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CHOQUET INTEGRALS IN POTENTIAL THEORY

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Abstract

This is a survey of various applications of the notion of the Choquet integral to questions in Potential Theory, i.e. the integral of a function with respect to a non-additive set function on subsets of Euclidean n-space, capacity. The Choquet integral is, in a sense, a nonlinear extension of the standard Lebesgue integral with respect to the linear set function, measure. Applications include an integration principle for potentials, inequalities for maximal functions, stability for solutions to obstacle problems, and a refined notion of pointwise differentiation of Sobolev functions.

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1. Introduction

The concept of capacity, as a generic set theoretic measuring device, is intimately associated to the idea of a function space —in much the same way that Lebesgue measure is related to the classical L^p spaces. The sets of capacity zero are the exceptional sets for representatives of the function space. The prime example of such function spaces are, of course, the classical Sobolev spaces $W^{m,p}(\Omega)$, m a positive integer, $1 \leq p \leq \infty$, and Ω a domain in Euclidean n-space $\mathbb{R}^n$: the distributional derivatives of orders $\leq m$ all belong to $L^p(\Omega)$. When $mp \leq n$, such spaces contain functions all of whose representatives are discontinuous somewhere. The associated capacity, in this case, is nontrivial; it is the variational capacity

$$C'_{m,p}(K) = \inf\{\|\phi\|^p_{W^{m,p}} : \phi \in C_0^\infty,\ \phi \geq 1 \text{ on } K\}^1$$

where K is a compact subset of $\mathbb{R}^n$ and $W^{m,p} = W^{m,p}(\mathbb{R}^n)$ with

$$\|\phi\|^p_{W^{m,p}(\Omega)} = \sum_{|\sigma|\leq m} \int_\Omega |D^\sigma \phi|^p \, dx$$

and σ is an n-tuple of non-negative integers $(\sigma_1, \dots, \sigma_n)$ such that $\sigma_1 + \cdots + \sigma_n \leq m$ i.e. $|\sigma| \leq m$. $D^\sigma\phi$ is the usual partial differentiation of ϕ, σ_k times in direction x_k, $x = (x_1, \dots, x_n)$. $C_0^\infty(\Omega)$ denotes the usual space of infinitely differentiable functions on Ω with compact support; $C_0^\infty = C_0^\infty(\mathbb{R}^n)$. It is easy to see that $C'_{m,p}$ is never zero when $mp > n$, and this coincides with the fact that each $u \in W^{m,p}$ now has an everywhere continuous representative.

It is well known how function spaces have become such an influencial concept in analysis, especially in the modern theory of partial differential equations. In particular, Sobolev spaces often provide just the right setting to prove existence, uniqueness and regularity results for solutions. Capacity has classically entered this picture through removable singularity results and boundary regularity criteria. But recently, the concept of capacity has become much more —a tool that is used in much the same way as measure is used. And one manifestation of this is a desire to "integrate with respect to capacity" as if it really were an additive set function —which it clearly is not. One way around this difficulty is to define such an integral using the distributional form of a Lebesgue

[1]Throughout this manuscript we shall endeavor to adhere to the notation and terminology of our basic reference [**AH**].

integral. This was first proposed by Choquet in his seminal work on capacities [**C**]. And so we define

$$\int_E f\,dC \equiv \int_0^\infty C(\{x \in E : f(x) > t\})\,dt$$

where f is a non-negative function and $C(\cdot)$ is a capacity. Such a concept allows us to at least attempt to treat capacity as a set function much like a measure for purposes of integrating representatives of our function space. And it is precisely to such uses that this note is dedicated: applications that have been around for some time but which can be rephrased in this language, or new uses of this tool. One such reinterpretation is to write the famous Wiener boundary regularity criterion for harmonic functions in domain Ω with specified continuous boundary values as: x_0 is a regular boundary point (i.e. the solution takes on it boundary value at x_0 continuously) if and only if

$$\int_{\Omega^c \cap B(x_0,1)} |x - x_0|^{2-n}\,dC'_{1,2}(x) = +\infty$$

for $n > 2$, $\Omega \subset \mathbb{R}^n$, $\Omega^c =$ complement of Ω, and $B(x_0, r)$ is a ball centered x_0 of radius r. See [**AH**, p. 165], [**H**, p. 220], and [**KM**].

For an extensive discussion of the development of the concept of capacity and related ideas like equilibrium potential, thinness, Hausdorff measures, energy etc. see the book [**AH**] and the survey paper [**A1**]. Many such concepts will be used or refered to in this presentation, and the reader should consult these as well as other references quoted in the bibliography.

The ideas presented here mainly reflect the interests of the author, but their origin can be traced back to two papers, in addition to the fundamental article [**C**] of Choquet. The first of these, [**Fu**], is Fuglede's 1971 interpretation of the concept of capacity as a nonlinear functional generalizing the well known idea of a measure as a linear functional. The second, is the author's 1978 paper [**A4**] where an interesting connection was made between the Choquet integral and a certain capacity functional through the "capacity strong type inequality", (3.1) below. This idea is expressed here as Theorem 2 of section 3(c). However, most of the results presented here are quite recent —since the mid to late 1980's— and often they have not generally been available to the mathematical community at large. Consequently, though this is a survey article, it still seems desirable to supply the proofs or at least some indication of the proofs in cases where these results have had little or no exposure.

The topics included are loosely tied together by their common use of the Choquet integral. The reader can regard each section as an independent topic after a few introductory remarks from section 2. The topics include a study of the Choquet space $L^q(C)$ for various choices of the capacity C. Here a special role is noted for the Hausdorff capacity (content) —section 4. And, as with the study of various function spaces, two topics often of general interest are: a discussion of the dual spaces and the embeddings of more familiar spaces into the new spaces. This we do for $L^p(C_{\alpha,p})$ in Theorems 3 and 4. In section 6, the Choquet integral is used to express a general stability criterion for solutions to an obstacle problem for the p-Laplace operator on a given bounded domain Ω of n-space, Theorem 11. The use of the Choquet integral here allows for a very broad class of obstacles, obstacles that need not be smooth or even bounded. Finally, in section 6, we use the Choquet integral as a tool to refine the usual pointwise differentiation theory for functions in Sobolev spaces.

It is hoped that this survey will provide an introduction, of a substantional nature, for this circle of ideas, ideas that often provide a very elegant and convenient language to express a complimentary notion to the traditional (linear) integral of a function with respect to an additive measure, i.e. the Choquet integral.

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2. Capacity and integrals

(a) Bessel and Riesz capacity.

Calderon's theorem ([**AH**, p. 13]) shows that every $u \in W^{m,p}$ can be represented as a Bessel potential: $u(x) = G_m * f(x)$, $x \in \mathbb{R}^n$ and $f \in L^p$. G_α is the L^1 function with Fourier transform $(1 + |\xi|^2)^{-\alpha/2}$, $\xi \in \mathbb{R}^n$, $\alpha > 0$. With this, we have the following potential theoretic capacity, equivalent to $C'_{m,p}$ when $\alpha = m$, an integer:

$$C_{\alpha,p}(K) = \inf\{\|f\|_{L^p}^p : G_\alpha * f \geq 1 \text{ on } K,\, f \geq 0 \text{ a.e.}\}$$

for $\alpha > 0$ and $1 < p < \infty$. This capacity is often refered to as Bessel capacity and denoted $B_{\alpha,p}$ in the literature. If we substitute the Riesz kernel $I_\alpha(x) = \gamma(\alpha)\ |x|^{\alpha-n}$, $0 < \alpha < n$, (see [**AH**, p. 9]) for the Bessel kernel $G_\alpha(x)$ above, then we write $\dot{C}_{\alpha,p}$ for the Riesz capacity. $\dot{C}_{\alpha,p}$ and $C_{\alpha,p}$ are locally comparable when $\alpha p < n$; see [**AH**, p. 131]. We can also replace the kernel G_α by any of several candidates $g(x,y)$, by replacing the convolution $G_\alpha * f$ by

$$\int_{\mathbb{R}^n} g(x,y) f(y)\, d\mu(y) \tag{2.1}$$

where μ is some Borel measure, say. For details here, the reader is refered to [**AH**, p. 24f].

One important feature of the theory of capacities associated with operators like (2.1) is a so called dual theory; (see [**AH**, p. 34]). For example, the Bessel capacity has the dual formulation:

$$C_{\alpha,p}(K)^{1/p} = \sup\{\mu(K) : \mu \in \mathcal{M}^+(K), \|G_\alpha * \mu\|_{L^{p'}} \le 1\}.$$

Here $\mathcal{M}(K)$ is the dual of $C(K)$, the continuous real valued functions on the compact set K, $\mathcal{M}^+(K)$ the cone of positive measures on K. Also, p' will denote the Hölder conjugate exponent to p throughout,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

If we let F^K and ν^K be the extremals for the two versions of $C_{\alpha,p}(K)$ above, then

$$\|F^K\|_{L^p}^p = C_{\alpha,p}(K), \tag{2.2}$$

$$F^K = (G_\alpha * \mu^K)^{p'-1}, \quad \mu^K = C_{\alpha,p}(K)^{1/p'} \nu^K, \tag{2.3}$$

$$\mu^K(K) = C_{\alpha,p}(K). \tag{2.4}$$

The capacitary potential or equilibrium potential is the function $G_\alpha * F^K = G_\alpha * (G_\alpha * \mu^K)^{p'-1}$. Generally, we refer to the function

$$V^\mu_{\alpha,p}(x) = G_\alpha * (G_\alpha * \mu)^{p'-1}$$

as the nonlinear potential of the measure μ. Note that when $p = 2$, $V^\mu_{\alpha,p}$ becomes the linear potential $G_{2\alpha} * \mu$. The capacitary potential also satisfies:

$$V^{\mu^K}_{\alpha,p} \ge 1, \quad C_{\alpha,p}\text{-a.e. on } K, \tag{2.5}$$

$$V^{\mu^K}_{\alpha,p} \le 1, \quad \text{on support of } \mu^K, \tag{2.6}$$

$$V^{\mu^K}_{\alpha,p} \le A, \quad \text{on all of } \mathbb{R}^n, \tag{2.7}$$

where A is a constant depending only on n, α, p. See [**AH**, p. 21, 40].

Similar statements can be made about the Riesz capacity $\dot{C}_{\alpha,p}$. We will denoted the Riesz nonlinear potentials by $\dot{V}^{\mu}_{\alpha,p}$. But note that $\dot{C}_{\alpha,p}(K) \equiv 0$ for all K when $\alpha p \geq n$ due to the fact that $I_\alpha(x) \notin L^{p'}$ near ∞ when $\alpha p \geq n$. Such, of course, is not the case for $G_\alpha(x)$, it decays exponentially as $|x| \to \infty$; see [**AH**, p. 12].

Finally, we should note an important variant of the nonlinear potentials, the Wolff potentials. These are:

$$W^{\mu}_{\alpha,p}(x) = \int_0^1 [r^{\alpha p - n}\mu(B(x,r))]^{p'-1}\frac{dr}{r}$$

and

$$\dot{W}^{\mu}_{\alpha,p}(x) = \int_0^\infty [r^{\alpha p - n}\mu(B(x,r)]^{p'-1}\frac{dr}{r},$$

the non-homogeneous and homogeneous versions. These potentials are associated to the Bessel and Riesz versions through the celebrated Wolff inequality:

$$\|G_\alpha * \mu\|^{p'}_{L^{p'}} \leq A \int W^{\mu}_{\alpha,p}\, d\mu.$$

See [**AH**, p. 109]. A similar estimate holds for $I_\alpha * \mu$ and $\dot{W}^{\mu}_{\alpha,p}$. Also one can easily show

$$V^{\mu}_{\alpha,p}(x) \geq A W^{\mu}_{\alpha,p}(x)$$

for all x (and similarly for $\dot{V}$ and $\dot{W}$). Hence it follows that

$$\|G_\alpha * \mu\|_{L^{p'}} \sim \int W^{\mu}_{\alpha,p}\, d\mu, \tag{2.8}$$

i.e. the first quantity is bounded by a constant multiple of the second and vice versa.

Note the simple expression

$$\dot{W}^{\mu}_{\alpha,2}(x) = \frac{1}{n-2\alpha}\int |x-y|^{2\alpha - n}\, d\mu(y),$$

for $2\alpha < n$.

Also, due to the equivalence (2.8), one can use the Wolff potentials in the dual formulation of $C_{\alpha,p}$ and then obtain Wolff potential analogues to (2.5)-(2.7) above; see [**AH**, p. 34, 108].

(b) Function spaces and capacity.

Two important function space scales are the Besov scale $B^{p,q}_\alpha = B^{p,q}_\alpha(\mathbb{R}^n)$ and the Lizorkin-Triebel scale $F^{p,q}_\alpha = F^{p,q}_\alpha(\mathbb{R}^n)$; α real and $0 < p, q \le \infty$. The reader is refered to [**AH**, chapter 4], for the necessary definitions and properties of these function spaces. Suffice it to say here that the space $F^{p,2}_\alpha$ coincides with the space of Bessel potentials of $L^p(\mathbb{R}^n)$ functions when $1 < p < \infty$, and the space of Bessel potentials of functions (distributions) belonging to the classical real Hardy spaces $H^p(\mathbb{R}^n)$ when $0 < p \le 1$. $F^{\infty,2}_0$ is the John-Nirenberg space of functions of bounded mean oscillation, BMO, on $\mathbb{R}^n$.

The spaces $B^{p,q}_\alpha$, for $\alpha > 0$, are often refered to in the literature as generalized Lipschitz classes and denoted by $\Lambda^{p,q}_\alpha$; cf. [**St**]. One important connection between these two scales of spaces is the relation: $B^{p,p}_\alpha = F^{p,p}_\alpha$. This will be exploited below in section 4(c).

An interesting idea is to define a potential theory and a theory of capacities based entirely on the Besov and Lizorkin-Triebel spaces. Such capacities are denoted by $C(\cdot; X)$ in [**AH**], (see p. 105), where X can stand for any one of the spaces given above (or others). For example, one can write:

$$C(K; B^{p,q}_\alpha) = \inf\{\|\phi\|^p_{B^{p,q}_\alpha} : \phi \in \mathcal{S},\ \phi \ge 1 \text{ on } K\}$$

where $\mathcal{S}$ denotes the Schwartz class of rapidly decreasing C^∞-functions on $\mathbb{R}^n$: cf. [**AH**, p. 105]. Also, we clearly have

$$C_{\alpha,p}(K) = C(K; F^{p,2}_\alpha).$$

It is a bit more subtle, however, to conclude that

$$C_{\alpha,p}(K) \sim C(K; F^{p,q}_\alpha) \tag{2.3}$$

for all $q \in (0, \infty)$; see [**A2**] and [**Ne2**]. Here again the symbol $\sim$ between two set functions, means that their ratio is bounded above and below by positive finite constants independent of the sets being measured. Relationship (2.3) does not hold, however, for the Besov spaces —i.e. the capacities based on the Besov spaces do depend very much on the second exponent q in $B^{p,q}_\alpha$.

And this brings up an important subject with regard to such multiple scaled collections of capacities: when are two such capacities equivalent (as in (2.3)) or when are they strictly different? Or perhaps there is no relationship between them. This general sorting out of the relationship between these various capacities is sometimes refered to as the "classification problem". See [**A2**], [**Ne1**], [**Ne2**].

Another capacity that we will discuss in detail in section 4 is the Hausdorff capacity (also called Hausdorff content). It is defined by

$$\Lambda_\alpha^{(\infty)}(K) = \inf\left\{\sum_i r_i^\alpha : K \subset \bigcup_{i=1}^\infty B(x_i, r_i)\right\}$$

for $\alpha > 0$, $K \subset \mathbb{R}^n$. An equivalent form can be obtained replacing balls by cubes in $\mathbb{R}^n$. Another equivalent form occurs when we use dyadic cubes to cover the compact K, as in

$$\tilde{\Lambda}_\alpha^{(\infty)}(K) = \inf\left\{\sum_i l_i^\alpha : K \subset \mathop{\cup}_i \tilde{Q}_i\right\}$$

where $\tilde{Q}_i$ are dyadic cubes, i.e. tessellate $\mathbb{R}^n$ with cubes all of whose verticies consist of integral multiples of 2^k, $k = 0, \pm 1, \pm 2, \ldots$. Here l_i is the edge length of Q_i. It is easy to show that $\Lambda_\alpha^{(\infty)} \sim \tilde{\Lambda}_\alpha^{(\infty)}$. Also it should be noted that the sets of zero $\Lambda_\alpha^{(\infty)}$ capacity coincide with the sets of α-dimensional Hausdorff measure zero; see [**AH**, p. 132].

More generally, one can set

$$\Lambda_\alpha^{(\epsilon)}(K) = \inf \sum_i r_i^\alpha$$

where now the infimum is over all countable covers of K, but with radii $r_i \leq \epsilon$. Again, $\Lambda_\alpha^{(\epsilon)} \sim \Lambda_\alpha^{(\infty)}$, for all $\epsilon > 0$. Hausdorff measure is:

$$\lim_{\epsilon \to 0} \Lambda_\alpha^{(\epsilon)}(K) = \Lambda_\alpha(K);$$

see [**Ca1**], or [**AH**, p. 132].

(c) Choquet capacity and capacitability.

