of the second assertion based on a theorem of Mycielski [14].

Observe first that for every open subset G of X with $G \cap W \neq \emptyset$ we have that $\dim\,(G \cap W) \geq \aleph_0$. Indeed, fix $w \in G \cap W$ and a linearly independent sequence (w_n) in W. Choose $\mu_n \in (0,1]$ with $\mu_n w_n + (1-\mu_n)w \in G$ for $n = 1, 2, \ldots$. Thus, $G \cap W$ contains a linearly independent sequence.

Define R_n to be the subset of W^n consisting of all linearly dependent sequences. We claim that $W^n \setminus R_n$ is dense in W^n. Indeed, fix $(w_1, ..., w_n) \in W^n$ and open subsets $G_1, ..., G_n$ of X with $w_i \in G_i$ for $i = 1, ..., n$. By the observation above, we can find $(v_1, ..., v_n) \in W^n$ such that

$$v_i \in G_i \cap W \quad \text{and} \quad v_i \notin \operatorname{lin}\{v_1, ..., v_{i-1}\} \quad \text{for} \quad i = 1, ..., n \ .$$

This establishes the claim. Moreover, R_n is closed, since $(w_1, ..., w_n) \in W^n$ is linearly dependent if and only if there exists $(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ with

$$\sum_{i=1}^{n} |\lambda_i| = 1 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i \, w_i = 0 \ .$$

In sum, R_n is meager in W^n for $n = 1, 2, ...,$ which implies that there exists a continuous function f from the Cantor set C into W with

$$(f(c_1), ..., f(c_n)) \notin R_n$$

whenever $(c_1, ..., c_n)$ is a sequence of distinct elements of C (see [14], Theorem 1). It follows that dim $f(C) = 2^{\aleph_0}$. $\square$

We note that the second part of Theorem 6 cannot be reduced directly to its classical special case where $W = X$. This is so because lin W need not be closed in X. In fact, if W is compact, then lin W is meager in X, by a theorem of Riesz.

Postscript. The second part of Theorem 1(b) already appears in Lemma 1 of the paper by M. A. Simões: Very strongly and very weakly convergent sequences in locally convex spaces, *Proc. Roy. Irish Acad.* **84A** (1984), 125–132.

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