

## WEAK CONDITIONS FOR INTERPOLATION IN HOLOMORPHIC SPACES

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*Abstract*

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An analogue of the notion of uniformly separated sequences, expressed in terms of extremal functions, yields a necessary and sufficient condition for interpolation in  $L^p$  spaces of holomorphic functions of Paley-Wiener-type when  $0 < p \leq 1$ , of Fock-type when  $0 < p \leq 2$ , and of Bergman-type when  $0 < p < \infty$ . Moreover, if a uniformly discrete sequence has a certain uniform non-uniqueness property with respect to any such  $L^p$  space ( $0 < p < \infty$ ), then it is an interpolation sequence for that space. The proofs of these results are based on an approximation theorem for subharmonic functions, Beurling's results concerning compactwise limits of sequences, and the description of interpolation sequences in terms of Beurling-type densities. Details are carried out only for Fock spaces, which represent the most difficult case.

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### 1. Introduction

This paper studies some consequences of the fact that interpolation sequences for Bergman, Fock, and Paley-Wiener spaces are invariant under certain natural group actions and corresponding compactwise limits. We will show how and when this invariance implies that the defining property of interpolation sequences is equivalent to apparently weaker statements about sequences. Two such weak conditions will be considered: The first is the analogue of Carleson's condition of uniform separation (cf. [11]); the second is a condition of "uniform non-uniqueness", in some sense dual to a condition of Beurling used to describe sampling sequences.

Our methods apply to all of the spaces mentioned above, but the main focus will be put on Fock spaces, because they represent the most difficult case. We begin by describing our results in that setting.

Let  $d\sigma$  denote Lebesgue area measure on  $\mathbb{C}$ . We define

$$\|f\|_{\alpha,p}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\alpha|z|^2} d\sigma(z)$$

for  $\alpha > 0$  and  $p < \infty$ , and  $\|f\|_{\alpha,\infty} = \sup_z |f(z)|e^{-\alpha|z|^2}$ . The Fock space  $F_\alpha^p$  consists of those entire functions  $f$  such that  $\|f\|_{\alpha,p} < \infty$ . Of basic importance is that the translation operator  $T_{\zeta,\alpha}$  ( $\zeta \in \mathbb{C}$ ) defined as

$$T_{\zeta,\alpha}f(z) = f(z - \zeta)e^{\alpha(2\bar{\zeta}z - |\zeta|^2)}$$

acts isometrically on  $F_\alpha^p$ . This fact will sometimes be referred to as the *translation invariance* of  $F_\alpha^p$ .

We say that a sequence  $\Gamma = \{\gamma_n\}$  of distinct points in  $\mathbb{C}$  is *interpolating* for  $F_\alpha^p$  if for every sequence  $\{a_n\}$  satisfying

$$\sum_n |a_n|^p e^{-p\alpha|\gamma_n|^2} < \infty,$$

there is an  $f \in F_\alpha^p$  such that  $f(\gamma_n) = a_n$  for all  $n$ . For the case  $p = \infty$ , we require the existence of an  $f \in F_\alpha^\infty$  with  $f(\gamma_n) = a_n$  whenever

$$\sup_n |a_n| e^{-\alpha|\gamma_n|^2} < \infty.$$

For a sequence  $\Gamma = \{\gamma_n\}$  of distinct points, set  $\Gamma_k = \{\gamma_n\}_{n:n \neq k}$  and let  $F_\alpha^p(\Gamma_k)$  denote the closed subspace of  $F_\alpha^p$  consisting of functions vanishing on  $\Gamma_k$ . (We will keep this notation when  $F_\alpha^p$  is replaced by other spaces of holomorphic functions.) Our first main theorem is the following:

**Theorem 1.** *Let  $\Gamma = \{\gamma_n\}$  be a sequence of distinct points in  $\mathbb{C}$ , and suppose  $0 < p \leq 2$ . Then  $\Gamma$  is interpolating for  $F_\alpha^p$  if and only if there is a  $\delta > 0$  such that*

$$(1) \quad \sup\{|f(\gamma_k)| : f \in F_\alpha^p(\Gamma_k), \|f\|_{\alpha,p} \leq 1\} \geq \delta e^{\alpha|\gamma_k|^2} \quad \text{for all } k.$$

A normal family argument shows that the supremum on the left-hand side of (1) is attained, and thus it is in fact a maximum. A simple example (see Section 3) shows that Theorem 1 fails for  $p > 2$ .

We note that condition (1) is indeed “apparently weaker” than the condition of being interpolating, because (1) says only that we can solve the interpolation problems  $f(\gamma_k) = 1$ ,  $f(\gamma_n) = 0$  for  $n \neq k$ , with control of the norms of the solutions. The fact that we can achieve such norm control when  $\Gamma$  is interpolating follows from the open mapping theorem in a standard way. Thus the necessity of (1) for  $\Gamma$  to be interpolating is trivial, modulo the open mapping theorem.