So far all the capacities discussed above have been defined only for compact subsets $K \subset \mathbb{R}^n$. It is natural to try to extend them to more general sets. The usual way is to define so called inner and outer capacities as:

$$C^I(E) = \sup\{C(K) : K \subset E,\ K \text{ compact}\}$$

$$C^0(E) = \inf\{C^I(U) : U \supset E,\ U \text{ open}\}.$$

And when these two set functions agree, we say the set E is capacitable and this common value will be the capacity of the set E. Note

$C^0(U) = C^I(U)$ for all open U. And so the question becomes: what conditions on C imply that the capacitable sets contain a significantly large family of sets —say all Borel sets? The answer was given by Choquet [**C**]: if C is a extended real valued set function on all subsets of $\mathbb{R}^n$, such that

$$C(\emptyset) = 0; \tag{2.9}$$

$$E_1 \subset E_2 \Rightarrow C(E_1) \leq C(E_2); \tag{2.10}$$

if K_i is a decreasing of compact sets, then

$$C(\bigcap_i K_i) = \lim_{i\to\infty} C(K_i); \tag{2.11}$$

if E_i is a increasing of sequence of arbitrary sets, then

$$C(\bigcup_i E_i) = \lim_{i\to\infty} C(E_i). \tag{2.12}$$

Then all Suslin sets, and in particular all Borel sets, are capacitable for C.

A set function C that satisfies (2.9)-(2.12) will be refered to as a Choquet capacity. However, not all set functions that we may want to consider satisfy (2.11) or (2.12) —especially (2.11), which is refered to by Choquet as "continuity on the right". Hence it is desirable to have another definition of capacity. We shall say that an extended real valued set function C defined on all subsets of $\mathbb{R}^n$ is a capacity if it satisfies: (2.9), (2.10) together with

$$C(\bigcup_i E_i) \leq \sum_i C(E_i), \tag{2.13}$$

i.e. C is monotone and countably subadditive. Of course, the reader will identify these three conditions with the usual definition of outer measure. In our case, however, most all of the set functions that we call capacities can not be restricted to a nontrivial sigma algebra so as to be an additive measure there. In fact they fail to be, what in measure theory is called, a "metric outer measure". Indeed, the "Newtonian capacity" $\dot{C}_{1,2}$ of $B_r \cup \partial B_1$ is the same as the capacity of ∂B_1 = boundary of B_1; B_r is the ball centered at the origin of radius r, $0 < r \leq 1$, in $\mathbb{R}^n$, $n \geq 3$.

It is well known that $C_{\alpha,p}, \dot{C}_{\alpha,p}$, and $\tilde{\Lambda}_\alpha^{(\infty)}$ are Choquet capacities which satisfy (2.13); see [**AH**, p. 28], [**A3**], [**Fe**].

(d) Choquet integrals.

Assume now that C is a capacity satisfying (2.12) on the given set $\Omega \subset \mathbb{R}^n$. We define the Choquet integral of a non-negative function $f : \Omega \to \mathbb{R}$ by

$$\int_\Omega f\, dC = \int_0^\infty C(\Omega \cap [f > t])\, dt.$$

Here the integral on the right is the usual Lebesgue integral. Indeed, $h(t) = C(\Omega \cap [f > t])$, $t \geq 0$, is upper semi-continuous and hence Lebesgue measureable. Note the following properties:

$$\int_\Omega \alpha f\, dC = \alpha \int_\Omega f\, dC, \quad \alpha \geq 0$$

$$\int_\Omega f\, dC = 0 \leftrightarrow f = 0 \qquad C\text{-a.e. on } \Omega$$

$$\int_\Omega \chi_E\, dC = C(E), \qquad E \subset \Omega,$$

where χ_E is the characteristic function on the set E.

The Choquet integral is not linear. One might, however, hope for sublinearity, i.e.

$$\int_\Omega (f + g)\, dC \leq \int_\Omega f\, dC + \int_\Omega g\, dC.$$

Choquet [**C**] has shown through, that sublinearity is equivalent, to strong subadditivity, i.e. for all E_1, E_2,

$$C(E_1 \cup E_2) + C(E_1 \cap E_2) \leq C(E_1) + C(E_2).$$

Of course, there is a slight difference here between our formulation and that of Choquet. He confines himself to upper semicontinuous integrands f and Choquet capacities C. Our treatment allows a more general function f but at a price: we may be giving up capacitability. See also an alternative approach in [**An**]. The following theorem holds:

Theorem 1. *The Choquet integral for a C satisfying (2.9), (2.10) and (2.12) is sublinear if and only if C is strongly subadditive.*

We will not pursue this point further, for it seems that there are only a few capacities that are actually known to be strongly subadditive. These include $C_{\alpha,p}$ and $\dot{C}_{\alpha,p}$ for $\alpha \leq 1$ as well as $C'_{1,p}$ and dyadic Hausdorff capacity $\tilde{\Lambda}_\alpha^{(\infty)}$. Hence the sublinearity of the Choquet integral will not play much of a role in the following. Note, however, one trivially has:

$$\int_\Omega (f+g)\,dC \le 2\left[\int_\Omega f\,dC + \int_\Omega g\,dC\right]$$

and the constant 2 replaced by N when N-summands are used in the integrand. Thus, although the Choquet integral may fail to define a norm, one can easily eke out a very useful version of Hölder's inequality here, using the above:

$$\int_\Omega fg\,dC \le 2\left(\int_\Omega f^p\,dC\right)^{1/p}\left(\int_\Omega g^{p'}\,dC\right)^{1/p'}.$$

(e) Quasi-additivity.

A capacity C is said to be quasi-additive with respect to a closed set $F \subset \mathbb{R}^n$ if there is a countable tessellation of $\mathbb{R}^n \setminus F = \underset{k}{\cup} M_k$ into nonoverlapping regions such that

$$C(E) \ge A \sum_k C(E \cap M_k)$$

for any $E \subset \mathbb{R}^n \setminus F$. A is a constant independent of E. This, with countable subadditivity, gives

$$C(E) \sim \sum_k C(E \cap M_k)$$

which implies

$$\int_{\underset{k}{\cup} E_k} f\,dC \sim \sum_k \int_{E_k} f\,dC.$$

If $F = \{x_0\}$, we can take the M_k to be annuli $\{2^k \le |x - x_0| < 2^{k+1}\}$, $k = 0, \pm 1, \pm 2, \dots$ with $C = \dot{C}_{\alpha,p}$; see [**A4**]. If F is a set of Minkowski dimension $d < n - \alpha p$, then M_k can be taken to be the Whitney cube decomposition of $\mathbb{R}^n \setminus F$ and again $C = \dot{C}_{\alpha,p}$; see [**Ai**] and [**AE**]. Note that for such a set F, $\dot{C}_{\alpha,p}(F) = 0$. This follows from the usual relations between Hausdorff capacity and Riesz capacity; see [**AH**, chapter 5].

Quasi-additivity has been useful in dealing with the concepts of thinness and the associated fine topology. In particular, it has been used in the classification of the fine topologies associated with the capacities $C_{\alpha,p}$; see [**AH**, 6.5.8]. Also we employ it below in the proof of Proposition 3 of section 6(a) and in Theorem 15 of section 6(b).

3. Spaces of quasi-continuous functions

(a) Quasi-continuity.

Often our interest is in the Choquet integral of a more regular class of functions. A natural such class is the quasi-continuous functions. A function $\phi : \Omega \to \mathbb{R}$ is termed C-quasi-continuous on Ω if given any $\epsilon > 0$ there is a relatively open set $G \subset \Omega$ such that $C(G) < \epsilon$ and the restriction of ϕ to the compliment of G, G^c, is continuous on G^c.

Note that the classical theorem of Egorov implies that any Lebesgue integrable function ϕ on Ω is $\mathcal{L}_n$-quasi-continuous there. Here $\mathcal{L}_n$ is the usual Lebesgue n-measure of subsets of $\Omega \subset \mathbb{R}^n$, Ω measureable. Notice also that the potential $G_\alpha * f$, $f \in L^p$, is $C_{\alpha,p}$-quasi continuous on $\mathbb{R}^n$; see [**AH**, p. 156].

Proposition 1. *If C is an outer capacity on subsets of Ω, an open subset of $\mathbb{R}^n$, (i.e. $C = C^0$), and ϕ_k a sequence of continuous functions on Ω with compact support with* $(p > 0)$

$$\int_\Omega |\phi_k - \phi|^p \, dC \to 0$$

as $k \to \infty$, then ϕ is C-quasi-continuous on Ω.

The proof can be found in [**AH**, Theorem 7.4.2].

We shall say that $\phi \in L^p(C;\Omega)$ if ϕ is the limit of continuous functions with compact support on Ω in the sense of Proposition 1. The space $L^p(C;\Omega)$ will be refered to as a *Choquet space* and will be studied below for various choices of C.

(b) Capacitary strong type inequalities (CSI).

From the definition of $C_{\alpha,p}$, it easily follows that

$$C_{\alpha,p}([G_\alpha * f > t]) \le t^{-p} \int_{\mathbb{R}^n} f(x)^p \, dx$$

for any $f \ge 0$ a.e. on $\mathbb{R}^n$. Such an estimate is often refered to as a “weak type inequality” in analogy to the situation of operators between measureable functions. Here, if we use the Lorentz space notation (see [**SW**]), we are thinking of the operator

$$G_\alpha : L^p \to L^{p,\infty}(C_{\alpha,p}),$$

i.e. L^p into the weak Choquet space. The corresponding "strong type inequality" should be

$$\int_{\mathbb{R}^n} (G_\alpha * f)^p \, dC_{\alpha,p} \le A \int_{\mathbb{R}^n} f(x)^p \, dx, \tag{3.1}$$

again for $f \ge 0$ a.e. and $1 < p < \infty$; A is a constant independent of f. For a proof of (3.1) see [**AH**, p. 187].

The history of such inequalities really begins with Maz'ya in [**Ma1**], where the CSI for first order variational capacity was given; see also [**Ma2**] and [**Ma3**]. Here, for simplicity, we adopt the notation

$$C_p(K) = \inf \left\{ \int |\nabla \phi|^p \, dx : \phi \in C_0^\infty(\mathbb{R}^n),\ \phi \ge 1 \text{ on } K \right\}.$$

The CSI for C_p is

$$\int |u|^p \, dC_p \le A \int |\nabla u|^p \, dx, \tag{3.2}$$

$1 < p < n$. We discuss extensions of (3.2) below. The technique involves the notion of "truncation", i.e. if $u \in W^{1,p}$, then $H \circ u \in W^{1,p}$ whenever H is a Lipschitz function with $H(0) = 0$. An attempt to extend (3.2) to higher order derivatives using "smooth truncation" has only been partially successful, i.e. replacing H by a smoother version. For a more complete account of the background for CSI, see [**AH**, section 7.6.2 and 7.7]. Also section 7.5 of [**AH**] gives an alternative approach to CSI, this time for the Lizorkin-Triebel spaces.

(c) Choquet spaces $L^p(C_{\alpha,p})$.

For $1 < p < \infty$, we see by (3.1) that any $\phi \ge 0$ and $C_{\alpha,p}$-quasi-continuous on $\mathbb{R}^n$, for which there is an $f \in L^p$ such that $\phi \le G_\alpha * f$, $C_{\alpha,p}$-a.e. in $\mathbb{R}^n$, belongs to $L^p(C_{\alpha,p})$. Also, by our earlier remarks about the sublinearity of the Choquet integral, we do not know whether or not the quantity

$$\left(\int_{\mathbb{R}^n} |f|^p \, dC_{\alpha,p} \right)^{1/p} \tag{3.3}$$

always defines a norm on $L^p(C_{\alpha,p})$. Note though that when $\alpha p > n$,

$$L^p(C_{\alpha,p}) \subset L^\infty(C_{\alpha,p})$$

and then (3.3) is just the $C_{\alpha,p}$-ess. sup. norm. This is because $C_{\alpha,p} > 0$ for all sets $\ne \emptyset$ when $\alpha p > n$.

Two questions arise:

(i) can we characterize the membership in $L^p(C_{\alpha,p})$ by the above domination principle, and,

(ii) is it possible to norm the space $L^p(C_{\alpha,p})$?

To answer both of these questions, we introduce a functional capacity

$$\Gamma_{\alpha,p}(u) = \inf\{\|f\|_{L^p}^p : f \in \mathbb{K}_u\}$$

where $\mathbb{K}_u = \{f \in L^p : f \geq 0,\ G_\alpha * f \geq |u| \text{ in } \mathbb{R}^n\}$. Clearly, $\Gamma_{\alpha,p}(\chi_K) = C_{\alpha,p}(K)$. We also have:

Theorem 2. *The functional $\Gamma_{\alpha,p}(\cdot)^{1/p}$ defines a norm on $C_0(\mathbb{R}^n)$, the continuous functions with compact support, and there exists a constant A such that*

$$\frac{1}{4}\Gamma_{\alpha,p}(u) \leq \int |u|^p\, dC_{\alpha,p} \leq A\Gamma_{\alpha,p}(u). \tag{3.4}$$

For a proof of Theorem 2, see [**AH**, p. 199].

Thus we see that if $\phi \in L^p(C_{\alpha,p})$, then there must exist an $f \in L^p$ such that $G_\alpha * f \geq |\phi|$, $C_{\alpha,p}$-a.e., and the domination principle is a characterization of membership in $L^p(C_{\alpha,p})$. Further, with Theorem 2, it follows that $L^p(C_{\alpha,p})$ is a Banach space.

Duality: The dual of the space $L^p(C_{\alpha,p})$ is known:

Theorem 3. *The dual of the space $L^p(C_{\alpha,p})$, $1 < p < \infty$, can be identified with the space of all $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that $G_\alpha * |\mu| \in L^{p'}$. Further, if $u \in L^p(C_{\alpha,p})$ and $\mu \in L^p(C_{\alpha,p})^*$, then $u \in L^1(|\mu|)$ and duality is given by*

$$\langle \mu, u\rangle = \int_{\mathbb{R}^n} u\, d\mu.$$

The norm of μ is

$$\|\mu\| = \|G_\alpha * |\mu|\ \|_{L^{p'}}.$$

For the proof of Theorem 3, see [**AH**, p. 202].

Embedding: The space of Bessel potentials can be embedded into the Choquet-Lorentz spaces. Such a result is given by:

Theorem 4. *(i) The map* $G_\alpha : L^p \to L^{p,q}(C_{\alpha,p})$ *is continuous for* $1 < p \le q \le \infty$, *i.e.*

$$\left\{\int_0^\infty (t^p C_{\alpha,p}[G_\alpha * f > t])^{q/p} \frac{dt}{t}\right\}^{1/q} \le A\|f\|_{L^p}$$

for $q < \infty$, *with the usual modifications for* $q = \infty$.

(ii) The map $G_\alpha : L^p \to L^{p,q}(C_{\alpha,r})$ *is compact for* $1 < r < p \le q \le \infty$, $\alpha > 0$.

Proof: (i) follows easily from the weak and strong inequalities mentioned above. Note that $L^{p,q}(C_{\alpha,p})$ is a Choquet space of the Lorentz type; see [**SW**]. For the proof of (ii) we follow the main ideas of [**AP**]. We will consider only the case $\alpha = m =$ positive integer. The reader is refered to [**RS**, p. 363] for the necessary results to make the general case work. The main idea is to first use the equivalent Sobolev norm and show

$$\|(G_m * f)^{p/r}\|_{W^{m,r}} \le A(\|G_m * f\|_{L^p}^{p/r-1}\|f\|_{L^p} + \|G_m * f\|_{L^p}^{p/r}). \tag{3.5}$$

Then, if $\{f_k\}$ is a sequence in L^p that tends weakly to zero, there is a subsequence $\{f_{k'}\}$ such that $G_m * f_{k'}$ tends to zero strongly in $L^{p,q}(C_{m,r})$, $r < p$, since

$$\begin{aligned}\|G_m * f\|_{L^{p,q}(C_{m,r})}^q &= \int_0^\infty \left(s^r C_{m,r}[G_m * f \ge s^{r/p}]\right)^{\frac{qr/p}{r}} \frac{dt}{t}\\ &\le A\|(G_m * f)^{p/r}\|_{W^{m,r}}^{qr/p}.\end{aligned}$$

Now just use (3.5).

To see (3.5), we compute

$$|D^m(G_m * f)^{p/r}| \le A(G_m * f)^{p/r-1}(|Kf| + Mf)$$

via the celebrated lemma of Hedberg [**He1**]: $0 < \theta < 1$,

$$I_{\alpha\theta} * f \le A(I_\alpha * f)^\theta (Mf)^{1-\theta}.$$

Here Kf is a Calderon-Zygmund singular integral operator and Mf is the usual Hardy-Littlewood maximal function. Thus (3.5) follows from the L^p boundedness of the operators K and M and Hölder's inequality. ■

One can also obtain a compact embedding result

$$G_\alpha : L^p \to L^r(C_{\alpha,r}), \quad r < p$$

from the above by multiplying $G_\alpha * f$ by a smooth cutoff function with compact support and estimating the part near infinity uniformly in f. See [**AP**], where this is done for bounded domains $\Omega \subset \mathbb{R}^n$, instead, as here, for the whole space $\mathbb{R}^n$.

One can also translate the results of Theorem 4 into continuous and compact embeddings of Bessel potentials (and hence Sobolev functions) into Lebesgue spaces taken with respect to a Borel measure μ supported somewhere in $\mathbb{R}^n$. The usual condition is

$$\mu(K) \le C_{\alpha,p}(K)^{q/p} \tag{3.6}$$

for all compact sets $K \subset \mathbb{R}^n$. This gives,

$$G_\alpha : L^p \to L^q(\mu)$$

is continuous for $1 < p \le q < \infty$. When $q > p$ condition (3.6) can be weakened to require only that K be merely a closed ball, any center and any radius ≤ 1. See [**AH**, section 7.2]. Also, the reader should be aware that there is a sharp compact embedding result into $L^q(\mu)$, in [**AH**, section 7.3], that does not follow from Theorem 4(ii).

(d) Choquet spaces $L^q(C_{\alpha,p})$, $q \ne p$.

Because the CSI seems to be so closely related to the space $L^p(C_{\alpha,p})$ —and even leads to a method of norming it— it is an interesting question to find an analogue of CSI for $L^q(C_{\alpha,p})$, $q \ne p$, that at least gives some sort of domination principle for membership, as we have noted earlier for the case $q = p$.

We begin with the variational capacity C_p rewritten as

$$C_p(K) = \inf\left\{\int |\nabla\phi|^p\,dx : \phi \in \dot{W}^{1,p}(\mathbb{R}^n),\ \phi \ge \chi_K,\ C_p\text{-a.e.}\right\}.$$

Here $\dot{W}^{1,p}(\mathbb{R}^n)$ is the homogeneous Sobolev space normed by

$$\|\phi\|_{L^{p^*}} + \|\nabla\phi\|_{L^p}$$

$p^* = np/(n-p)$, $1 \le p < n$. Modifying this a bit, we define

$$C_p^q(K) = \inf\left\{\int |\nabla\phi|^p\phi^{q-p}\,dx : \phi^{q/p} \in \dot{W}^{1,p}(\mathbb{R}^n),\ \phi \ge \chi_K,\ C_p\text{-a.e.}\right\}.$$

Also, for $\psi \ge 0$, set

$$\Gamma_p^q(\psi) = \inf\left\{\int |\nabla\phi|^p\phi^{q-p}\,dx : \phi^{q/p} \in \dot{W}^{1,p}(\mathbb{R}^n),\ \phi \ge \psi,\ C_p\text{-a.e.}\right\}.$$

Note that $C_p^p(K) = C_p(K)$, $C_p(K) \sim \dot{C}_{1,p}(K)$, and $\Gamma_p^q(\chi_K) = C_p^q(K)$, and that

$$\left(\int |\psi|^q\,dC_p\right)^{1/q}$$

is a norm for $1 \le q < \infty$ since C_p is strongly subadditive. We now have:

Theorem 5. *The following equivalences hold, for $1 < p < n$ and all $q > 0$:*

$$C_p(K) \sim C_p^q(K), \tag{3.7}$$

$$\Gamma_p^q(\psi) \sim \int \psi^q \, dC_p. \tag{3.8}$$

Proof: We first note that

$$C_p^q(K) \le A\dot{C}_{1,p}(K).$$

Let $\phi = (I_1 * f)^{p/q}$, $f =$ capacitary extremal for $\dot{C}_{1,p}(K)$. Then

$$|\nabla\phi| = \frac{p}{q}(I_1 * f)^{p/q-1}|\nabla I_1 * f|,$$

hence

$$\int |\nabla\phi|^p \phi^{q-p} \, dx \le \left(\tfrac{p}{q}\right)^p \int |\nabla I_1 * f|^p \, dx \le A \int f^p \, dx = A\dot{C}_{1,p}(K).$$

Next, we show

$$C_p(K) \le A \cdot C_p^q(K).$$

This proof gives the basics for the CSI (3.2). In fact, it gives

$$\int \psi^q \, dC_p \le A \cdot \Gamma_p^q(\psi).$$

Take $\phi \ge \psi$, $\phi^{q/p} \in \dot{W}^{1,p}$, then

$$\begin{aligned}
\int \phi^q \, dC_p &\le (2^q - 1) \sum_k C_p([\phi > 2^k]) 2^{kq} \\
&\le (2^q - 1) \sum_k \int |H'\left(\frac{\phi}{2^k}\right)|^p |\nabla\phi|^p 2^{k(q-p)} \, dx \\
&= (2^q - 1) 2^p \sum_k \int_{[2^k < \phi < 2^{k+1}]} |\nabla\phi|^p 2^{k(q-p)} \, dx \\
&\le A \int |\nabla\phi|^p \phi^{q-p} \, dx
\end{aligned}$$

where $H(t)$ is the Lipschitz function

$$\begin{cases} 0, & t \leq 1/2 \\ 2t-1, & 1/2 < t < 1 \\ 1, & t \geq 1. \end{cases}$$

Thus it remains to show

$$\Gamma_p^q(\psi) \leq A \cdot \int \psi^q \, dC_p.$$

Without loss of generality, we can assume that the right side is finite. Then $\psi < \infty$, C_p-a.e., hence

$$\Gamma_p^q(\psi) = \Gamma_p^q(\psi \cdot \chi_{\cup A_k})$$

where $A_k = [2^k \leq \psi < 2^{k+1}]$, $k = 0, \pm 1, \pm 2, \dots$. We now show

$$\Gamma_p^q(\psi \cdot \chi_{\cup A_k}) \leq \sum_k \Gamma_p^q(\psi \chi_{A_k}), \tag{3.9}$$

$$\Gamma_p^q(\psi \chi_{A_k}) \leq 2^{(k+1)q} C_p^q(A_k). \tag{3.10}$$

With these, we argue as follows:

$$\Gamma_p^q(\psi) \leq A \sum_k 2^{kq} C_p(A_k) \leq A \int \psi^q \, dC_p$$

using (3.7).

For (3.9) let $\phi_k \geq \psi \chi_{A_k}$, C_p-a.e. and set $\Phi = \sup_k \phi_k$. Then $\Phi \geq \psi \chi_{\cup A_k}$, C_p-a.e. and

$$\begin{aligned} \int |\nabla \Phi|^p \Phi^{q-p} \, dx &\leq \sum_k \int_{[\Phi = \phi_k]} |\nabla \Phi|^p \Phi^{q-p} \, dx \\ &= \sum_k \int_{[\Phi = \phi_k]} |\nabla \phi_k|^p \phi_k^{q-p} \, dx \\ &\leq \sum_k \int |\nabla \phi_k|^p \phi_k^{q-p} \, dx. \end{aligned}$$

The proof of (3.10) is just a homogeneity argument. ■

The results of Theorem 5 suggest that the right CSI for the spaces $L^q(\dot{C}_{\alpha,p})$ might be

$$\int (I_\alpha * f)^q \, d\dot{C}_{\alpha,p} \leq A \int f^p (I_\alpha * f)^{q-p} \, dx. \tag{3.11}$$

Indeed, according to the above argument, we could get

$$\dot{\Gamma}^q_{\alpha,p}(I_\alpha * f) \leq A \cdot \int (I_\alpha * f)^q \, d\dot{C}_{\alpha,p}$$

provided we can show

$$\dot{C}^q_{\alpha,p}(K) \leq A \cdot \dot{C}_{\alpha,p}(K). \tag{3.12}$$

Here, we have set

$$\dot{\Gamma}^q_{\alpha,p}(\psi) = \inf \left\{ \int f^p \cdot (I_\alpha * f)^{q-p} \, dx : f \geq 0, \, I_\alpha * f \geq \psi, \, \dot{C}_{\alpha,p}\text{-a.e.} \right\}$$

and $\dot{C}^q_{\alpha,p}(K) = \dot{\Gamma}^q_{\alpha,p}(\chi_K)$. Indeed, for any smooth non-negative function ϕ of compact support,

$$\phi \leq A \cdot I_m * |D^m \phi|$$

hence

$$\phi^{q-p} \geq A (I_m * |D^m \phi|)^{q-p}$$

for $q \leq p$. Thus

$$\int |D^m \phi|^p \phi^{q-p} \, dx \geq A \cdot \dot{C}^q_{m,p}(K).$$

But if we take $\phi = (I_m * f)^{p/q}$, where f is the $\dot{C}_{m,p}(K)$ capacity extremal, then

$$\int |D^m \phi|^p \phi^{q-p} \, dx \leq A \int (I_m * f)^{(p/q-1)p} (|Kf| + Mf)(I_m * f)^{p/q(q-p)} \, dx$$

$$\leq A \int f^p \, dx = A \cdot C_{m,p}(K)$$

by our earlier argument using the Hedberg lemma. Thus (3.12) follows for $q \leq p$.

This suggests that (3.11) might be correct. The following verifies the conjecture for $1 \le q < p + \frac{n}{n-m}$, $m < n$.

$$\begin{aligned}\int (I_m * f)^q \, d\dot{C}_{m,p} &= \int [(I_m * f)^{q/p}]^p \, d\dot{C}_{m,p} \\ &\le A \int |D^m (I_m * f)^{q/p}|^p \, dx \\ &\le A \int (|Kf| + Mf)^p (I_m * f)^{q-p} \, dx.\end{aligned}$$

So what is needed here is for $(I_m * f)^{q-p}$ to be an A_p-weight. This is true for $1 \le q \le p$ by the argument of [**APe**]. Furthermore, one can easily verify that $(I_m * f)^{q-p} \in A_1$ for $p < q < \frac{n}{n-m}$. Note that we have shown

$$\int \psi^q \, d\dot{C}_{\alpha,p} \sim \dot{\Gamma}^q_{\alpha,p}(\psi)$$

for $1 \le q \le p$.

(e) An L^q-integration principle for potentials.

In potential theory, whether we are dealing with Riesz potentials, Bessel potentials or even Green potentials —of a positive measure with compact support in $\mathbb{R}^n$— there are several basic principles that allow one to think that the worst possible behavior of such potentials actually occurs on the support of the measure. Below, we shall look only at Riesz potentials for simplicity, but similar things can be said about other potentials of positive measures. So, for $0 < \alpha < n$ and μ a positive measure with compact support, consider the Riesz potential $I_\alpha * \mu(x)$. The Boundedness Principle (BP) can be stated as:

(BP) If $I_\alpha * \mu(x)$ is bounded by A for x in the support of μ, then $I_\alpha * \mu(x)$ is bounded for all x in $\mathbb{R}^n$ (by $2^{n-\alpha} A$). Note that the BP applies as well to the Wolff potentials.

The Continuity Principle (CP) states:

(CP) If $I_\alpha * \mu(x)$ is continuous for x restricted to the support of μ, then $I_\alpha * \mu(x)$ is continuous for all x in $\mathbb{R}^n$.

The (CP) is sometimes called the Evans-Vasillesco continuity principle; see [**H**]. There are also higher order vesions —Hölder continuity and Hölder continuity of the first derativatives; for a excellent presentation of this, see [**CK**].

However, in this section, we are interested in a lower order version of these ideas, a so called L^q-Integration Principle. Roughly, it should state:

(IP) If $I_\alpha * \mu(x)$ is L^q-integrable on the support of μ, then it is in L^q on the whole space.

But the difficulty here is what to use as the integrating measure? Ideally one would like to take some connonical measure on $\mathbb{R}^n$. But Lebesgue n-measure will not work here since the support of μ could have Lebesgue n-measure zero. Lower dimensional Hausdorff measure may be finite on the support of μ, but it is well known to be infinite on balls in $\mathbb{R}^n$, where the potential may in fact be very well behaved. Hence, it would seem natural to try some sort of capacity here, Hausdorff or potential theoretic, and interpret L^q-integration as Choquet integration. The Integration Principle can be formally stated as:

Theorem 6. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ with compact support and such that μ is absolutely continuous with respect to the capacity $\dot{C}_{\alpha,p}$ (i.e. $\dot{C}_{\alpha,p}(e) = 0$ implies $\mu(e) = 0$ for e Borel: $\mu \ll \dot{C}_{\alpha,p}$). If $0 < p-1 < q < \infty$, then there is a constant $A = A(\alpha, p, n, q)$ such that*

$$\int_{\mathbb{R}^n} (\dot{W}^\mu_{\alpha,p})^q \, d\dot{C}_{\alpha,p} \leq A \int_{S_\mu} (\dot{W}^\mu_{\alpha,p})^q \, d\dot{C}_{\alpha,p}.$$

Furthermore, if $q = p-1 > 0$,

$$\dot{C}_{\alpha,p}([\dot{W}^\mu_{\alpha,p} > t]) \leq \frac{A}{t^{p-1}} \int_{S_\mu} (\dot{W}^\mu_{\alpha,p})^{p-1} \, d\dot{C}_{\alpha,p}.$$

S_μ = support of μ.

Here $\dot{W}^\mu_{\alpha,p}$ is the homogeneous Wolff potential and, when $p = 2$, this includes the case of Riesz potentials $I_\alpha * \mu$ as noted earlier. Other versions of Theorem 6 can be given that include weighted capacity (see section 6(b)) or the various versions of nonlinear potentials for other capacities, i.e. those associated to other function spaces.

For a proof of Theorem 6, we need two facts from potential theory:

$$\begin{aligned} &\dot{C}_{\alpha,p}(K) \geq A \cdot \mu(K), \text{ for any } \mu \in \mathcal{M}^+ \text{ such that} \\ &\dot{W}^\mu_{\alpha,p}(x) \leq 1 \text{ on support of } \mu. \end{aligned} \tag{3.13}$$

(See [**AH**, Theorem 2.5.5]. Here we have freely used the fact that an equivalent $\dot{C}_{\alpha,p}(K)$ can be obtained by replacing $\int (I_\alpha * \mu)^{p'}\,dx$ by $\int \dot{W}^\mu_{\alpha,p}\,d\mu$ in the dual definition, via Wolff's inequality.)

$$\dot{C}_{\alpha,p}([\dot{W}^\mu_{\alpha,p} > t]) \le A\mu(\mathbb{R}^n)\cdot t^{1-p}. \tag{3.14}$$

(See [**AH**, Proposition 6.3.12].)

Lemma 1. *Let $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ with $\mu \ll \dot{C}_{\alpha,p}$. Then there is a constant $A = A(\alpha, p, n)$ such that*

$$\mu(e) \le A \int_e (\dot{W}^\mu_{\alpha,p})^{p-1}\,d\dot{C}_{\alpha,p},$$

for all Borel sets e.

Proof: Set $E_k = [2^k \le (\dot{W}^\mu_{\alpha,p})^{p-1} < 2^{k+1}] \cap e$ and

$$\mu_k = 2^{-k-1}\mu^{E_k}$$

i.e. μ restricted to the set E_k times 2^{-k-1}. Then

$$(\dot{W}^{\mu_k}_{\alpha,p})^{p-1} \le 2^{-k-1}(\dot{W}^\mu_{\alpha,p})^{p-1} \le 1$$

on E_k. Thus by (3.13) above,

$$\dot{C}_{\alpha,p}(E_k) \ge A\mu(E_k)2^{-k}.$$

By (3.14), we have $\dot{C}_{\alpha,p}[\dot{W}^\mu_{\alpha,p} = +\infty] = 0$, so $\mu[\dot{W}^\mu_{\alpha,p} = +\infty] = 0$ by hypothesis. Hence

$$\begin{aligned}\mu(e) = \sum_k \mu(E_k) &\le A\sum_\alpha 2^k \dot{C}_{\alpha,p}(E_k)\\ &\le A\sum_k \dot{C}_{\alpha,p}(E \cap [(\dot{W}^\mu_{\alpha,p})^{p-1} \ge 2^k])2^k\\ &\le A\int_0^\infty \dot{C}_{\alpha,p}(e \cap [(\dot{W}^\mu_{\alpha,p})^{p-1} > t])\,dt. \ \blacksquare\end{aligned}$$

Proof of Theorem 6:

Step 1: We first show

$$\int (\dot{W}_{\alpha,p})^q \, d\dot{C}_{\alpha,p} \le A \int (\dot{W}^{\mu}_{\alpha,p})^{q-p+1} \, d\mu$$

for $q > p-1$.