It is instructive to restate and discuss condition (1) in a more general setting. If  $\mathcal{B}$  is a normed or quasi-normed linear space of holomorphic functions defined on some domain  $\Omega$ , we say that a sequence  $\Gamma$  of distinct points from  $\Omega$  is a *weak interpolation sequence for  $\mathcal{B}$*  if there exists a  $\delta > 0$  such that

$$(2) \quad \sup\{|f(\gamma_k)| : f \in \mathcal{B}(\Gamma_k), \|f\|_{\mathcal{B}} \leq 1\} \\ \geq \delta \sup\{|f(\gamma_k)| : f \in \mathcal{B}, \|f\|_{\mathcal{B}} \leq 1\}$$

for all  $k$ . Thus (1) says that  $\Gamma$  is a weak interpolation sequence for  $F_{\alpha}^p$ . The connection between interpolation and weak interpolation is part of Carleson's classical interpolation theorem (cf. Chapter IX of [3]): A sequence is interpolating for  $H^p$  if and only if it is a weak interpolation sequence for  $H^p$ , where  $H^p$  denotes the Hardy space of the open unit disk  $\mathbb{D}$  of  $\mathbb{C}$  and  $0 < p \leq \infty$ . (In this case, interpolating Blaschke products are extremal functions for the left-hand side of (2).) In [11], we showed that the same statement is true for the Bergman spaces  $A_{\beta}^p$  of the unit disk (see Section 4 for definition) when  $0 < p < \infty$ . (The proof technique of [11] is different from the one used here and does not carry over to the Fock and Paley-Wiener spaces.) In Section 4, we show that for the Paley-Wiener spaces  $PW_{\tau}^p$ , this characterization holds only when  $0 < p \leq 1$ .

The  $p$ -dependence in these results is quite curious. It can be attributed directly to the underlying geometry of our spaces, respectively to the line (Paley-Wiener), the plane (Fock), and the disk (Bergman). However, in general, it is less clear why in some cases weak interpolation implies interpolation (like for the Hardy spaces), and in some it does not (like for the Dirichlet space; see Section 5).

Theorem 1 will be obtained as a consequence of a closely related result, which we will now describe. For this we need Beurling's notion of compactwise limits. A sequence  $Q_j$  of closed sets converges strongly to  $Q$ , denoted  $Q_j \rightarrow Q$ , if  $[Q, Q_j] \rightarrow 0$ ; here  $[Q, R]$  denotes the Fréchet distance between two closed sets  $Q$  and  $R$ , i.e.,  $[Q, R]$  is the smallest number  $t$  such that  $Q \subset \{z : d(z, R) \leq t\}$  and  $R \subset \{z : d(z, Q) \leq t\}$ , where  $d(\cdot, \cdot)$  denotes Euclidean distance in  $\mathbb{C}$ .  $Q_j$  converges compactwise to  $Q$ , denoted  $Q_j \rightharpoonup Q$ , if for every compact set  $D$ ,  $(Q_j \cap D) \cup \partial D \rightarrow (Q \cap D) \cup \partial D$ . For a closed set  $\Gamma$ , we let  $W(\Gamma)$  denote the collection of sets  $\Gamma'$  such that  $\Gamma + a_j \rightarrow \Gamma'$  for some sequence  $\{a_j\}$ . If  $\Gamma$  is interpolating for  $F_{\alpha}^p$ , then using the translation invariance of  $F_{\alpha}^p$  and a normal family argument, we see that all sequences in  $W(\Gamma)$  are also interpolating for  $F_{\alpha}^p$ .

A sequence  $\Gamma$  is a *uniqueness sequence* for  $F_\alpha^p$  if the only element of  $F_\alpha^p$  vanishing on  $\Gamma$  is the zero function, and otherwise we say that  $\Gamma$  is a *non-uniqueness sequence* for  $F_\alpha^p$ . A sequence is called *uniformly discrete* if the infimum of the Euclidean distances between distinct points is strictly positive. With this terminology, our second main theorem can be stated as follows:

**Theorem 2.** *Let  $\Gamma = \{\gamma_n\}$  be a sequence of distinct points in  $\mathbb{C}$  and suppose  $p < \infty$ . Then  $\Gamma$  is interpolating for  $F_\alpha^p$  if and only if  $\Gamma$  is uniformly discrete and every  $\Gamma' \in W(\Gamma)$  is a non-uniqueness sequence for  $F_\alpha^p$ .*

This uniform non-uniqueness condition is known to be necessary for a sequence to be interpolating for  $F_\alpha^p$  (see Section 2), and so the problem is to prove that this apparently rather weak condition implies that a sequence is interpolating. Our proof of this implication is easily transferred to Paley-Wiener and Bergman spaces, and in fact to all the weighted spaces described in [15].

Theorem 2 appears particularly interesting when contrasting it with the  $H^p$  case: The analogue of Theorem 2 fails for  $H^p$ , as can be seen by constructing a suitable “uniform” Blaschke sequence which does not meet the Carleson condition.

The next two sections contain the proofs of Theorem 2 and Theorem 1. In Section 4, we will summarize our findings in the Paley-Wiener and Bergman spaces, with only indications of proofs. Finally, in Section 5, the general question about the relationship between interpolation and weak interpolation is discussed briefly in the case that  $\mathcal{B}$  is a Hilbert space. In particular, we show that weak interpolation does not imply interpolation in the Dirichlet space.

In what follows, we write  $f \lesssim g$  whenever there is a constant  $K$  such that  $f \leq Kg$ , and  $f \simeq g$  if both  $f \lesssim g$  and  $g \lesssim f$ .