To see this set $\mu^t = \mu^{G_t}$, where $G_t = [\dot{W}^{\mu}_{\alpha,p} > t/2]$ and $\mu_t = \mu - \mu^t$. Note $\dot{W}^{\mu_t}_{\alpha,p}(x) \le t/2$ on S_{μ_t}. The Boundedness Principle applied to $\dot{W}^{\mu_t}_{\alpha,p}$ yields: $\dot{W}^{\mu_t}_{\alpha,p}(x) \le At$ for all x, some constant A. Hence by (3.14), there are constants A_1 and A_2 such that

$$\dot{C}_{\alpha,p}[W^{\mu}_{\alpha,p} > A_1 t] \le A_2 \mu(G_t) \cdot t^{1-p}.$$

Consequently,

$$\int (\dot{W}^{\mu}_{\alpha,p})^q \, d\dot{C}_{\alpha,p} \le A \int_0^{\infty} t^{1-p} \int_{[\dot{W}^{\mu}_{\alpha,p} > t]} d\mu dt^q = A \int (\dot{W}^{\mu}_{\alpha,p})^{q-p+1} \, d\mu.$$

Step 2: We show now that

$$\int (\dot{W}^{\mu}_{\alpha,p})^{q-p+1} \, d\mu \le A \int_{S_\mu} (\dot{W}^{\mu}_{\alpha,p})^q \, d\dot{C}_{\alpha,p}.$$

Indeed, from Lemma 1, it follows that

$$\begin{aligned} &\int_0^{\infty} \mu[\dot{W}^{\mu}_{\alpha,p} > t] \, dt^{q-p+1} \\ &\quad \le A \int_0^{\infty} \int_0^{\infty} \dot{C}_{\alpha,p}(S_\mu \cap [\dot{W}^{\mu}_{\alpha,p} > s] \cap [\dot{W}^{\mu}_{\alpha,p} > t]) \, ds^{p-1} \, dt^{q-p+1} \\ &\qquad\qquad = A' \int_0^{\infty} \dot{C}_{\alpha,p}(S_\mu \cap [\dot{W}^{\mu}_{\alpha,p} > t]) \, dt^q. \end{aligned}$$

Finally, we note that the weak type inequality, for $q = p-1$ of Theorem 6, is just a combination of Lemma 1 (with $e = S_\mu$) and (3.14) above. ■

4. Hausdorff capacity

(a) A capacity for $p = 1$.

So far we have considered capacities only for $p > 1$, Bessel, Riesz and variational capacities. Of course, there are also the function space capacities $C(\cdot; X)$ as mentioned earlier where X might be a Besov or Lizorkin-Triebel space. In this latter setup, especially with regard to the Lizorkin-Triebel spaces $F_\alpha^{p,q}$, where $F_\alpha^{p,2}$ coincides with the space of Bessel potentials of L^p-functions, $1 < p < \infty$, it is of interest to identify $F_\alpha^{1,2}$. As noted earlier, it can be characterized as the space of Bessel potentials of $f \in H^1(\mathbb{R}^n)$, the real Hardy space.

$$H^1(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n) : R_j * f \in L^1(\mathbb{R}^n),\ j = 1, \dots , n\}$$

where R_j is the j^{th} Riesz transform, i.e. the singular integral operator $f \to \dfrac{x_j}{|x|^{n+1}} * f$. Similarly, $\dot{F}_\alpha^{1,2}$ is the space of Riesz potentials of $f \in H^1(\mathbb{R}^n)$. What makes these cases so interesting is that with the machinery the H^1-BMO duality, one can show (see [**A3**]) that the capacities $C_{\alpha,1}$ and $\dot{C}_{\alpha,1}$ are equivalent to $\Lambda_{n-\alpha}^{(1)}$ and $\Lambda_{n-\alpha}^{(\infty)}$, respectively. In particular, they have the same null sets.

Also, set

$$\Gamma_{\alpha,1}(\psi) = \inf\{\|f\|_{H^1} : f \in \mathcal{S}_{00},\ I_\alpha * f \geq \psi \text{ on support of } \psi\},$$

where $\|f\|_{H^1} = \|f\|_{L^1} + \sum_k \|R_j * f\|_{L^1}$ and ψ is a non-negative bounded function with compact support. $\mathcal{S}_{00}$ is the subclass of the Schwartz rapidly decreasing C^∞-functions on $\mathbb{R}^n$ such that the Fourier transform has compact support disjoint from the origin; see [**St2**]. Then we have an analogue of the Choquet integral characterization for $p > 1$, namely

$$\dot{\Gamma}_{\alpha,1}(\psi) \sim \int \psi \, d\Lambda_{n-\alpha}^{(\infty)},$$

for $0 < \alpha < n$; see [**A3**].

It appears to be a bit easier to approach the case $p = 1$ through variational capacity, i.e. something like $C'_{m,1}$ (using our earlier notation from section 1) or what we might call $\dot{C}'_{m,1}$ by replacing $\|\phi\|_{W^{m,1}}$ by

$$\int_{\mathbb{R}^n} |D^m \phi| \, dx,$$

i.e. all derivatives of order m on ϕ only. But again one can show that $\dot{C}'_{m,1}$ is just $\Lambda_{n-m}^{(\infty)}$; see [**T**] for a nice account of this approach.

Since $\tilde{\Lambda}^{(\infty)}_{n-\alpha}$, the dyadic Hausdorff capacity, is strongly subadditive,

$$\left[\int |\psi|^q \, d\tilde{\Lambda}^{(\infty)}_{n-\alpha}\right]^{1/q}$$

is a norm for $q \geq 1$. However, what is not yet developed is a characterization of membership via a domination principle as we have discuss earlier in the cases $p > 1$. Perhaps via the variational capacity methods one can determine such a principle using the integral $\displaystyle\int |D^m\phi|\phi^{q-1}\,dx$. See section 3(d) above.

(b) Maximal functions.

Here we use the Choquet integral with respect to Hausdorff capacity to extend some well known estimates for various maximal operators. For $0 \leq \alpha < n$, set

$$M_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |f(y)|\,dy,$$

the fractional maximal function of f of order α. The barred integral denotes the usual integral average, in this case, over the ball $B(x,r)$. Notice that when $\alpha = 0$, $M_0 = M$, the usual Hardy-Littlewood maximal operator. Our interest here lies with

Theorem 7. *Let $0 < d \leq n$, $0 \leq \alpha < n$.*

(a) For $p \leq q$,

(i) *let $d/n < p < d/\alpha$ and set $\delta = q(d-\alpha p)/p$, then there is a constant A_1 (independent of f) such that*

$$\|M_\alpha f\|_{L^{q,p}(\Lambda^{(\infty)}_\delta)} \leq A_1 \|f\|_{L^p(\Lambda^{(\infty)}_d)}; \tag{4.1}$$

(ii) *for $p = d/n$, there is a constant A_2 such that*

$$\|M_\alpha f\|_{L^{q,\infty}(\Lambda^{(\infty)}_\delta)} \leq A_2 \|f\|_{L^{d/n}(\Lambda^{(\infty)}_d)}, \tag{4.2}$$

with $q = \delta/(n-\alpha)$, $\delta \geq \dfrac{d}{n}(n-\alpha)$;

(iii) *for $p = d/\alpha$, there is a constant A_3 such that*

$$\|M_\alpha f\|_{L^\infty} \leq A_3 \|f\|_{L^{d/\alpha}(\Lambda^{(\infty)}_d)}. \tag{4.3}$$

(b) For $q < p$, there is a constant A_4 such that

$$\|M_\alpha f\|_{L^q(wd\Lambda^{(\infty)}_{d-\alpha p})} \le A_4 \|f\|_{L^p(\Lambda^{(\infty)}_d)} \tag{4.4}$$

if and only if

$$\|w\|_{L^{p/(p-q)}(\Lambda^{(\infty)}_{d-\alpha p})} < \infty \tag{4.5}$$

when $\dfrac{d}{n} < p < \dfrac{d}{\alpha}$. Estimates similar to parts (ii) and (iii) of part (a) also hold here.

In the above, we are employing the Choquet-Lorentz space notation as used earlier in Theorem 4. Note that the meaning of the left side of (4.4) is

$$\int (M_\alpha f)^q w \, d\Lambda^{(\infty)}_{d-\alpha p} = \int_0^\infty \Lambda^{(\infty)}_{d-\alpha p}([M_\alpha f)^q \cdot w > t]) \, dt.$$

Some special cases, including when $d = n$ and/or $\alpha = 0$, occur in the literature. For example, when $d = n$, (4.1) contains a result originally due to E. Sawyer [**Sa**], though not phrased there interms of Choquet integrals. In [**A5**], it was stated simply as

$$\int (M_\alpha f)^p \, d\Lambda^{(\infty)}_{n-\alpha p} \le A\|f\|^p_{L^p}$$

i.e. for $q = p$ and $d = n$; here $1 < p < n/\alpha$. The special case $p = 1$ here is a consequence of the simple Vitali covering lemma; see [**AH**, Theorem 1.4.1]. This gives (4.2) with $\delta = n - \alpha$, $d = n$. See also [**BZ**]. The case $p = n/\alpha$, $d = n$ is just a simple consequence of Hölder inequality.

When $\alpha = 0$ and $d = n$, (4.1) is the classical estimate of Hardy-Littlewood. But when $\alpha = 0$ and $d < n$, the result is a recent one due to Orobitg-Verdera [**OV**]. Their result came out of their effort to understand the special case $\alpha = 0$, $d < n$, and $p = 1$ that first appeared in [**A3**] —a result of the H^1-BMO duality theory applied to the characterization of the Riesz capacities $\dot{C}_{\alpha,1}$, as discussed above. The Orobitg-Verdera proof is a modification of arguments due to Carleson [**Ca2**] and Hormander [**Ho**]. The key to all of these is a covering lemma. The version we need is also the one employed by Orobitg-Verdera, and is due to Melnikov [**Me**].

Lemma 2. *Let $\{Q_j\}$ be a family of non-overlapping dyadic cubes. There exists a subfamily $\{Q_{jk}\}$ such that*

(i) $\sum_{Q_{jk}\subset Q} l(Q_{jk})^d \le 2l(Q)$, *$Q$ dyadic,*

(ii) $\Lambda_d^{(\infty)}(\cup_j Q_j) \le 2\sum_k l(Q_{jk})^d$

for $0 < d \le n$ and $l(Q)$ the edge length of Q.

The method of proof of Theorem 7 follows [**OV**], with minor modifications. It is important there to work first with the dyadic version of $M_\alpha f$,

$$\tilde{M}_\alpha f(x) = \sup_{Q\ni x} l(Q)^\alpha \fint_Q |f(y)|\, dy$$

where Q is a dyadic cube containing x. Then to get to $M_\alpha f$ one needs the obvious $\tilde{M}_\alpha f(x) \le A \cdot M_\alpha f(x)$ for some constant A, and from [**GF**, p. 136]:

$$\Lambda_d^{(\infty)}([M_\alpha f > t]) \le A_1 \Lambda_d^{(\infty)}([\tilde{M}_\alpha f > A_2 \cdot t])$$

for some constants A_1 and A_2.

We modify the proof in [**OV**] with

Lemma 3. *Let χ_Q be the characteric function of the cube Q, then*

$$\|M_\alpha(\chi_Q)\|^p_{L^{q,p}(\Lambda_\delta^{(\infty)})} \le A \cdot l(Q)^d$$

for some constant $A = A(\alpha, p, d, n)$.

In fact, what is needed is the computation:

$$M_\alpha(\chi_Q)(x) \le \begin{cases} A \cdot l(Q)^n |x - x_Q|^{\alpha - n}, & |x - x_Q| > 2l(Q) \\ A \cdot l(Q)^\alpha, & |x - x_Q| \le 2l(Q) \end{cases}$$

where x_Q is the center of Q.

And finally, we need the following inequality to pass from integration with Lebesgue n-measure to the Hausdorff capacities:

$$\int f(x)\, dx \le \frac{n}{d} \left(\int f^{d/n}\, d\Lambda_d^{(\infty)} \right)^{n/d}$$

for $0 < d \le n$; see [**OV**]. In fact, it is clear that something like this is necessary to prove (4.3).

To see (4.4) first apply Hölder's inequality to the left side, reducing it to the desired integral in w and an application of the case (4.1) with $q = p$. For the converse, we argue in much the same way as we did when characterizing membership in the Choquet spaces using the functionals Γ. We begin by setting

$$H^d_{\alpha,p}(K) = \inf\left\{\int f^p \, d\Lambda^{(\infty)}_d : M_\alpha f \ge \chi_K\right\}.$$

We first note that $H^d_{\alpha,p} \sim \Lambda^{(\infty)}_{d-\alpha p}$ for $\alpha p < d$. Indeed, since $M_\alpha \chi_{B(x,r)} \sim r^\alpha$ on $B(x,r)$,

$$H^d_{\alpha,p}(B(x,r)) \sim r^{d-\alpha p},$$

hence $H^d_{\alpha,p} \le A \cdot \Lambda^{(\infty)}_{d-\alpha p}$. For the other direction, we see from, (4.1) that

$$\Lambda^{(\infty)}_{d-\alpha p}([M_\alpha f > t]) \le A t^{-p} \|f\|^p_{L^p(\Lambda^{(\infty)}_d)},$$

hence $\Lambda^{(\infty)}_{d-\alpha p} \le A \cdot H^d_{\alpha,p}$.

Now set

$$\xi^d_{\alpha,p}(\psi) = \inf\left\{\int f^p \, d\Lambda^{(\infty)}_d : M_\alpha f \ge \psi\right\}.$$

Then as before, one easily argues that

$$\xi^d_{\alpha,p}(\psi) \le A \cdot \int \psi^p \, d\Lambda^{(\infty)}_{d-\alpha p}.$$

Consequently, if (4.4) holds, then take f such that $M_\alpha f \ge w^{1/(p-q)}$. This gives

$$\int w^{p/(p-q)} \, d\Lambda^{(\infty)}_{d-\alpha p} \le A_4 \xi^d_{\alpha,p}(w^{1/(p-q)})^{q/p}$$

which gives the result. ∎

(c) Bessel and Riesz capacity for $p < 1$.

In section 3(a), we noted that

$$C_{\alpha,1}(K) = C(K; F^{1,2}_\alpha) \sim \Lambda^{(1)}_{n-\alpha}(K)$$

and

$$\dot{C}_{\alpha,1}(K) = C(K; \dot{F}^{1,2}_\alpha) \sim \Lambda^{(\infty)}_{n-\alpha}(K).$$

Here we wish to make a few remarks about the case $p < 1$, i.e. identify $C(K; F^{p,2}_\alpha)$ and $C(K; \dot{F}^{p,2}_\alpha)$. As mentioned earlier, these spaces correspond to potentials of functions (distributions) that belong to the real

Hardy spaces $H^p(\mathbb{R}^n)$, $p < 1$; see [**St2**] for a full discussion of these spaces.

In the middle 1980's, it was conjectured that

$$\dot{C}_{\alpha,p}(K) \sim \Lambda^{(\infty)}_{n-\alpha p}(K)$$

for all compact K whenever $0 < p \leq 1$ and $\alpha p < n$. In 1988, J. Orobitg verified this for $p \in (n/(n+\alpha), 1]$. However, it wasn't until 1989 that Yu. V. Netrusov completely settled this question, in [**Ne1**]. But what is rather interesting was how it was done. He did this by studying the Besov capacities $C(\cdot; B^{p,q}_\alpha)$. The connection is: $F^{p,p}_\alpha = B^{p,p}_\alpha$ and the fact that the capacities $C(\cdot; F^{p,q}_\alpha)$ do not depend on q, $0 < q < \infty$. Thus

$$C_{\alpha,p} = C(\cdot; F^{p,2}_\alpha) \sim C(\cdot; F^{p,p}_\alpha) = C(\cdot; B^{p,p}_\alpha).$$

Hence it suffices to identify $C(\cdot; B^{p,p}_\alpha)$ for $0 < p \leq 1$. This follows from [**Ne1**]; see also [**Ne2**].

The important observation is:

$$C(K; B^{p,q}_\alpha) \sim \Lambda^{(1)}_{n-\alpha p, q/p}(K)$$

where either $0 < q < 1$, $0 < p < \infty$ or $0 < p \leq 1$, $0 < q < \infty$, when $\alpha p < n$. Here, the set function on the right is the Netrusov capacity:

$$\Lambda^{(\epsilon)}_{d,\theta}(K) = \inf \left(\sum_{i=0}^{\infty} (m_i 2^{-id})^\theta \right)^{1/\theta}$$

where the infimum is taken over all countable coverings of K by balls whose radii r_j do not exceed ϵ, while m_i is the number of balls from this covering whose radii r_j belong to the interval $(2^{-i-1}, 2^{-i}]$, $i = 0, 1, 2, \dots$. Also,

$$C(K; \dot{B}^{p,q}_\alpha) \sim \Lambda^{(\infty)}_{n-\alpha p, q/p}(K).$$

Note that $\Lambda^{(\epsilon)}_{d,1} = \Lambda^{(\epsilon)}_{d}$, $0 < \epsilon \leq \infty$, and $\Lambda^{(\epsilon)}_{d,\theta} \leq \Lambda^{(\epsilon)}_{d,\sigma}$ for $\sigma \leq \theta$, $\epsilon > 0$. This verifies the conjecture.

Embedding: An interesting estimate for Riesz potentials of $f \in H^p(\mathbb{R}^n)$, the Hardy space $0 < p < 1$, is the Choquet integral inequality

$$\int (M_\Phi(I_\alpha * f))^p \, d\Lambda^{(\infty)}_{n-\alpha p} \leq A \cdot \|f\|^p_{H^p} \tag{4.6}$$

for all $f \in \mathcal{S}_{00}$, the rapidly decreasing C^∞-functions on $\mathbb{R}^n$ with Fourier transforms compactly supported away from the origin, and a suitable $\Phi \in C_0^\infty(\mathbb{R}^n)$. In the notation of [**St2**, Chapter III],

$$M_\Phi(g)(x) = \sup_{t>0} |\Phi_t * g(x)|.$$

The reader can verify (4.6) using the techniques of [**K**]; here, of course, $0 < p \leq 1$ and $\alpha p < n$. In particular, (4.6) gives

$$I_\alpha(H^p) \subset L^p(\Lambda^{(\infty)}_{n-\alpha p}).$$

Duality: We introduce the Morrey space of signed Borel measures $\mathcal{L}^{1,\lambda}(\mathbb{R}^n)$, for $0 < \lambda < n$, as those μ satisfying the condition

$$\sup_{\substack{r>0 \\ x\in\mathbb{R}^n}} |\mu|(B(x,r)) \cdot r^{-\lambda} < \infty.$$

Then using the ideas of [**A3**, Proposition 1], we have

$$L^p(\Lambda^{(\infty)}_{n-\alpha p})^* \cong \mathcal{L}^{1,n/p-\alpha}$$

for $0 < \alpha < n$, and $n/(n+\alpha) \leq p \leq 1$. The key idea here is to observe the estimate

$$\left| \int f \, d\mu \right| \leq A \left(\int |f|^p \, d\Lambda^{(\infty)}_{n-\alpha p} \right)^{1/p}.$$

This follows by writing

$$\begin{aligned} \int |f| \, d|\mu| &\leq \sum_k 2^{k+1} |\mu|(E_k) \leq \sum_k \sum_j |\mu|(B_j) \\ &\leq A \sum_k \sum_j r_j^{n/p-\alpha} \leq A \sum_k \left(\sum_j r_j^{n-\alpha p} \right)^{1/p} \\ &\leq A \sum_k (\Lambda^{(\infty)}_{n-\alpha p}(E_k) + \epsilon 2^{-|k|})^{1/p} \\ &\leq A \left(\int |f|^p \, d\Lambda^{(\infty)}_{n-\alpha p} + \epsilon \right)^{1/p}, \end{aligned}$$

where $E_k = [2^k < |f| \leq 2^{k+1}]$, $k = 0, \pm 1, \pm 2, \ldots$, and the cover $\{B_j\}$ is chosen so that

$$\sum_j r_j^{n-\alpha p} < \Lambda^{(\infty)}_{n-\alpha p}(E_k) + \epsilon 2^{-|k|}.$$

When $0 < p < n/(n+\alpha)$ each bounded linear functional on $L^p(\Lambda^{(\infty)}_{n-\alpha p})$ vanishes identically. This follows from the standard argument that one uses to deal with the case of n-dimensional Lebesgue measure in $\mathbb{R}^n$.

The simplier case $p = 1$ was treated in [**A3**]. But this can be considered as a special case of $p \geq 1$ as well. This is treated next.

(d) $L^p(\Lambda^{(\infty)}_d)$, $1 \leq p < \infty$: duality.

As we have seen, the dual of $L^1(\Lambda^{(\infty)}_{n-\alpha})$ is a Morrey space of signed measures. The condition can be rephrased using the maximal function

$$\mathcal{M}_d\mu(x) \equiv \sup_{r>0} r^{-d}|\mu|(B(x,r)),$$

$0 < d < n$. Thus, $\mu \in \mathcal{L}^{1,d}$ iff $\mathcal{M}_d\mu \in L^\infty$.

This leads to

Theorem 8. *The dual space to $L^p(\Lambda^{(\infty)}_d)$ is the set of all $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that*

$$\left\{\int (\mathcal{M}_d\mu(x))^{p'}\, d\Lambda^{(\infty)}_d\right\}^{1/p'} < \infty,$$

$1 \leq p < \infty$. *Furthermore, if l is a bounded linear functional on $L^p(\Lambda^{(\infty)}_d)$, then*

$$l(\phi) = \int \phi\, d\mu$$

for all $\phi \in C_0(\mathbb{R}^n)$. The norm of μ is

$$\|\mathcal{M}_d\mu\|_{L^{p'}(\Lambda^{(\infty)}_d)} \leq A\|l\|,$$

for some constant A depending only on d.

Proof: We use techniques similar to those of Theorem 6 to show that there is a constant A such that

$$\int (\mathcal{M}_d\mu)^{p'}\, d\Lambda^{(\infty)}_d \leq A \int (\mathcal{M}_d\mu)^{p'-1}\, d|\mu|. \tag{4.7}$$

Consequently, if l is a bounded linear functional on $L^p(\Lambda^{(\infty)}_d) \supset C_0(\mathbb{R}^n)$, then $l(\phi) = \displaystyle\int \phi\, d\mu$ for all $\phi \in C_0(\mathbb{R}^n)$. Hence

$$\left|\int \phi\, d\mu\right| \leq \|l\| \left(\int |\phi|^p\, d\Lambda^{(\infty)}_d\right)^{1/p}$$

for all such ϕ, and, by passing to a limit, for all lower semicontinuous ϕ. Then setting $\phi = (\mathcal{M}_d\mu)^{p'-1}$ gives

$$\int (\mathcal{M}_d\mu)^{p'-1}\, d|\mu| \le \|l\| \left(\int (\mathcal{M}_d\mu)^{p'}\, d\Lambda_d^{(\infty)}\right)^{1/p}$$

and the bound for $\|\mathcal{M}_d\mu\|_{L^{p'}(\Lambda_d^{(\infty)})}$ in terms of $\|l\|$ is evident.

Thus turning our attention to (4.7), we set $G_t = [\mathcal{M}_d\mu > t/2]$ and $\mu^t = \mu^{G_t}$, $\mu_t = \mu - \mu^t$. Then

$$\Lambda_d^{(\infty)}[\mathcal{M}_d\mu > 2^{d+1}t] \le \Lambda_d^{(\infty)}[\mathcal{M}_d\mu^t > 2^d t] + \Lambda_d^{(\infty)}[\mathcal{M}_d\mu_t > 2^d t].$$

But, in analogy to the boundedness principle (section 3(e)), we have

$$\mathcal{M}_d\mu_t(x) \le 2^d\mathcal{M}_d\mu_t(x') \le 2^d\mathcal{M}_d\mu(x') \le 2^{d-1}t$$

where $x' \in \operatorname{supp}\mu_t = [\mathcal{M}_d\mu \le t/2]$. Hence

$$\Lambda_d^{(\infty)}[\mathcal{M}_d\mu > 2^{d+1}t] \le \frac{A}{t}|\mu|([\mathcal{M}_d\mu > t/2])$$

by the weak type inequality —i.e. a direct consequence of the simple Vitali lemma (as noted earlier). So

$$\begin{aligned}\int (\mathcal{M}_d\mu)^{p'}\, d\Lambda_d &= \int_0^\infty \Lambda_d^{(\infty)}[\mathcal{M}_d\mu > t]\, dt^{p'} \\ &\le A\int_0^\infty t^{-1}|\mu|([\mathcal{M}_d\mu > t/2])\, dt^{p'} \\ &\le A\int (\mathcal{M}_d\mu)^{p'-1}\, d|\mu|.\end{aligned}$$

For the converse, we note that for all $\phi \in C_0(\mathbb{R}^n)^+$,

$$\int \phi\, d|\mu| \le A\int \phi\mathcal{M}_d\mu\, d\Lambda_d^{(\infty)}, \tag{4.8}$$

for some constant A depending only on n and d. The result follows from (4.8) via Hölder's inequality. So to see (4.8), we first work with a "linear capacity", i.e. a measure ν such that on all balls $\nu(B(x,r)) \le r^d$, $r > 0$. Then it easily follows that

$$\int r^{-d}\int_{B(x,r)} \phi(y)\, d\nu(y)\, d\mu(x) \le \int \mathcal{M}_d\mu(y)\phi(y)\, d\Lambda_d^{(\infty)}(y).$$

Now we apply Frostman's lemma (see [**Ca1**, p. 7]): there is a constant A depending only on the dimension such that for every compact set $K \subset \mathbb{R}^n$, there is a measure ν —as above— such that $\nu(K) \geq A \cdot \Lambda_d^{(\infty)}(K)$. Hence, we have

$$A \int r^{-d} \int_{B(x,r)} \phi(y)\, d\Lambda_d^{(\infty)}(y)\, d\mu(x) \leq \int \mathcal{M}_d \mu \phi \, d\Lambda_d^{(\infty)}.$$

Now use the lower semi-continuity of the integral as $r \to 0$. ∎

(e) $L^q(C_{\alpha,p})$, $1 \leq q < \infty$: duality.

It is easy to see that the dual to $L^1(C_{\alpha,p})$ is

$$\{\mu \in \mathcal{M} : |\mu|(K) \leq A \cdot C_{\alpha,p}(K),\ K = \text{ compact set in } \mathbb{R}^n\}$$

for some constant A, since it easily follows that

$$\left| \int \phi \, d\mu \right| \leq A \int |\phi| \, dC_{\alpha,p}.$$

For $1 < q < \infty$, we have

Theorem 9. *The dual to the space $L^q(C_{\alpha,p})$ is the set of all $\mu \in \mathcal{M}$, $|\mu| \ll C_{\alpha,p}$ and such that*

$$\left\{ \int (W_{\alpha,p}^{|\mu|})^{(p-1)q'} \, dC_{\alpha,p} \right\}^{1/q'} < \infty,$$

$1 < q < \infty$. Furthermore, if l is a bounded linear functional on $L^q(C_{\alpha,p})$, then

$$l(\phi) = \int \phi \, d\mu$$

for all $\phi \in C_0(\mathbb{R}^n)$. The norm of μ is

$$\|(W_{\alpha,p}^{|\mu|})^{p-1}\|_{L^{q'}(C_{\alpha,p})} \leq A\|l\|$$

for some constant A depending only on α, p and n.

Proof: We use the techniques of earlier results. For the necessity, we follow Theorem 8 replacing (4.7) by

$$\int (W_{\alpha,p}^{|\mu|})^{(p-1)q'} \, dC_{\alpha,p} \leq A \int (W_{\alpha,p}^{|\mu|})^{(p-1)(q'-1)} \, d|\mu|$$

with almost an identical argument as in Theorem 8. For the sufficiency use Lemma 1 of section 2(e). ∎

Notice that the result of Theorem 9 agrees with Theorem 3 when $q = p$ since one has

$$W^{\mu}_{\alpha,p}(x) \leq A \cdot V^{\mu}_{\alpha,p}(x)$$

for all x; see section 2(a). Thus

$$\int (W^{\mu}_{\alpha,p})^p \, dC_{\alpha,p} \leq A \|G_\alpha * \mu\|^{p'}_{L^{p'}}$$

by (3.1). The reverse inequality follows from Lemma 1.

5. Obstacle problems

A problem intimately connected to the concept of the Choquet integral is the well studied "harmonic obstacle problem": find the smallest superharmonic function u on the bounded domain Ω, anchored at $\partial\Omega$ —i.e. u fixed at $\partial\Omega$— and such that it dominates a given function ψ on Ω; $\psi <$ boundary values of u at $\partial\Omega$. This can also be interpreted as "stretching a thin elastic membrane" over a fixed obstacle in Ω. See [**R**, p. 2-6] or [**KS**, p. 6-7]. It can be achieved by minimizing the Dirichlet integral $\int_\Omega |\nabla v|^2 \, dx$ over all $v \in W^{1,2}(\Omega)$ such that $v - \theta \in W^{1,2}_0(\Omega)$ and $v \geq \psi$ on Ω. Here θ represents the fixed boundary values of the solution —extended into Ω so that $\theta \in W^{1,2}(\Omega)$; $\psi < \theta$ on $\partial\Omega$.

If the obstacle function is of class $C^2(\bar{\Omega})$, then it can be shown ([**KS**, p. 129]) that the solution has bounded second derivatives throughout Ω, but is generally not of class C^2 in Ω. Also, Ω can be written as the disjoint union $C \cup N$, where the closed set C is the coincidence set $\{x \in \Omega : u(x) = \psi(x)\}$ and $N = \Omega \setminus C$, the non-coincidence set. On N, u is harmonic and u and ∇u agree with ψ and $\nabla\psi$ on ∂C, the free boundary.

But such obstacle problems are also well-defined for a much larger class of obstacles. In particular, if we let $\psi = \chi_K =$ characteristic function of the compact subset $K \subset \Omega$, then the solution is just the capacitary potential function —in this case $G\mu$, the Green function of a measure μ supported on K. Here we have taken $\theta \equiv 0$. Actually, all that is needed to get existence of a solution to the obstacle problem is a version of Theorem 5, restricted to bounded domains Ω, i.e.

$$\int_\Omega \psi^+ \, dC^{\Omega}_2 < \infty.$$

See Theorem 10 below. Also, the differential operator —Laplacian here— can be replaced by any other linear or nonlinear elliptic second order operator. Below we consider the special case of the p-Laplace operator Δ_p, a prototype for many quasi-linear second order elliptic operators. See especially [**HKM**] for a full discussion of the traditional questions for the Laplace operator $\Delta = \Delta_2$ treated for the p-Laplace operator, $p \neq 2$. But our main concern will be to show how the Choquet integral enters the picture, especially with regard to the tricky problem of the stability of the solution under a perturbation in the obstacle data.

After consideration of the Δ_p-obstacle problem, we say a few brief words along these same lines for other such problems, namely a biharmonic obstacle problem and then an obstacle problem for second order systems. All of these problems make important uses of the Choquet integral.

(a) The p-harmonic problem.

Let Ω be a bounded domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega$. Further, let $\theta \in W^{1,p}(\Omega)$, $1 < p < \infty$, and ψ an extended real valued function defined on Ω. Set

$$\mathbb{K}_{\psi,\theta}(\Omega) = \{u \in W^{1,p}(\Omega) : u - \theta \in W_0^{1,p}(\Omega),\ u \geq \psi,\ C_{1,p}\text{-a.e. on } \Omega\}.$$

Here, as usual, $W^{1,p}(\Omega)$ is the Sobolev space of functions on Ω, p^{th} power integrable on Ω along with their first order distribution derivatives. $W_0^{1,p}(\Omega)$ is the usual closure of $C_0^\infty(\Omega)$ with the norm of $W^{1,p}(\Omega)$, i.e.

$$\|u\|_{L^p(\Omega)} + \sum_{j=1}^{n} \|D_j u\|_{L^p(\Omega)}.$$

The p-harmonic obstacle problem, with data θ and ψ, is the problem

$$\min \int_\Omega |\nabla u|^p \, dx$$

where the minimum is taken over all $u \in \mathbb{K}_{\psi,\theta}$. When $p = 2$, it is refered to simply as the "harmonic obstacle problem" on Ω. The p-harmonic problem can also be written as a variational inequality:

$$\text{find } u \in \mathbb{K}_{\psi,\theta} \text{ such that } \langle -\Delta_p u, v - u\rangle \geq 0 \text{ for all } v \in \mathbb{K}_{\psi,\theta}.$$

Here $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, the p-Laplace operator, and the brackets $\langle , \rangle$ denote the duality paring.

The connection between the Choquet integral and the p-harmonic problem is made explicit by:

Theorem 10. *With the above notation, the following are equivalent:*

1. $\mathbb{K}_{\psi,\theta}$ *is non-empty;* (5.1)
2. *there exists a unique* $u_{\psi,\theta} \in \mathbb{K}_{\psi,\theta}$ *such that*

$$\int_\Omega |\nabla u_{\psi,\theta}|^p \, dx = \inf_{u \in \mathbb{K}_{\psi,\theta}} \int_\Omega |\nabla u|^p \, dx; \tag{5.2}$$

3. $\displaystyle\int_\Omega [(\psi - \theta)^+]^p \, dC_p^\Omega < \infty.$ (5.3)

Furthermore,

$$\inf \int_\Omega |\nabla u|^p \, dx \sim \|(\psi - \theta)^+\|^p_{L^p(C_p^\Omega)}. \tag{5.4}$$

Here, we have modified the capacity C_p *of section 3(b) and (d), to*

$$C_p^\Omega(K) = \inf \left\{ \int_\Omega |\nabla \phi|^p \, dx : \phi \in C_0^\infty(\Omega), \, \phi \geq \chi_K \right\},$$

i.e. everything is now defined relative to Ω*.*

The proof of this result can be found in [**AN**], but it is, in fact, a simple varient of the corresponding result for $p = 2$ found in [**A6**], and hence will not be discussed here in detail. What we wish to concentrate on here is the use of the Choquet spaces in characterizing the stability of solutions to the p-harmonic obstacle problem under a change in the obstacle. This result is the main thrust of [**AN**]. A similar but weaker result appears in [**LM**].

We shall say that solutions to the p-harmonic obstacle problem are *stable* if the solution operator $S_\theta(\psi) = u_{\psi,\theta}$ is continuous from $L^p(C_p^\Omega)$ into $W^{1,p}(\Omega)$. We intend to show a bit more, namely that S_θ is locally Hölder continuous with exponent

$$\alpha_p = \min\left(\frac{1}{p}, \frac{1}{2}\right)$$

for $1 < p < \infty$.

Theorem 11. *There is a constant* A *that depends on the quantities* $n, p,\ \|\theta_i\|_{W^{1,p}(\Omega)},\ \|\psi_i\|_{L^p(C_p^\Omega)},\ i = 1, 2,$ *such that*

$$\|S_\theta(\psi_1) - S_\theta(\psi_2)\|_{W^{1,p}(\Omega)} \leq A \cdot \|\psi_1 - \psi_2\|^{\alpha_p}_{L^p(C_p^\Omega)}.$$

Furthermore, the exponent α_p *is sharp for* $p \geq 2$*.*

Proof: The proof proceeds via several steps, the first of which is to solve an approximate problem (the so called "penalized problem").