## 2. Proof of Theorem 2

We comment first on the necessity of the two conditions of Theorem 2. The easy proof that an interpolation sequence is uniformly discrete can be found in [9] (cf. Proposition 3). The proof of the fact that the uniform non-uniqueness condition necessarily holds can be found in that same paper for  $p \geq 1$  (cf. Theorem 2). The case  $p \leq 2$  follows from Lemma 6.2 of [12] with  $L^2$  replaced by  $L^p$ ,  $p \leq 2$ . (Cf. also the proof of Theorem 2 in Section 3.)

We describe next the basic ingredients for the proof of the converse implication.

The sequence  $\Gamma = \{\gamma_n\}$  of distinct points in  $\mathbb{C}$  is a *sampling sequence* for  $F_\alpha^p$  if

$$\|f\|_{\alpha,p} \simeq \|\{f(\gamma_n)e^{-\alpha|\gamma_n|^2}\}\|_{\ell^p}$$

for  $f \in F_\alpha^p$ . We will need the description of sampling and interpolation sequences in terms of lower and upper uniform densities. To this end, fix  $\Gamma$  and denote by  $n(z, r)$  the number of points of the sequence  $\Gamma$  in  $D(z, r)$ , which is the disk of centre  $z$  and radius  $r$ . The *lower uniform density* of  $\Gamma$  is defined as

$$D^-(\Gamma) = \liminf_{r \rightarrow \infty} \min_{z \in \mathbb{C}} \frac{n(z, r)}{\pi r^2},$$

and the *upper uniform density* of  $\Gamma$  is

$$D^+(\Gamma) = \limsup_{r \rightarrow \infty} \max_{z \in \mathbb{C}} \frac{n(z, r)}{\pi r^2}.$$

Note that  $0 \leq D^-(\Gamma) \leq D^+(\Gamma) < \infty$  when  $\Gamma$  is uniformly discrete.

We shall need the following theorem:

**Theorem A.** *Suppose  $\Gamma$  is uniformly discrete and  $0 < p \leq \infty$ . Then  $\Gamma$  is sampling for  $F_\alpha^p$  if and only if  $D^-(\Gamma) > (2\alpha)/\pi$ , and  $\Gamma$  is interpolating for  $F_\alpha^p$  if and only if  $D^+(\Gamma) < (2\alpha)/\pi$ .*

The results for  $p = 2$  and  $p = \infty$  are proved in [12] and [16]; the methods of those papers extend directly to the case  $p \leq 2$  and with minor modifications to the case  $2 < p < \infty$ . A different approach for the case  $p \geq 1$  can be found in [9]. (The ‘‘sampling part’’ of that paper contains a proof valid for all  $p$ .) Note also that in fact Theorem 1 gives an interesting new proof of the necessity of the density condition for interpolation when  $p \leq 2$ .

The second ingredient is a result from [13], which is an analogue of a theorem of Beurling [2]. This theorem, which reflects the translation invariance of sampling sequences, is crucial for proving the sampling part of Theorem A.

**Theorem B.** *Suppose  $\Gamma$  is uniformly discrete. Then  $\Gamma$  is sampling for  $F_\alpha^\infty$  if and only if every  $\Gamma' \in W(\Gamma)$  is a uniqueness sequence for  $F_\alpha^\infty$ .*

The third auxiliary result is a special case of an interesting approximation principle of Lyubarskii and Malinnikova [5]. It states that any subharmonic function can be approximated by the logarithm of the modulus of an entire function outside a ‘‘small’’ exceptional set. We need the following special version of it, in essence proved earlier by Lyubarskii and Sodin [7] and stated as Theorem 3 in [5]:

**Theorem C.** *Suppose  $\phi$  is a subharmonic function in  $\mathbb{C}$  such that  $\Delta\phi \simeq 1$ , where  $\Delta$  is the Laplacian  $\frac{\partial^2}{\partial\bar{\partial}}$ . Then there exists an entire function  $G$ , with uniformly discrete zero sequence  $\Gamma$ , such that*

$$(3) \quad |G(z)| \simeq e^{\phi(z)}d(z, \Lambda).$$

Suppose now that  $\Gamma$  is uniformly discrete and  $W(\Gamma)$  contains only non-uniqueness sequences for  $F_\alpha^p$ . Let us first explain informally the plan of our proof. We will use Theorem C to construct another sequence  $\Lambda$ , which “completes”  $\Gamma$  in the sense that the union of  $\Gamma$  and  $\Lambda$  has a uniform distribution just like a lattice. We shall then transform the problem in such a way that it is solved by applying Theorem B to the “dual” sequence  $\Lambda$  and an appropriate space  $F_{\alpha'}^\infty$ .