Step 1: Let $\eta(t) \in C^\infty(\mathbb{R})$ such that $\eta(t) \equiv 0$ for all $t \geq 0$, $\eta(t) < 0$ for $t < 0$, and $0 \leq \eta'(t) \leq 2$ for all t. Set $\eta_\epsilon(t) = \epsilon^{-1}\eta(t)$ for $\epsilon > 0$. The penalized problem is

$$\begin{cases} -\Delta_p u_\epsilon = -\eta_\epsilon(u_\epsilon - \psi), & \text{in } \Omega \\ u_\epsilon = \theta, & \text{in } \partial\Omega. \end{cases} \tag{5.5}$$

A unique classical solution to (5.5) exists by standard fixed point arguments together with the estimates given below.

We first estimate u_ϵ in $W^{1,p}(\Omega)$ independent of ϵ.

$$\begin{aligned} \|\nabla u_\epsilon\|^p_{L^p(\Omega)} &= \int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla(u_\epsilon - \theta)\,dx + \int_\Omega |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon \nabla\theta\,dx \\ &\leq -\int_\Omega \eta_\epsilon(u_\epsilon - \psi)(u_\epsilon - \theta)\,dx + \|\nabla u_\epsilon\|^{p-1}_{L^p(\Omega)}\|\nabla\theta\|_{L^p(\Omega)}. \end{aligned}$$

The first term does not exceed

$$\begin{aligned} -\int_\Omega \eta_\epsilon(u_\epsilon - \psi)(u_\epsilon - \psi)\,dx - \int_\Omega \eta_\epsilon(u_\epsilon - \psi)(\psi - \theta)\,dx \\ \leq -\int_\Omega \eta_\epsilon(u_\epsilon - \psi)w\,dx \leq \|\nabla u_\epsilon\|_{L^p}\|\nabla w\|_{L^p(\Omega)} \end{aligned}$$

where $w \in W^{1,p}_0(\Omega)$ satisfies: $w \geq (\psi - \theta)^+$ on Ω. Thus we get

$$\|\nabla u_\epsilon\|_{L^p(\Omega)} \leq A\left\{\left(\int_\Omega (\psi - \theta)^+\,dC^\Omega_p\right)^{1/p} + \|\nabla\theta\|_{L^p(\Omega)}\right\},$$

from Theorem 10.

Step 2: Since $\{u_\epsilon\}_{\epsilon>0}$ is a bounded set in $W^{1,p}((\Omega))$, it follows from weak compactness that there is a subsequence $\{u_{\epsilon'}\}$ for which $u_{\epsilon'}$ tends weakly in $W^{1,p}(\Omega)$ to some $u \in W^{1,p}(\Omega)$. But from the well known Rellich compactness theorem (see [**KS**, p. 62]), $W^{1,p}((\Omega)$ can be compactly embedded in $L^p(\Omega)$. Hence, there is a further subsequence (still denoted by $u_{\epsilon'}$) such that $u_{\epsilon'}$ converges strongly to u in $L^p(\Omega)$. Furthermore, it follows that $u \geq \psi$ a.e. on Ω. To see this last fact, we write

$$-\int \eta(u_{\epsilon'} - \psi)\phi\,dx \leq \epsilon'\|\nabla u_{\epsilon'}\|_{L^p(\Omega)}\|\nabla\phi\|_{L^p(\Omega)}$$

which tends to zero with ϵ'. On the otherhand, the left side tends to

$$-\int \eta(u - \psi)\phi\,dx$$

by the L^p-convergence of $u_{\epsilon'}$. Here $\phi \in C^\infty_0(\Omega)$. Thus $\eta(u - \psi) = 0$ a.e. on Ω and the result follows.

Step 3: $\nabla u_{\epsilon'}$ tends to ∇u strongly in $L^p(\Omega)$. To see this, write

$$\int |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla u|^{p-2}\nabla u)\nabla(u_\epsilon - u)\,dx$$
$$= -\int \eta_\epsilon(u_\epsilon - \psi)(u_\epsilon - u)\,dx - \int |\nabla u|^{p-2}\nabla u\nabla(u_\epsilon - u)\,dx.$$

The first term is ≤ 0 since $u_\epsilon \le \psi \le u$ when η is non-zero, and the second term tends to zero by weak convergence.

Now to get our desired estimate on the difference $\nabla(u_\epsilon - u)$, we need two basic inequalities associated with the p-Laplace operator:

for ξ, $\zeta \in \mathbb{R}^n$, and $p \ge 2$, there is a constant $\lambda > 0$ such that

$$(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)\cdot(\xi - \zeta) \ge \lambda|\xi - \zeta|^p; \tag{5.6}$$

for $1 < p < 2$,

$$(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)\cdot(\xi - \zeta) \ge (p-1)|\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2}. \tag{5.7}$$

For a proof of (5.6) see [**BI**] and for (5.7) see [**AN**].

Clearly when $p \ge 2$, our result follows. However, when $1 < p < 2$, we estimate as follows:

$$\int_\Omega |\nabla(u_\epsilon - u)|^p\,dx \le \left(\int_\Omega |\nabla(u_\epsilon - u)|^2(|\nabla u_\epsilon| + |\nabla u|)^{p-2}\,dx\right)^{p/2}$$
$$\cdot\left(\int_\Omega (|\nabla u_\epsilon| + |\nabla u|)^p\,dx\right)^{1-p/2}$$

via Hölder's inequality. And now the result for $p < 2$ follows.

Step 4: u is the unique solution to the p-harmonic problem. We show that u is a solution, uniqueness follows in a standard way. In fact, if $V \in \mathbb{K}_{\psi,\theta}$, then

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla(V - u)\,dx = \lim_{\epsilon\to 0}\int_{\Omega j} |\nabla u_\epsilon|^{p-2}\nabla u_\epsilon\nabla(V - u_\epsilon)\,dx$$
$$= -\lim_{\epsilon\to 0}\int_\Omega \eta_\epsilon(u_\epsilon - \psi)(V - u_\epsilon)\,dx \ge 0,$$

since $V \ge \psi \ge u_\epsilon$ when η_ϵ is non-zero.

Step 5: Let $u = S_\theta(\psi_1)$, $v = S_\theta(\psi_2)$. Then we will produce the Hölder estimate in the statement of the Theorem. We begin with

$$\begin{aligned}
&\int_\Omega (|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon)\cdot\nabla(u_\epsilon - v_\epsilon)\,dx \\
&\quad = -\int \eta_\epsilon(u_\epsilon-\psi_1)(u_\epsilon-\psi_1)\,dx - \int \eta_\epsilon(v_\epsilon-\psi_2)(v_\epsilon-\psi_2)\,dx \\
&\qquad - \int \eta_\epsilon(u_\epsilon-\psi_1)(\psi_2 - v)\,dx - \int \eta_\epsilon(v_\epsilon-\psi_2)(\psi_1-u)\,dx \\
&\qquad - \int \eta_\epsilon(u_\epsilon-\psi_1)(\psi_1-\psi_2)\,dx - \int \eta_\epsilon(v_\epsilon-\psi_2)(\psi_2-\psi_1)\,dx \\
&\qquad - \int \eta_\epsilon(u_\epsilon-\psi_1)(v-v_\epsilon)\,dx - \int \eta_\epsilon(v_\epsilon-\psi_2)(u-u_\epsilon)\,dx.
\end{aligned}$$

The first 4 terms are non-positive. The 5th and 6th can be estimated by

$$\|\nabla u_\epsilon\|_{L^p}^{p-1}\|\nabla w_1\|_{L^p} + \|\nabla v_\epsilon\|_{L^p}^{p-1}\|\nabla w_2\|_{L^p}$$

where $w_i \in W_0^{1,p}(\Omega)$, $w_1 \geq (\psi_1-\psi_2)^+$, $w_2 \geq (\psi_2-\psi_1)^+$. The last two terms are estimated by

$$\|\nabla u_\epsilon\|_{L^p}^{p-1}\|\nabla(v-v_\epsilon)\|_{L^p} + \|\nabla v_\epsilon\|_{L^p}:\|\nabla(u-u_\epsilon)\|_{L^p}.$$

Now let ϵ tend to zero. The result follows using Theorem 10 and (5.6) and (5.7).

To see that the exponent $\alpha_p = 1/p$, $p \geq 2$, is sharp, let

$$h_{r,R}(t) = \begin{cases} 1, & 0 \leq t \leq r \\ At^{(p-n)/(p-1)} + B, & r \leq t \leq R \\ 0, & t \geq R \end{cases}$$

with constants A and B determined so that $h_{r,R}(t)$ is continuous on $[0,\infty)$. Now set $\psi_\epsilon(x) = h_{(1-\epsilon)r,R}(|x|)$, for $0 \leq \epsilon < 1/2$, and $u_\epsilon = S_0(\psi_\epsilon)$. Then a short calculation gives

$$\|\nabla(u_\epsilon - u_0)\|_{L^p(B(0,R))} \geq A\cdot\epsilon^{1/p}$$

and

$$|\psi_\epsilon(x) - \psi_0(x)| \leq A'\cdot\epsilon$$

for all $x : |x| \leq R$; A and A' are two constants that depend only on n, p, r, R. ■

We close this subsection with an application of the ideas of Theorem 6 applied to solutions to the equation

$$\text{(5.8)} \qquad \begin{cases} -\Delta_p u = \mu, & \text{in } \Omega \\ u \in W_0^{1,p}(\Omega) \end{cases}$$

μ a Borel measure compactly supported in Ω. In fact by [**KM**], we know that any such u must satisfy

$$A_1 W_p^\mu(x,R) \le u(x) \le A_2 \inf_{B(x,R)} u + A_3 \cdot W_p^\mu(x,2R),$$

where $B(x,3R) \subset \Omega$ and

$$W_p^\mu(x,R) = \int_0^R [r^{p-n}\mu(B(x,r)]^{1/(p-1)} \frac{dr}{r},$$

a non-homogeneous Wolff potential of the measure μ. Of course when $p=2$, such an estimate is well known —in fact in that case u is exactly a Green potential of μ on Ω. Thus, because $u = S_0(\psi)$ satisfies (5.8), where μ is a measure supported on the coincidence set —assuming $\psi \in C(\Omega)$ and $\psi < 0$ near $\partial\Omega$ implies $[u=\psi] \Subset \Omega$— we can derive local $L^q(C_p^\Omega)$ integrability result for such u. In fact, one can easily deduce, in the spirit of Theorem 6, that for any $\Omega' \Subset \Omega$, the solution to the p-harmonic obstacle problem $u = S_0(\psi)$ must satisfy

$$\int_{\Omega'} u^q \, dC_p^\Omega \le A \cdot \int_\Omega (\psi^+)^q \, dC_p^\Omega$$

for any $q \ge p$; A depends only on n, p, q and the distance Ω' to $\partial\Omega$.

(b) The biharmonic problem.

One version of the biharmonic obstacle problem is:

$$\min\left\{ \int_\Omega (\Delta u)^2 \, dx : u \in W_0^{1,2}(\Omega) \cap W^{2,2,}(\Omega),\ u \ge \psi \text{ a.e. on } \Omega \right\}.$$

Here ψ is the given obstacle, an extended real valued function on Ω, and Δ is the Laplacian in $\mathbb{R}^n$; $\Omega \subset \mathbb{R}^n$ a bounded domain. One might refer to this version of the biharmonic obstacle problem as the "hinged" case, since a natural boundary condition that appears, assuming that the data ψ is smooth —say $C^2(\Omega)$— is $\Delta u = 0$ on $\partial\Omega$. The "clamped" case corresponds to replacing $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ by $W_0^{2,2}(\Omega)$; this forces u and its normal derivative to vanish at $\partial\Omega$. For simplicity, we will discuss only the hinged case here. For a full treatment of this problem, the author should consult [**CF**], [**A7**], [**A8**]. This last reference is relevant here since one can view the hinged problem as a 2×2 second order system with the obstacle in the first component.

The appearance of the Choquet integral in the context of the biharmonic obstacle problem involves both the question of existence (as for the p-harmonic obstacle problem —Theorem 10) and of regularity of solutions. Of course, if the obstacle is of class $C^2(\Omega)$ and negative near the $\partial\Omega$, then there is generally no need for Choquet integrals. In fact in that case, one can easily show that the solution u is in $W^{2,\infty}(\Omega)\cap W^{3,2}(\Omega)$; see [**CF**]. However, for more general ψ one needs the concept of capacity and Choquet integrals. For this we set

$$C^{\Omega}_{2,2}(K) = \inf\left\{\int_\Omega |\Delta\phi|^2\,dx : \phi\in C_0^\infty(\Omega),\right.$$
$$\left.\phi \text{ superharmonic, } \phi\geq 1 \text{ on } K\right\}.$$

Then from [**Ha**] one has

$$\int u^2\,dC^{\Omega}_{2,2} \sim \inf\left\{\int |\Delta\phi|^2\,dx : \phi\in W^{2,2}(\Omega)\cap W_0^{1,2}\Omega,\right.$$
$$\left.\phi \text{ superharmonic, } \phi\geq \psi C^{\Omega}_{2,2}\text{-a.e. } \Omega\right\}.$$

Our result is:

Theorem 12. *(a) The hinged biharmonic obstacle problem has a unique solution if and only if*

$$\int_\Omega (\psi^+)^2\,dC^{\Omega}_{2,2} < \infty.$$

(b) The solution of part (a) belongs to the Sobolev space $W^{3,2}(\Omega)$ if

$$\int_\Omega [(-\Delta\psi)^+]^2\,dC^{\Omega}_{2} < \infty.$$

We outline the proof of (b). As in the proof of Theorem 11, we must first solve a penalized problem, obtain estimates on its solution u_ϵ independent of ϵ and then pass to the limit for a suitable subsequence, $\epsilon' \to 0$. This proceedure will then show that the solution u to the hinged biharmonic obstacle problem satisfies

$$\int_\Omega |\nabla \Delta u|^2\,dx \le A \int [(-\Delta\psi)^+]^2\,dC_2^\Omega$$

which is the conclusion of (b). Thus we solve

$$\Delta^2 u_\epsilon = -\eta_\epsilon(u_\epsilon - \psi), \text{ in } \Omega$$

$$\left.\begin{aligned} u_\epsilon &= 0 \\ \Delta u_\epsilon &= 0 \end{aligned}\right\} \text{ on } \partial\Omega,$$

where η_ϵ is as in Theorem 11. Then

$$\begin{aligned}
\int_\Omega |\nabla\Delta u_\epsilon|^2\,dx &= \int \eta_\epsilon(u_\epsilon - \psi)\Delta(u_\epsilon - \psi)\,dx \\
&\quad + \int \eta_\epsilon(u_\epsilon - \psi)\Delta\psi\,dx \\
&\le -\int \eta'_\epsilon(u_\epsilon - \psi)|\nabla(u_\epsilon - \psi)|^2\,dx \\
&\quad - \int \eta_\epsilon(u_\epsilon - \phi)(-\Delta\psi)^+\,dx \\
&\le -\int \eta_\epsilon(u_\epsilon - \psi)w\,dx \le \|\nabla\Delta u_\epsilon\|_{L^2(\Omega)} \cdot \|\nabla w\|_{L^2(\Omega)}
\end{aligned}$$

where $w \ge (-\Delta\psi)^+$ on Ω and $w \in W_0^{1,2}(\Omega)$. Thus

$$\int_\Omega |\nabla\Delta u_\epsilon|^2\,dx \le A \int [(-\Delta\psi)^+]^2\,dC_2^\Omega,$$

with A independent of ϵ. Now we use lower-semi-continuity with respect to weak convergence in $W^{3,2}(\Omega)$, since the weak limit will be the unique solution to the problem. ■

Results extending these ideas together with stability questions for the biharmonic and polyharmonic obstacle problems will appear in the forthcoming paper [**AV**]. Such stability questions have previously been considered by Schild in [**S**], but only in very special cases —with real analytic obstacles given on lower dimensional subsets in $\mathbb{R}^2$, the so called smooth thin obstacle case. But using Choquet integrals, the more general “rough” situation will be treated in [**AV**].

(c) Systems of second order obstacle problems.

There are many papers on systems of variational inequalities, and, in particular, on the obstacle problem, with obstacles in each component direction; see esp. [**HW**] and [**F**] among others referenced in [**To**] or [**R**]. But our main concern here is with the appearance of the Choquet integral in some natural way in these problems. One such curious appearence was noted in [**AL**]. There a 2×2 nonlinear systems was investigated, for existence, uniqueness, and regularity —a sort of nonlinear version of "cooperative systems". The basic operators involved are a version of the p-Laplace operator. Without going into the details of the existence proof here, the main relevant point is that one would expect existence of a solution to a second order system to somehow only depend on information about no more that two derivatives of the data —the obstacle in this case. However, it was found that using standard Sobolev space techniques that it was necessary to know $(-\Delta_p\psi)^+ \in L^2(\Omega)$ and $\nabla(-\Delta_p\psi)^+ \in L^p(\Omega)$, to get existence. This seemed unnatural. Indeed, it turns out that one can replace this third derivative statement by a weaker statement, one that depends only on second derivatives:

$$\int_\Omega [(-\Delta_p\psi)^+]^p \, dC_p^\Omega < \infty.$$

Notice that because of the CSI, such an integral is finite when the gradient of $(-\Delta_p\psi)^+$ belongs to $L^p(\Omega)$.