We turn to the details. Let  $F$  be some entire function vanishing precisely on  $\Gamma$ . We claim that for any  $\beta > \pi D^+(\Gamma)/2$  we can find a uniformly discrete sequence  $\Lambda$  and an entire function  $G$  vanishing on  $\Lambda$  such that

$$(4) \quad |F(z)G(z)| \simeq d(z, \Gamma)d(z, \Lambda)e^{\beta|z|^2}.$$

(Thus the zeros of  $FG$  are distributed in the same regular way as the points in a lattice of density  $(2\beta)/\pi$ .) This is done in the following fashion (cf. [1, pp. 113–114]). Set

$$\chi_r(z) = \begin{cases} 1/(\pi r^2), & |z| < r \\ 0, & |z| \geq r, \end{cases}$$

and let

$$\nu = \sum_n \delta_{\gamma_n}$$

be the measure consisting of a point mass at each point of our sequence  $\Gamma$ . Choose  $r$  so big that  $2\beta - \pi\nu * \chi_r(z) \geq \epsilon$ , where  $\epsilon = \beta - \pi D^+(\Gamma)/2$  and  $\nu * \chi_r$  denotes the convolution of  $\nu$  and  $\chi_r$  over  $\mathbb{C}$ . We set

$$v = (\nu - \nu * \chi_r) * E,$$

where  $E = \log|z|$ . Actually, to be precise, this function is defined as  $v(z) = \lim_{t \rightarrow \infty} v_t(z)$ , where  $v_t$  corresponds to the finite sequence consisting of those  $\gamma_n$  which satisfy  $|\gamma_n| < t$ . The function  $v_t$  is clearly well-defined, and the limit exists because  $v_t(z) = v_\tau(z)$  if  $|z| + r < t < \tau$ , by the mean value property for harmonic functions. We define

$$\phi(z) = \beta|z|^2 + v(z) - \log|F(z)|$$

and observe that  $\epsilon \leq \Delta\phi(z) \leq \beta$ . It remains to apply Theorem C and use the estimate

$$|v(z) - \log d(z, \Gamma)| \lesssim 1,$$

which follows from the definition of  $v$ .

It is essential to note that  $\Lambda = \{\lambda_n\}$  is constructed in such a way that we obtain the identity

$$D^+(\Gamma) = (2\beta)/\pi - D^-(\Lambda).$$

Namely, the Riesz measure  $\Delta\phi(z) d\sigma(z)$  of  $\phi$  is atomized by splitting the plane into ‘‘cells’’, each of mass 1 with respect to  $(2/\pi)\Delta\phi(z) d\sigma(z)$ , and placing one  $\lambda_n$  in each ‘‘cell’’; we refer to [5] for details.

To show that  $\Gamma$  is an interpolation sequence for  $F_\alpha^p$ , it therefore suffices to prove that  $D^-(\Lambda) > 2(\beta - \alpha)/\pi$ . In view of Theorem A, we need only show that  $\Lambda$  is a sampling sequence for  $F_{\beta-\alpha}^\infty$  and by Theorem B, this reduces to showing that  $W(\Lambda)$  contains only uniqueness sequences for  $F_{\beta-\alpha}^\infty$ .

Suppose then that  $\Lambda' \in W(\Lambda)$ . Recall that it is obtained as a compactwise limit of a sequence  $\Lambda + a_k$ . Choose a subsequence of  $\{a_k\}$ , say  $\{a_{k_j}\}$ , such that also  $\Gamma + a_{k_j} \rightarrow \Gamma' \in W(\Gamma)$ . By a normal family argument applied to the functions  $T_{a_{k_j}, \beta}(FG)$ , it is clear that there exist functions  $\tilde{F}$  and  $\tilde{G}$  vanishing respectively on  $\Gamma'$  and  $\Lambda'$  such that

$$(5) \quad |\tilde{F}(z)\tilde{G}(z)| \simeq d(z, \Gamma')d(z, \Lambda')e^{\beta|z|^2}.$$

This proves that  $\Gamma' \cup \Lambda'$  is a uniqueness sequence for  $F_\beta^p$ , because otherwise there would have to exist an entire function  $f$  such that  $f\tilde{F}\tilde{G} \in F_\beta^p$ . However, using (5) and the subharmonicity of  $|f|^p$  in small disks around each point in  $\Gamma'$  and  $\Lambda'$ , we obtain

$$\|f\tilde{F}\tilde{G}\|_{\beta,p}^p \simeq \int_{\mathbb{C}} |f(z)|^p d\sigma(z) < \infty,$$

which is a contradiction unless  $f \equiv 0$ . On the other hand, since  $\Gamma'$  is a non-uniqueness sequence for  $F_\alpha^p$ , there is a nontrivial function  $g \in F_\alpha^p$  which vanishes on  $\Gamma'$ . If  $\Lambda'$  is likewise a non-uniqueness sequence for  $F_{\beta-\alpha}^\infty$ , there exists a function  $h \in F_{\beta-\alpha}^\infty$  vanishing on  $\Lambda'$  such that  $gh \in F_\beta^p$ , which we have seen is impossible. So  $\Lambda'$  is a uniqueness sequence for  $F_{\beta-\alpha}^\infty$ , and the proof of Theorem 2 is complete.

Note that an application of Theorem C with  $\phi(z) = \alpha|z|^2$  shows that Theorem 2 fails when  $p = \infty$ . Since the sequence  $\Lambda$  satisfies (3), it is a non-uniqueness sequence for  $F_\alpha^\infty$ . A normal family argument shows that the same is true of every member of  $W(\Lambda)$ . On the other hand, it is clear by construction that  $D^+(\Lambda) = (2\alpha)/\pi$ , so that by Theorem A,  $\Lambda$  is not interpolating.

Theorem 2 does have an analogue for  $p = \infty$ , however, if one considers instead the space  $F_\alpha^{\infty,0}$ , which consists of those entire functions satisfying  $|f(z)|e^{\alpha|z|^2} \rightarrow 0$  as  $|z| \rightarrow \infty$ . In this setting,  $\Gamma$  is interpolating if the problem  $f(\gamma_n) = a_n$  is solvable whenever  $a_n e^{\alpha|\gamma_n|^2} \rightarrow 0$  as  $n \rightarrow \infty$ . The statement and proof of the result are then identical to that of Theorem 2, with  $F_\alpha^p$  replaced by  $F_\alpha^{\infty,0}$ .

### 3. Proof of Theorem 1

Several remarks are in order before we turn to the proof of the sufficiency of the weak interpolation condition of Theorem 1. First, note that when  $0 < p \leq 1$ , one can solve the interpolation problem explicitly:

$$f(z) = \sum_k a_k \frac{G_k(z)}{G_k(\gamma_k)},$$

where  $G_k$  is an extremal function of the left-hand side of (1). It is clear that this series converges locally uniformly to a function in  $F_\alpha^p$  and solves the interpolation problem when the  $a_k$ 's satisfy the compatibility condition.

Second, since  $F_\alpha^p$  is continuously embedded into  $F_\alpha^2$  for  $p \leq 2$ , and since the interpolation sequences are independent of  $p$ , it suffices to prove the theorem for the case  $p = 2$ , which is what we will do.

Third, we note that the theorem fails when  $p > 2$ . To show this, we again appeal to Theorem C to find an entire  $G$ , with uniformly discrete zero sequence  $\Gamma$ , such that  $G(z) \simeq e^{\alpha|z|^2} d(z, \Gamma)$ . Then setting  $G_k(z) = G(z)/(z - \gamma_k)$ , we have

$$\sup_k \|G_k\|_{\alpha,p} < \infty$$

and

$$G_k(\gamma_k) \gtrsim e^{\alpha|\gamma_k|^2}.$$

However,  $\Gamma$  is not interpolating because  $D^+(\Gamma) = (2\alpha)/\pi$ , as follows from the construction of  $\Gamma$ .

To prove Theorem 1, we will show that  $\Gamma$  is uniformly discrete and that  $W(\Gamma)$  contains only non-uniqueness sequences for  $F_\alpha^p$ . The result will then follow from an application of Theorem 2. Throughout the proof, we will let  $G_k$  denote an extremal function of the left-hand side of (1).

If  $f \in F_\alpha^p$  vanishes at some point  $w$ , we find that  $f(z)/(z - w) \in F_\alpha^p$  with norm controlled by  $\|f\|_{\alpha,p}$ , and so

$$e^{-\alpha|z|^2} |f(z)| \lesssim |z - w| \|f\|_{\alpha,p}.$$

We set  $f = G_k$ ,  $z = \gamma_k$  and  $w = \gamma_n$ , to see that this inequality becomes  $1 \lesssim |\gamma_k - \gamma_n|$ , whence  $\Gamma$  is uniformly discrete.

It remains to prove that  $W(\Gamma)$  contains only non-uniqueness sequences. To begin with, we note that the translation invariance of  $F_\alpha^p$  along with a normal family argument implies that if  $\Gamma$  meets (1), then so does every member of  $W(\Gamma)$ . Thus it suffices to demonstrate that every sequence  $\Gamma$  satisfying (1) is a non-uniqueness sequence for  $F_\alpha^p$ .

Suppose then, that  $\Gamma$  is a uniqueness sequence. If  $D^+(\Gamma) < (2\alpha)/\pi$ , then  $\Gamma$  will be an interpolation sequence and hence a non-uniqueness sequence, so we will assume that  $D^+(\Gamma) \geq (2\alpha)/\pi$  and obtain a contradiction.

Fix an arbitrary  $k$  and consider the unique function  $g_k \in F_\alpha^p$  with

$$g_k(\gamma_n) = \begin{cases} e^{\alpha|\gamma_k|^2}, & n = k \\ 0, & n \neq k. \end{cases}$$

Setting  $g(z) = (z - \gamma_k)g_k(z)$ , we observe that  $g(z)/(z - \gamma_n) \in F_\alpha^p$  for arbitrary  $n$ . Hence we must have

$$(6) \quad \int_{\mathbb{C}} \left| \frac{g(z)}{z - \gamma_n} \right|^p e^{-p\alpha|z|^2} d\sigma(z) \lesssim |g'(\gamma_n)|^p e^{-p\alpha|\gamma_n|^2}.$$

By the subharmonicity of  $|g(z)/(z - \gamma_n)|^p$ , we also have

$$(7) \quad \begin{aligned} |g'(\gamma_n)|^p e^{-p\alpha|\gamma_n|^2} &\lesssim \epsilon^{-2} \int_{\epsilon < |z - \gamma_n| < 2\epsilon} |g(z)|^p e^{-p\alpha|z|^2} d\sigma(z) \\ &\leq \epsilon^{-2} \int_{D(\gamma_n, 2\epsilon)} |g(z)|^p e^{-p\alpha|z|^2} d\sigma(z), \end{aligned}$$

independently of  $\epsilon > 0$ .