The reader, interested in systems of second order variational inequalities —and the use of the Choquet integral— might also be interested in the recent paper [**A11**] on $N\times N$ systems. This paper has a curious mixture of Choquet integral techniques together with abstract group theoretic devices to determine existence, regularity and stability of solutions to a unilateral obstacle problem that among other things, generalizes the "hinged plate" biharmonic problem. See also [**A8**].

6. Differentiation

(a) The differentiation of functions.

(i) Introduction. There are several ways of saying that a given real valued function, defined at $x_0 \in \mathbb{R}^n$, is differentiable or even continuous at x_0. Here we consider three such notions for Sobolev functions and present a unified approach —ultra fine differentiability. One advantage to this approach is that it produces a stronger notion of differentiability when $\alpha p < n$, and uses the ideas of "maximal smoothing operators" of [**CFR**], i.e. the existence of the limit is controlled by a maximal function.

So suppose we are given a function $F(x) = F(x; x_0)$, then we say:

(1) $F(x; x_0)$ tends to zero at x_0 *pointwise* if $F(x; x_0) \to 0$ as $x \to x_0$ in the usual sense;

(2) $F(x; x_0)$ tends to zero at x_0 *in the mean* with exponent $q > 0$ if

$$⨍_{B(x_0,r)} |F(x; x_0)|^q \, dx \to 0$$

as $r \to 0$;

(3) $F(x; x_0)$ tends to zero at x_0 *in the fine topology* if there is a set E, thin at x_0, such that

$$F(x; x_0) \to 0 \text{ as } x \to x_0, \quad x \notin E.$$

Our notion of thinness will be that introduced by N. G. Meyers (see [**AH**, sec. 6.3]), so called (α, p)-thinness, i.e. a set E is termed (α, p)-thin at x_0 if

$$\int_0^1 [r^{\alpha p - n} C_{\alpha,p}(E \cap B(x_0, r))]^{p'-1} \frac{dr}{r} < \infty.$$

This can be rephrased using Choquet-Lorentz space notation as

$$\chi_E(x) \cdot |x - x_0|^{\alpha p - n} \in L^{1,p'-1}(C_{\alpha,p})$$

locally. A set G is an open neighborhood of x_0 in the (α, p)-fine topology iff the complement of G is (α, p)-thin at x_0. Thus it follows that

Proposition 2. *F tends to zero in the (α, p)-fine topology iff for all $\epsilon > 0$ the set $\{x : F(x; x_0) \geq \epsilon\}$ is (α, p)-thin at x_0.*

For a proof see [**AH**, sec. 6.4].

The idea now is to set

$$F(x; x_0) = \frac{|u(x) - P(x - x_0)|}{|x - x_0|^k}$$

where u is a given Sobolev function, say $u \in W^{m,p}_{\text{loc}}$, at least in some neighborhood of x_0, $m \geq k$, and P a polynomial in $x - x_0$ (with coefficients depending on x_0) of degree $\leq k$; $m, k \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$. With this, we have three notions of differentiability of degree k at x_0.

In the literature, the notion of L^q-differentiation has drawn some attention; see [**CZ**] and [**Z**]. In fact an extensive collection of results and terminology has arisen, especially the so called T_k^q and t_k^q spaces. The first indicates that the averages

$$r^{-kq} \fint_{B(x_0,r)} |u(x) - P(x-x_0)|^q \, dx$$

are bounded as $r \to 0$, where as the second implies that these averages tends to zero as $r \to 0$. It is easy to see that being k^{th} order differentiable in the mean with exponent q implies that the function belongs to the space t_k^q at x_0.

The notion of (α, p)-fine differentiability is due to Mizuta; see [**M1**] and [**M2**].

(ii) Thin functions. To introduce our notion of differentiability (using Choquet integrals), we first need an analogue of the Brelot-Meyers idea of thinness for sets, extended to functions. Thus we say that the extended real valued function F, defined in a neighborhood of x_0, is (α, p)-thin at x_0 if

$$\int_0^1 \left(r^{\alpha p - n} \int_{B(x_0,r)} |F|^p \colon dC_{\alpha,p} \right)^{p'-1} \frac{dr}{r} < \infty.$$

Clearly, χ_E is (α, p)-thin at x_0 iff E is (α, p)-thin at x_0. Also, it follows that if F is (α, p)-thin at x_0, then not only does F tends to zero in the (α, p)-fine topology, but the averages

$$\fint_{B(x_0,r)} |F|^p \, dC_{\alpha,p} \to 0$$

as $r \to 0$, which in turn implies that F tends to zero in the mean with exponent $p^* = np/(n - \alpha p)$, when $\alpha p < n$. This is due to the "isoperimetric inequality"

$$|E|^{1-\alpha p/n} \leq A \cdot C_{\alpha,p}(E)$$

where $|E|$ denotes the Lebesgue n-measure of $E \subset \mathbb{R}^n$; see [**AH**, Chapter 5]. When $\alpha p = n$, it is even possible to show that the exponential averages

$$\fint_{B(x_0,r)} [\exp(b|F|^{p'}) - 1] \, dx \to 0$$

as $r \to 0$, for some constant $b > 0$; see [**A5**, Chapter III].

Notice that the function $F(x;x_0) = \chi_K(x)\cdot|x-x_0|^\sigma$ is thin at x_0 when $\sigma > 0$, even if K is not thin there. For any $\sigma \in \mathbb{R}$, the (α,p)-thinness of this F is equivalent to

$$\int_0^1 [r^{\alpha p-n} C_{\alpha,p}(K\cap B(x_0,r))]^{p'-1}\cdot r^{\sigma p'}\frac{dr}{r} < \infty.$$

(iii) Ultra-fine differentiability (Ufd). The mode of differentiability that will supersede the three mentioned above, is the following:

> an extended real valued function u defined in a neighborhood of the point x_0 is (α,p)-ufd of order k at x_0 if there exists a function ϕ differentiable of order k at x_0 in the usual pointwise sense such that
> $$|u(x)-\phi(x)|\cdot|x-x_0|^{-k}$$
> is (α,p)-thin at x_0.

Thus if we write

$$\frac{u-P}{|x-x_0|^k} = \frac{u-\phi}{|x-x_0|^k} + \frac{\phi-P}{|x-x_0|^k}$$

and note the previous discussion, then taking P to be the polynomial determined by the differentiability of ϕ at x_0, we see that the notion of (α,p)-ufd is stronger than the three modes considered above.

(iv) Maximal smoothing operators. For our estimates below, we will use certain maximal operators that are known to control differentiation in L^p; see [**CFR**]. We set

$$M_k^* u(x) = \sup_{|h|>0} \frac{|\Delta_h^k u(x)|}{|h|}$$

where $\Delta_h^k u$ denotes the k^{th} difference operator: $\Delta_h u(x) = u(x+h)-u(x)$, $\Delta_h^k u = \Delta_h^{k-1}\Delta_h u$, $k = 1,2,3,\dots$. In particular from [**CFR**], we have $(k \le m)$

$$\|M_k^* u\|_{L^p} \le A\|u\|_{W^{m,p}}.$$

Employing the same methods, yields

$$C_{m-k,p}([M_k^* u > t]) \le At^{-p}\|u\|^p_{W^{m,p}}$$

for $m \geq k$; for $m = k$, replace the capacity by Lebesgue measure. By standard methods, such estimates immediately give the required differentiability, $C_{m-k,p}$-a.e. Our main results below is an attempt to upgrade this capacitary weak type inequality to a strong type inequality.

(v) Ufd of Sobolev functions. Since every $u \in W^{m,p}(\mathbb{R}^n)$ can be represented as the Bessel potential of order m of an L^p function, $u = G_m * f$, $C_{m,p}$-a.e., we can assume throughout, without loss of generality, that $f \geq 0$. Hence $G_m * f$ is defined everywhere. With this, notice that we can write

$$u(x) = g(x; x_0) + b(x; x_0)$$

the "good and bad" parts of u, with $g(x_0; x_0) = u(x_0)$ and $b(x_0; x_0) = 0$. In fact just take

$$g(x; x_0) = \int_{|x-y| \geq \frac{1}{2}|h|} G_m(x-y) f(y)\, dy$$

$$b(x; x_0) = \int_{|x-y| < \frac{1}{2}|h|} G_m(x-y) f(y)\, dy,$$

where $h = x - x_0$.

With this in mind, we state our main result

Theorem 13. *For $C_{m,p}$-a.e. x_0 each $u \in W^{m,p}(\mathbb{R}^n)$ can be written as*

$$u(x) = g(x) + b(x)$$

with $g(x_0) = u(x_0)$, $b(x_0) = 0$ $C_{m,p}$-a.e., and

I. *if $mp < n$, $0 \leq k \leq m$,*

$$\|M_k^* g\|_{L^p(C_{m-k,p})} \leq A \cdot \|u\|_{W^{m,p}}, \tag{6.1}$$

$$|x - x_0|^{-k} b(x) \text{ is } (m,p)\text{-thin at } x_0; \tag{6.2}$$

II. *if $mp = n$, $0 \leq k \leq m$,*

$$\text{(6.1) again holds,} \tag{6.3}$$

$$\int_0^1 \left\{ r^{(m-k)q-n} \int_{B(x_0,r)} |b(x)|^q \, dC_{m,q}(x) \right\}^{p'/q} \frac{dr}{r} < \infty, \tag{6.4}$$

for $1 < q \leq p$;

III. *if $n < mp \leq kp + n$, $0 \leq k \leq m$,*

$$\|M_k^* u\|_{L^p(C_{m-k,p})} \leq A \|u\|_{W^{m,p}}. \tag{6.5}$$

It is interesting to note the difference in condition (6.2) when $mp = n$ and condition (6.4) when $q = p$. As we shall see, they are fundamentally different. But first note

Proposition 3. *For $\alpha p < n$ and $0 < r \le 1$, set*

$$J_1(x_0, r) = \int_{B(x_0,r)} |\,|x - x_0|^{-k} b(x)|^p \, dC_{\alpha,p}(x)$$

and

$$J_2(x_0, r) = \int_0^r s^{-kp} \left(\int_{B(x_0,s)} |b(x)|^p \, dC_{\alpha,p}(x) \right) \frac{ds}{s}.$$

Then $J_1(x_0, r) \sim J_2(x_0, r)$ for every x_0.

From this result, we have

Proposition 4. *For $\alpha p \le n$ and $k < \alpha$, set*

$$J_3(x_0) = \int_0^1 \left(r^{\alpha p - n} \int_{B(x_0,r)} |\,|x - x_0|^{-k} b(x)|^p \, dC_{\alpha,p}(x) \right)^{p'-1} \frac{dr}{r}$$

and

$$J_4(x) = \int_0^1 \left(r^{(\alpha - k)p - n} \int_{B(x_0,r)} |b(x)|^p \, dC_{\alpha,p}(x) \right)^{p'-1} \frac{dr}{r}.$$

Then for $\alpha p < n$, $J_3(x_0) \sim J_4(x_0)$, but when $\alpha p = n$, $J_4(x_0) \le A \cdot J_3(x_0)$ and $J_3(x_0)$ can be $+\infty$ when $J_4(x_0) < \infty$.

Proof of Proposition 3: Writing the ball $B(x_0, r)$ as the disjoint union of annuli $[2^{-j-1} r \le |x - x_0| < 2^{-j} r]$ and using the countable subadditivity of $C_{\alpha,p}$, we have

$$J_1(x_0, r) \le 2^p \sum_{j=0}^{\infty} (2^{-j} r)^{-kp} \int_{B(x_0, 2^{-j} r)} |b(x)|^p \, dC_{\alpha,p}(x).$$

But on the other hand,

$$J_2(x_0, 2r) \ge \left(\frac{1 - 2^{-p}}{p} \right) \sum_{j=0}^{\infty} (2^{-j} r)^{-kp} \int_{B(x_0, 2^{-j} r)} |b(x)|^p \, dC_{\alpha,p}(x).$$

To go the other way, we use quasi-additivity; see section 2(e). So with annuli again, we can write

$$J_1(x_0,r) \ge A\sum_{j=0}^{\infty}(2^{-j}r)^{-kp}(a_j - a_{j+1})$$

where $a_j = \int_{B(x_0,2^{-j}r)} |b(x)|^p\, dC_{\alpha,p}(x)$. If b is smooth, it is obvious that $\sum 2^{jkp}a_j < \infty$ for $k < \alpha$. Hence, our lower bound can be written as

$$J_1(x_0,r) \ge A'\sum_{j=0}^{\infty}(2^{-j}r)^{-kp}a_j.$$

A similar upper bound holds for $J_2(x_0,r)$. ■

Proof of Proposition 4: For simplicity set

$$Q(x_0,r) = \fint_{B(x_0,r)} |b|^p\, dC_{\alpha,p},$$

then by Proposition 3,

$$\begin{aligned} J_3(x_0) &\sim \int_0^1 \left(r^{\alpha p-n}\int_0^r s^{-kp}Q(x_0,s)\frac{ds}{s}\right)^{p'-1}\frac{dr}{r} \\ &= \int_0^1 \left(r^{\alpha p-n}\int_0^r s^{-kp}Q(x_0,s)\cdot \chi_{[s\le 1]}(s)\frac{ds}{s}\right)^{p'-1}\frac{dr}{r}. \end{aligned}$$

We now apply Hardy's inequality for $p \le 2$, and get

$$J_3(x_0) \le A\cdot J_4(x_0).$$

For $p > 2$, we write the integral over $[0,r]$ as a sum of integrals over $[2^{-j-1}r, 2^{-j}r]$, $j = 0,1,2,\dots$, and apply Jensen's inequality. The reverse inequality is trivial.

When $\alpha p = n$, neither Hardy's inequality nor Jensen's inequality applies, though clearly $J_3(x_0) \ge AJ_4(x_0)$ always holds. But notice that if $p = 2$, then

$$J_3(x_0) = \int_0^1 Q(x_0,s)s^{-2k}\log\ 1/s\frac{ds}{s},$$

whereas

$$J_3(x_0) = \int_0^1 Q(x_0,s)s^{-2k}\frac{ds}{s}. \quad ■$$

Proof of Theorem 13: We shall confine ourselves to parts I and III and, for simplicity, treat only the case $k = 1$; the other part as well as $k > 1$ follow in a similar manner. So write $u \in W^{m,p}$ as $u(x) = g(x; x_0) + b(x; x_0)$, as indicated earlier. Then for $mp < n$,

$$\Delta_h u(x_0) = u(x) - u(x_0) = g(x; x_0) - g(x_0; x_0) + b(x; x_0)$$

where $x = x_0 + h$. Hence

$$\begin{aligned} |\Delta_h g| &\le \int_{|x-y| \ge \frac{1}{2}|h|} |G_m(x-y) - G_m(x_0 - y)| f(y)\, dy \\ &\qquad + \int_{|x-y|<2|h|} G_m(x_0 - y) f(y)\, dy \\ &\le A|h| \cdot G_{m-1} * f(x_0) \end{aligned}$$

Thus

$$M_1^* g(x_0) \le A \cdot G_{m-1} * f(x_0).$$

Applying the CSI (section 3(b)), we get (6.1). Next,

$$\begin{aligned} |x - x_0|^{-1} b(x) &\le |h|^{-1} \int_{|x_0 - y| \le 2|h|} G_m(x-y) f(y)\, dy \\ &\le A \int_{B(x_0, 2r)} G_m(x-y) f(y) |x_0 - y|^{-1}\, dy \end{aligned}$$

when $x \in B(x_0, r)$. And again using CSI, we have

$$\begin{aligned} \int_{B(x_0,r)} [|x - x_0|^{-1} b(x)]^p\, dC_{m,p}(x) &\le A \int_{B(x_0,2r)} f(y)^p |x_0 - y|^{-p}\, dy \\ &= A \int_0^{2r} s^{-p} \int_{B(x_0,s)} f(y)\, dy \frac{ds}{s}. \end{aligned}$$

So (6.2) follows via Hardy's inequality and the non-homogeneous Wolff potential estimate

$$C_{m,p}([W^{\mu}_{m,p} > t]) \le A\mu(\mathbb{R}^n) \cdot t^{1-p}$$

analogous to (3.14). See [**AH**, p. 169]. Here the measure μ is played by $f(y)\, dy$.

For $mp > n$, we argue as in [**CFR**]. Take $n/m < q < p$, then

$$b(x) \le A|h|^{m-n/q} \left(\int_{|x_0-y|<2|h|} f(y)^q \, dy \right)^{1/q}$$

So

$$\sup_{|h|>0} |h|^{-1} b(x) \le A(M_{(m-1)q} f^q(x_0))^{1/q},$$

and when $(m-1)p < n$,

$$\int \left(\sup_{|h|>0} |h|^{-1} b(x) \right) dC_{m-1,p}(x) \le A \int (M_{(m-1)q} f^q(x_0))^{p/q} \, d\Lambda^{(\infty)}_{n-(m-1)p}$$
$$\le A \|f\|^p_{L^p},$$

using Theorem 7(a) and the standard estimate for $C_{\alpha,p}$ in terms of $\Lambda^{(\infty)}_{n-\alpha p}$; see [**AH**, p. 134]. When $(m-1)p = n$ the estimate on $\sup |h|^{-1} b(x)$ is simple. ■

(vi) The Kellogg property for thin functions. The idea here is to apply the standard Kellogg property for thin sets —see [**AH**, p. 175]— to produce an analogue for thin functions. Our result states:

$$C_{\alpha,p}(\{x : F(x) \ne 0\} \cap \{x : F \text{ is } (\alpha,p)\text{-thin at } x\}) = 0,$$

for $1 < p \le n/\alpha$. Indeed, if we set $E_\lambda = [F > \lambda]$, $\lambda > 0$, then it easily follows that E_λ is (α,p)-thin at points where F is (α,p)-thin. Hence by the standard Kellogg property

$$C_{\alpha,p}(E_\lambda \cap [E_\lambda \text{ is } (\alpha,p)\text{-thin}]) = 0.$$

Thus

$$\sum_{j=-\infty}^{\infty} C_{\alpha,p}(E_{2^j}; \cap [E_{2^j}; \text{ is } (\alpha,p)\text{-thin}]) = 0.$$

(b) The differentiation of weighted capacity.