For the arguments to be used next, it is convenient to apply a theorem of Landau [4], which implies that we may write instead

$$D^+(\Gamma) = \limsup_{r \rightarrow \infty} \max_{z \in \mathbb{C}} \frac{m(z, r)}{r^2},$$

where  $m(z, r)$  is the number of points of  $\Gamma$  found in a square of centre  $z$ , side length  $r$ , and having sides parallel to the coordinate axes. Letting  $d = D^+(\Gamma)$ , we can for every  $\epsilon > 0$  choose an  $r$  sufficiently large so that

$$(8) \quad \sup_{t>r} m^+(t)/t^2 \leq d + \epsilon,$$

where  $m^+(t) = \max_{z \in \mathbb{C}} m(z, t)$ . Then we choose a much bigger  $R$  such that

$$(9) \quad d \leq m^+(R)/R^2,$$

and denote by  $Q$  one of the squares of side length  $R$  containing  $m^+(R)$  points from  $\Gamma$ . We split  $Q$  into 4 squares, each of side length  $R/2$ . By (8) each of these squares contains at most  $(d + \epsilon)R^2/4$  points, so by (9) none of them contain fewer than  $(d - 3\epsilon)R^2/4$  points.

In general, we can write  $Q$  as the union of  $4^k$  squares of side length  $R/2^k$ , each of which contains at most  $(d + \epsilon)\frac{R^2}{4^k}$  points by (8). Again, (9) implies that  $Q$  contains at least  $dR^2$  points, so that as long as  $R/2^k \geq r$ , each square of side length  $R/2^k$  contains at least

$$dR^2 - (4^k - 1)(d + \epsilon)\frac{R^2}{4^k} = \frac{R^2}{4^k}(d - (4^k - 1)\epsilon)$$

members of  $\Gamma$ . Let  $k(R, r, \epsilon)$  denote the largest  $k$  for which both  $R/2^k \geq r$  and  $d/2 \geq (4^k - 1)\epsilon$  hold.

We sum (6) over  $\Gamma \cap Q$ . By the uniform discreteness and (7) with a sufficiently small  $\epsilon$ , we obtain

$$\begin{aligned} & \int_{\mathbb{C}} \sum_{\gamma_n \in Q} |\gamma_n - z|^{-2} |g(z)|^2 e^{-2\alpha|z|^2} d\sigma(z) \\ &= \sum_{\gamma_n \in Q} \int_{\mathbb{C}} |\gamma_n - z|^{-2} |g(z)|^2 e^{-2\alpha|z|^2} d\sigma(z) \\ &\lesssim \sum_{\gamma_n \in Q} |g'(\gamma_n)|^2 e^{-2\alpha|\gamma_n|^2} \\ &\lesssim \epsilon^{-2} \sum_{\gamma_n \in Q} \int_{D(\gamma_n, 2\epsilon)} |g(z)|^2 e^{-2\alpha|z|^2} d\sigma(z) \\ &\lesssim \epsilon^{-2} \int_{Q^+} |g(z)|^2 e^{-2\alpha|z|^2} d\sigma(z), \end{aligned}$$

where  $Q^+$  denotes the set of points at distance at most 1 from  $Q$ . This implies that

$$\inf_{z \in Q^+} \sum_{\gamma_n \in Q} |\gamma_n - z|^{-2} \lesssim 1.$$

On the other hand, letting  $\{Q_l\}_{l=1}^{4^k}$  be the partition of  $Q$  obtained at the  $k$ -th stage, we see that

$$\begin{aligned} \inf_{z \in Q^+} \sum_{\gamma_n \in Q} |\gamma_n - z|^{-2} &= \inf_{z \in Q^+} \sum_{l=1}^{4^k} \sum_{\gamma_n \in Q_l} |\gamma_n - z|^{-2} \\ &\geq \frac{R^2}{4^k} (d - (4^k - 1)\epsilon) \inf_{z \in Q^+} \sum_{l=1}^{4^k} \frac{1}{(d(z, Q_l))^2} \gtrsim k = k(R, r, \epsilon), \end{aligned}$$

which is a contradiction, because  $k(R, r, \epsilon)$  can be made arbitrarily large.

We remark that the case  $p = 2$  is more subtle than  $p < 2$ . In the latter case, the dividing of  $Q$  into smaller squares is not necessary to obtain the desired contradiction.

### 4. Paley-Wiener and Bergman spaces

The Paley-Wiener space  $PW_\tau^p$  ( $\tau > 0$ ) consists of all entire functions  $f$  of exponential type at most  $\tau$  such that the restriction of  $f$  to the real line  $\mathbb{R}$  belongs to  $L^2(\mathbb{R})$ . A sequence  $\Gamma = \{\gamma_n\}$  of distinct complex numbers  $\gamma_n = \xi_n + i\eta_n$  is *interpolating for  $PW_\tau^p$*  if for each sequence  $\{a_n\}$  satisfying  $a_n(1 + |\eta_n|)^{1/p} e^{-\tau|\eta_n|} \in \ell^p$  (here  $1/\infty = 0$ ), there exists a solution  $f \in PW_\tau^p$  to the interpolation problem  $f(\gamma_n) = a_n$ .

The counterpart to Theorem 1 is easy in the Paley-Wiener case. We state it as

**Proposition 1.** *Let  $\Gamma = \{\gamma_n\}$  be a sequence of distinct points in  $\mathbb{C}$ , and suppose  $0 < p \leq 1$ . Then  $\Gamma$  is interpolating for  $PW_\tau^p$  if and only if there is a  $\delta > 0$  such that*

$$\max\{|f(\gamma_k)| : f \in PW_\tau^p(\Gamma_k), \|f\|_p \leq 1\} \geq \delta(1 + |\eta_k|)^{-1/p} e^{\tau|\eta_k|} \quad \text{for all } k.$$

The necessity of the weak interpolation property is again clear by the open mapping theorem, while the sufficiency follows by a direct interpolation formula, as explained in the beginning of the previous section.

To see that the proposition fails when  $1 < p < \infty$ , we appeal to the results about complete interpolating sequences in [6]. Set  $\gamma_0 = 0$  and  $\gamma_k(p) = k + \delta_k(p)$ ,  $k \in \mathbb{Z} \setminus \{0\}$ , where  $\delta_k(p) = \text{sgn}(k)/(2p')$  and  $p' = \max(p, q)$ ,  $1/p + 1/q = 1$ . Then the infinite product

$$G(z) = z \prod_{k=1}^{\infty} (1 - z/\gamma_k)(1 - z/\gamma_{-k})$$

converges locally uniformly to an entire function of exponential type  $\pi$  and satisfies the estimate

$$G(x) \simeq d(x, \Gamma)(1 + |x|)^{-1/p'},$$

where, as before,  $\Gamma = \{\gamma_k\}$ . By Theorem 2 of [6],  $\Gamma$  is not interpolating for  $PW_\pi^p$ , but the functions  $g_k(z) = G(z)/(z - \gamma_k)$  satisfy

$$|g_k(\gamma_k)|^p \simeq \int_{-\infty}^{\infty} |g_k(x)|^p dx,$$

so the sequence is weakly interpolating. Obviously, this example can be scaled to serve any  $PW_\tau^p$ ,  $1 < p < \infty$ . When  $p = \infty$ , Beurling's interpolation theorem [2] shows that the proposition fails.

In this setting, one defines  $W(\Gamma)$  as before, except that complex translates are replaced by real translates. Euclidean distance is replaced by the "mixed" distance function

$$\delta(z, \zeta) = \frac{|z - \zeta|}{1 + |z - \bar{\zeta}|},$$

which is used when defining uniformly discrete sequences. Moreover,  $\Gamma$  is said to satisfy the *two-sided Carleson condition* if for any square  $Q$  of side-length  $l(Q)$  and with one of its sides sitting on the real axis, we have

$$\sum_{\gamma_n \in Q} |\Im \gamma_n| \leq Cl(Q),$$

with  $C$  independent of  $Q$ . We then have:

**Theorem 3.** *Let  $\Gamma = \{\gamma_n\}$  be a sequence of distinct points in  $\mathbb{C}$  and suppose  $p < \infty$ . If  $\Gamma$  is uniformly discrete, satisfies the two-sided Carleson condition, and every  $\Gamma' \in W(\Gamma)$  is a non-uniqueness sequence for  $PW_\tau^p$ , then  $\Gamma$  is interpolating for  $PW_\tau^p$ .*

This time we have not stated the uniform non-uniqueness condition as both necessary and sufficient. This is because it fails to be necessary when  $1 < p < \infty$ , as follows from the existence of complete interpolating sequences (see [6]). The result is stated in this way to underline the main point, namely that the condition is always sufficient.

The proof uses Beurling's original variant of Theorem B and the results of [10], which contains the appropriate analogues of Theorem A and Theorem C. We omit the details, which are essentially identical to those in Section 2.

Let us now describe the situation for the Bergman spaces. We fix a  $\beta > 0$  and set

$$\|f\|_{\beta,p}^p = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^{p\beta-1} d\sigma(z)$$

for  $p < \infty$ . We denote by  $A_\beta^p$  ( $0 < p < \infty$ ) the set of all functions  $f$  analytic in  $\mathbb{D}$  such that  $\|f\|_{\beta,p} < \infty$ . A sequence  $\Gamma = \{\gamma_n\}$  in  $\mathbb{D}$  is an *interpolation sequence for  $A_\beta^p$*  if for every sequence  $\{a_n\}$  satisfying

$$\sum |a_n|^p (1 - |\gamma_n|^2)^{p\beta+1} < \infty,$$

there is an  $f \in A_\beta^p$  with  $f(\gamma_n) = a_n$ .

Now it is appropriate to use pseudo-hyperbolic distance

$$\rho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{\zeta}z} \right|,$$

in terms of which uniformly discrete sequences are defined. One defines  $W(\Gamma)$  as for the Fock space, except that in the definition of  $[\cdot, \cdot]$ , one uses the pseudohyperbolic metric  $\rho$  instead of Euclidean distance  $d$  and Möbius transformations instead of complex translations. We have:

**Theorem 4.** *Let  $\Gamma = \{\gamma_n\}$  be a sequence of distinct points in  $\mathbb{D}$  and suppose  $p < \infty$ . Then  $\Gamma$  is interpolating for  $A_\beta^p$  if and only if  $\Gamma$  is uniformly discrete and every  $\Gamma' \in W(\Gamma)$  is a non-uniqueness sequence for  $A_\beta^p$ .*

The proof of Theorem 4 is also in essence identical to that of Theorem 2. The roles of Theorem A and Theorem B are played by the analogous description of interpolation and sampling sequences obtained in [13]. Instead of Theorem C, one uses the completely analogous Theorem 2 of [14]. Again we omit the details.

A remark similar to the one given at the end of Section 2 can be made about the case  $p = \infty$ . This means that Theorems 3 and 4 fail when  $p = \infty$ , but there exist analogues for the counterparts of the space  $F_\alpha^{\infty,0}$ .

For the Bergman spaces, it is easy to see that the extremal function condition (2) implies that  $\Gamma$  is a non-uniqueness sequence, so that the analogue of Theorem 1, which was proved in [11], is an immediate consequence of Theorem 4. In  $F_\alpha^p$  this is much less obvious, since multiplication by  $z$  does not take  $F_\alpha^p$  into itself.

## 5. The weak interpolation property in Hilbert spaces of holomorphic functions

In the introduction, we defined weak interpolation sequences for a normed or quasi-normed space  $\mathcal{B}$ . In this generality, it is not clear what the definition of an interpolation sequence should be, and so it is not meaningful to discuss the general question about the connection between interpolation and weak interpolation.

Suppose instead that  $\mathcal{H}$  is a Hilbert space of holomorphic functions on a domain  $\Omega$  in  $\mathbb{C}$ , with the additional property that the point evaluation functional is bounded. This implies that we are guaranteed the existence of a reproducing kernel for each  $a \in \Omega$ , that is, a function  $k_a \in \mathcal{H}$  such that

$$\langle f, k_a \rangle = f(a)$$

for all  $f \in \mathcal{H}$ . The reproducing kernel is directly related to the extremal problem on the right-hand side of (2) through the identity

$$\|k_a\|_{\mathcal{H}} = \sup\{|f(a)| : f \in \mathcal{H}, \|f\|_{\mathcal{H}} \leq 1\}.$$

A sequence  $\Gamma = \{\gamma_n\}$  in  $\Omega$  is an *interpolation sequence for  $\mathcal{H}$*  if for every sequence  $\{a_n\}$  such that

$$\sum |a_n|^2 \|k_{\gamma_n}\|^{-2} < \infty,$$

there is an  $f \in \mathcal{H}$  such that  $f(\gamma_n) = a_n$  for all  $n$ . The open mapping theorem shows that if  $\Gamma$  is interpolating, we can solve the interpolation problem with norm control of the solution, so that an interpolation sequence is in particular a weak interpolation sequence.

An interesting question is which Hilbert spaces  $\mathcal{H}$  have the property that every weak interpolation sequence is interpolating. This property will be called the *weak interpolation property*. We have seen that the spaces  $H^2$ ,  $A_{\beta}^2$ , and  $F_{\alpha}^2$  all have the weak interpolation property, while  $PW_{\tau}^2$  does not. We will now point out that it follows from an example in Marshall and Sundberg's paper [8], that the Dirichlet space also fails to have the weak interpolation property.

The Dirichlet space  $D$  consists of functions analytic in  $\mathbb{D}$ , whose derivatives are in the unweighted Bergman space  $A_{1/2}^2$ . A suitable norm for  $D$  is

$$\|f\|_D^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta + \frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 d\sigma(z),$$

where  $f(e^{i\theta})$  denotes the non-tangential limit of  $f$  at  $e^{i\theta}$ . With this norm, we have  $k_a(z) = \frac{1}{\bar{a}z} \log \frac{1}{1-\bar{a}z}$ .

We denote by  $M_D$  the space of multipliers of  $D$ , that is, the collection of functions  $\phi$  with the property that  $\phi f \in D$  for all  $f \in D$ . The norm on  $M_D$  is given by  $\|\phi\|_{M_D} = \sup_{\|f\|_D=1} \|\phi f\|_D$ . The space  $M_D$  is characterized by Stegenga [17]. A sequence  $\Gamma = \{\gamma_n\}$  in  $\mathbb{D}$  is an *interpolation sequence for  $M_D$*  if for every bounded sequence  $\{a_n\}$ , there is a  $\phi \in M_D$  with  $\phi(\gamma_n) = a_n$  for all  $n$ .

In [8], interpolation sequences for both  $D$  and  $M_D$  are characterized in terms of Carleson measures, but we need only the following two facts proved there:

- (1) The interpolation sequences for  $D$  coincide with those for  $M_D$ .
- (2) There exists a sequence  $\Gamma$  which is weakly interpolating but not interpolating for  $M_D$ .

Statement (2) means that there exist a sequence  $\Gamma$  and functions  $\phi_n$  in  $M_D$  such that  $\phi_n(\gamma_m) = \delta_{nm}$  and  $\|\phi_n\|_{M_D} \leq C$ , and such that  $\Gamma$  is not interpolating for  $M_D$ . By (1), this  $\Gamma$  is not interpolating for  $D$ , yet it is clear that  $\Gamma$  is weakly interpolating for  $D$ . Indeed, letting  $f_n(z) = \phi_n(z)k_{\gamma_n}(z)/\|k_{\gamma_n}\|_D$ , we see that  $|f_n(\gamma_n)| = \|k_{\gamma_n}\|_D$ , while

$$\|f_n\|_D = \left\| \phi_n \frac{k_{\gamma_n}}{\|k_{\gamma_n}\|_D} \right\|_D \leq \|\phi_n\|_{M_D} \leq C.$$

We conclude that  $D$  does not have the weak interpolation property.

**Acknowledgement.** We would like to thank Yurii Lyubarskii, John McCarthy, Artur Nicolau, and Joaquim Ortega-Cerdà for helpful conversations.

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Primera versió rebuda el 30 d'abril de 1999,  
darrera versió rebuda el 21 de setembre de 1999.