A form of weighted capacity that naturally occurs is variational weighted capacity, given here by

$$C^\Omega_p(K; w) = \inf \left\{ \int |\nabla\phi|^p w \, dx : \phi \in C^\infty_0(\Omega),\ \phi \ge 1 \text{ on } K \right\}$$

where K is a compact subset of the (bounded) domain $\Omega \subset \mathbb{R}^n$. w is the weight, a non-negative locally integrable function on $\mathbb{R}^n$. This set function is needed in the study of the local regularity properties of certain degenerate second order elliptic partial differential equations on Ω,

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}(a_{ij}(x)u_{x_i}) = 0. \tag{6.6}$$

Here the assumption on the coefficients is generally

$$\lambda \cdot w(x)|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq \Lambda \cdot w(x)|\xi|^2,$$

for all $x \in \Omega$ and $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$; $\lambda > 0$ and $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$. Generally, more than local integrability of w is needed to build a complete theory —a theory of local regularity reproducing as much of the theory of harmonic functions as can be expected. One usually makes some assumption like $w^{-1} \in A_\infty$, i.e. it is an A_∞ weight in the sense of Muckenhoupt; see [**GF**]. Throughout, for simplicity we shall usually assume $w \in A_1$, i.e. on any cube $Q \subset \mathbb{R}^n$ with sides parallel to the coordinates axes,

$$⨍_Q w\,dx \leq A \cdot \inf_Q w.$$

The weighted variational capacity closely associated with this degenerate equation is, of course, $C_2^\Omega(\cdot; w)$. When $p \neq 2$, the prototype equation is a degenerate form of the p-Laplace equation, e.g.

$$\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = 0.$$

We shall restrict our attention here mainly to $p = 2$, although most results discussed have $p \neq 2$ analogues; see [**KM**], [**M**], [**MZ**].

The case $p = 2$, $w \equiv 1$ is treated in [**LSW**], whereas the case $p = 2$, w a general weight, in [**FJK**] —for the regularity theory of equation (6.6). In Choquet space notation, the Wiener boundary regularity test can be phrased as: if $n > 2$,

$$\sigma(x,s) = s^{2-n}\left(⨍_{B(x,s)} w^{-1}\right),$$

with

$$\int_0^1 \sigma(x_0,s)\frac{ds}{s} = +\infty,$$

then the point $x_0 \in \partial\Omega$ is a regular point for solutions to equation (6.6) iff

$$\int_{E\cap B(x_0,1)} \Phi(x_0,y)\, dC_2^\Omega(y;w) = +\infty$$

where

$$\Phi(x,y) \equiv \int_{|x-y|}^\infty \sigma(x,s)\frac{ds}{s}.$$

However, our task here is not to give a discription of this rich boundary behavior theory, but to try to understand the nature of weighted capacity: is it possible to find an alternate form (possibly in terms of Choquet integrals) that more clearly reveals how weighted capacity acts? The conjecture might be:

$$C_p^\Omega(K;w) \sim \int_K w\, dC_p^\Omega,$$

i.e. to represent weighted capacity as an equivalent capacity obtained by integrating the weight with respect to unweighted capacity —sort of a "Radon-Nikodym type" result. Our first result is of this type; it has appeared in [**A9**]. Here, we consider only $\Omega = \mathbb{R}^n$, $n > 2$.

Theorem 14. *(a) If $w \in L^2(C_2)^+$, then*

$$\lim_{r\to 0} ⨍_{B(x,r)} w\, dC_2 = w(x) \tag{6.7}$$

for C_2-a.e. $x \in \mathbb{R}^n$;

(b) if $w \in A_2 \cap L^2(C_2)^+$, then

$$\lim_{r\to 0} \frac{C_2(B(x,r);w)}{C_2(B(x,r))} = w(x) \tag{6.8}$$

for C_2-a.e. x;

(c) if w is superharmonic in $\mathbb{R}^n$, then there is a constant A such that

$$C_2(K;w) \le A \cdot \int_K w\, dC_2 \tag{6.9}$$

for all compact K;

(d) if $w = I_2 * v$, $v \in A_1$, then there is a constant A such that

$$\int_K w\, dC_2 \le A \cdot C_2(K;w). \tag{6.10}$$

Proof: We prove (6.9) and (6.10) here; see [**A9**] for the rest. For $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\int |\nabla\phi|^2 w\,dx = \int (-\Delta\phi)\cdot\phi w\,dx + \frac{1}{2}\int \phi^2\Delta w$$

after integration by parts. Passing to a limit, we may replace ϕ by the Newtonian potential $I_2 * \mu$, where μ is the $\dot{C}_{1,2}$ capacitary extremal measure for K. Then $\phi = 1$ on K, $\phi \le 1$ everywhere, and $-\Delta\phi = \mu$. Hence

$$\int |\nabla\phi|^2 w\,dx \le \int_K w\,d\mu$$

since $\Delta w \le 0$. The result (6.9) easily follows now since $\mu(e) \le \dot{C}_{1,2}(e) \sim C_2(e)$, for all Borel sets e.

For (6.10), we use CSI for C_2 (or $\dot{C}_{1,2}$):

$$\int_K w\,dC_2 \le \int [w^{1/2}\phi]^2\,dC_2 \le A\int\left(\frac{|\nabla w|^2}{w}\phi^2 + |\nabla\phi|^2 w\right)dx,$$

for some $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi \ge 1$ on K. Since $w = I_2 * v$, the Hedberg inequality (cf. section 3(c)) gives

$$\frac{|\nabla w|^2}{w} \le AMv \le Av,$$

since $v \in A_1$. Thus

$$\int \frac{|\nabla w|^2}{w}\phi^2\,dx \le A\int \phi^2 v\,dx = 2A\int \phi\nabla\phi\nabla I_2 * v\,dx$$
$$\le A'\left(\int \phi^2\frac{|\nabla w|^2}{w}\,dx\right)^{1/2}\left(\int |\nabla\phi|^2 w\right)^{1/2}.$$

So

$$\int \frac{|\nabla w|^2}{w}\phi^2\,dx \le A\int |\nabla\phi|^2 w\,dx. \quad \blacksquare$$

An alternate approach to such a "Radon-Nikodym type" result is to work with weighted Riesz capacity

$$\dot{C}_{\alpha,p}(K;w) = \inf\left\{\int f(x)^p w(x)\,dx : f \ge 0 \text{ and } I_\alpha * f \ge 1 \text{ on } K\right\}.$$

To insure that $\dot{C}_{\alpha,p}(\cdot;w)$ is non-trivial, one must make some assumptions on w, say $w \in A_p$ and for $\alpha p < n$

$$\int_1^\infty \left[\frac{t^{\alpha p}}{W(B(x,t))}\right]^{p'-1} \frac{dt}{t} < \infty. \tag{6.11}$$

Actually, another way to deal with this is to modify w near ∞ so that (6.11) holds. And since we are usually only interested in local properties, this is quite reasonable. This can be accomplished using [**GF**, Theorem IV.5.6]: modify w near ∞ so that $w \sim 1$ there.

With this we prove the following result:

Theorem 15. *Let $w \in A_p$ and choose an $x_0 \in \mathbb{R}^n$ such that*

(6.12) $\alpha p < n - d_w(x_0)$,

$$\text{where } d_w(x_0) = \inf\left\{d : \sup_{0<r\le 1} r^d \fint_{B(x_0,r} w < \infty\right\};$$

$$\lim_{r\to 0} \frac{\dot{C}_{\alpha,p}(B(x_0,r);w)}{r^{-\alpha p} w(B(x_0,r))} > 0; \tag{6.13}$$

there is a constant A_σ such that

$$\sup_{A_{k;\sigma}(x_0)} w \le A_\sigma \cdot \inf_{A_{k;\sigma}(x_0)} w \tag{6.14}$$

$$A_{k;\sigma}(x_0) = \{x : 2^{-k-\sigma} \le |x - x_0| < 2^{-k+1}\}$$

for each fixed $\sigma > 1$ and all $k \ge k_0$. Then

$$\dot{C}_{\alpha,p}(K;w) \sim \int_K w\, d\dot{C}_{\alpha,p} \tag{6.15}$$

for all compact K sets in some fixed neighborhood of x_0.

Note that if w is (α,p)-quasi-continuous, then $d_w(x_0) = 0$ for $\dot{C}_{\alpha,p}$-a.e. x_0. Also, it is not hard to see that one also has (6.13) holding for $\dot{C}_{\alpha,p}$-a.e. x_0 —note the weighted capacity of the ball is computed, at least asymptotically as $r \to 0$, in [**A10**]. Thus the real determination of the validity of the result is whether or not (6.14) holds at almost all points x_0. For the weight $w(x) = |x|^{-\sigma}$, $-n(p-1) < \sigma < n$, $d_w(0) = \sigma$.

Recall: $w \in A_p$ iff for coordinate cubes Q

$$\sup_Q ⨍_Q w \cdot \left(⨍_Q w^{-1/(p-1)} \right)^{p-1} < \infty.$$

Proof: Using subadditivity,

$$\begin{aligned}
\dot{C}_{\alpha,p}(K; w) &\le \sum_{k=0}^{\infty} \dot{C}_{\alpha,p}(A_{k;\sigma} \cap K; w) \\
&\le A \sum_k (\sup_{A_{k;\sigma}} w) \cdot \dot{C}_{\alpha,p}(A_{k;\sigma} \cap K) \\
&\le A \sum_k (\inf_{A_{k;\sigma}} w) \cdot \dot{C}_{\alpha,p}(A_{k,\sigma} \cap K) \\
&\le A \sum_k \int_{A_{k;\sigma} \cap K} w \, d\dot{C}_{\alpha,p} \le A \int_K w \, d\dot{C}_{\alpha,p},
\end{aligned}$$

where $\alpha p < n - d_w(x_0)$ allows us to use Lemma 8.1 of [**A10**] —which effectively factors out the weight— together with the quasi-additivity of $\dot{C}_{\alpha,p}$; see [**A4**]. For the reverse inequality, we need (6.13) to guarantee that the weighted Riesz capacity is quasi-additive in a neighborhood of x_0 —the proof is similar to that given in [**A4**], although an adaptation of the methods of [**Ai**] also seem possible. The reverse inequality is now established by a similar argument. ■

7. Additional remarks

(a) Poincare type inequalities.

The simplest form of an inequality that we wish to describe here is: let $u \in W^{1,p}(\Omega)$, Ω a bounded domain in $\mathbb{R}^n$ with smooth boundary, then there exists constants a_0 and A_0 such that

$$\left(\int_\Omega |u - a_0|^p \, dx \right)^{1/p} \le A_0 \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}, \tag{7.1}$$

with

$$|a_0| \le ⨍_K |u| \, dC_{1,p}, \tag{7.2}$$

K a compact subset of $\bar{\Omega}$, and

$$A_0 \le A \cdot C_{1,p}(K)^{-1/p} \cdot |\Omega|^{1/p}. \tag{7.3}$$

The barred integral in (7.2) is the integral average —divide by $C_{1,p}(K)$. Thus in particular, if u vanishes on K, $C_{1,p}(K) > 0$, then $a_0 = 0$. The classical Poincare inequality follows by taking $K = \partial\Omega$. And if $\Omega = B(x_0, r)$, then it follows that $A_0 \le Ar$, as is well known.

Higher order versions of (7.1) are given in [**AH**, sec. 8.3]. There a_0 is replaced by a polynomial P of degree $\le m-1$ when the right side of (7.1) becomes the norm of $|D^m u|$, the pure m^{th} order derivatives. But again, the Choquet integrals appear, to estimate $\sup_\Omega P$ in terms of capacity integral averages.

(b) Bounded point evaluations.

Let $P(x,D) = \sum_{|\sigma|\le m} a_\sigma(x) D^\sigma$ be an elliptic operator of order $m < n$ with infinitely differentiable coefficients defined in an open set $\Omega \subset \mathbb{R}^n$. Let K be a compact subset of Ω and $\eta(K)$ the set of solutions u, defined in some neighborhood of K which satisfy the equation $P(x,D)u = 0$ in this neighborhood. A question studied by several authors (see in particular [**FP**] and [**He2**]) is to determine when $\eta(K)$ is dense in $L^p(K)$, $1 \le p < \infty$. An answer can be given in terms of bounded point evaluations (BPE). The point $x_0 \in K$ is a BPE for $\eta(K) \subset L^p(K)$ if there is a constant A such that

$$|u(x_0)| \le A \left(\int_K |u|^p \, dx \right)^{1/p}$$

for all $u \in \eta(K)$. See [**FP**, Theorem 5]. Our interest here is noting that the necessary and sufficient condition for a BPE given in [**FP**] can be expressed via Choquet integrals rather nicely. It is: x_0 is a BPE for $\eta(K) \subset L^p(K)$ iff

$$\int_{K^c \cap B(x_0,1)} |x_0 - x|^{(m-n)p'} \, dC_{m,p'}(x) < \infty;$$

see Theorem 3 of [**FP**].

(c) Mosco convergence.

Let $\{\mathbb{K}_j\}$ be a sequence of convex sets in $W_0^{1,p}(\Omega)$. We say $\mathbb{K}_j \to \mathbb{K}$ in the sense of Mosco (see [**Mo**]) if both M_s and M_w hold:

(M_s) for all $v \in \mathbb{K}$ there is $\{v_j\} \subset \mathbb{K}_j$ such that

$$v_j \to v \text{ strongly in } W_0^{1,p};$$

(M_w) if $\{v_{j_k}\}$ is a subsequence, $v_{j_k} \in \mathbb{K}_{j_k}$ with $v_{j_k} \to v$ weakly in $W_0^{1,p}$, then $v \in \mathbb{K}$.

We apply this notion to the convex unilateral sets that occur with obstacle problems:

$$\mathbb{K}_j = \{v \in W_0^{1,p}(\Omega) : v \geq \psi_j,\ C_{1,p}\text{-a.e. on } \Omega\}$$

and

$$\mathbb{K} = \{v \in W_0^{1,p}(\Omega) : v \geq \psi,\ C_{1,p}\text{-a.e. on } \Omega\}.$$

From [**AP**] one can use the Choquet integral to determine Mosco convergence: let ψ_j and ψ be $(1,p)$-quasi-continuous on Ω such that

$$(H_s) \qquad \int [(\psi_j - \psi)^+]^p \, dC_{1,p} \to 0, \quad j \to \infty,$$

$$(H_w) \qquad \text{for all } t > 0, \quad C_{1,p}([(\psi - \psi_j)^+ > t]) \to 0, \quad j \to \infty.$$

It then follows that

$$(H_s) \text{ implies } (M_s)$$

$$(H_w) \text{ implies } (M_w).$$

Furthermore, if $\psi \in L^p(C_{1,p})$, then (M_s) implies (H_s).

Thus one has a very nice set of sufficient conditions for the Mosco convergence of convex unilateral sets in obstacle problems. And this would seem like a good tool to establish stability results for obstacle problems —as noted in section 5 above. Indeed, from [**Mo**] one can say that given a differential operator —linear and coercive— and the Mosco convergence of the unilateral convex sets, then the solution u_j to a variational problem for obstacle ψ_j converges strongly (in $W_0^{1,p}$) to the solution to the "limit problem". The key here, however, is: what is the limit problem? With stability, we demand that the limit problem be of the same type. What can happen with Mosco convergence is the "appearence of strange terms" in the limit problem; see e.g. [**CM**]. It seems that Mosco convergence is a good tool when one is dealing with a sequence of problems that converges to something, but does not seem to be strong enough to guarantee stability as noted in section 5 above. Mosco convergence has also been useful in dealing with questions of homogenization in partial differential equations —see e.g. [**BM**].

(d) Probability.

Functionals like $\Gamma_{\alpha,p}$, of section 3(c) above, are appearing in the theory of probability —as capacities on function spaces, e.g. Wiener space. See [**FLP**] or the short survey article [**Mn**]. Hence, using these functionals, Feyel and de La Pradelle have considered Choquet spaces in a probability setting.

References

[AH] D. R. ADAMS AND L. I. HEDBERG, "*Function Spaces and Potential Theory*," Grundlehren **314**, Springer-Verlag, 1996.

[A1] D. R. ADAMS, Potential theory before and after Wiener, *Proc. Sympos. Appl. Math.* **52** (1994), 63–83.

[A2] D. R. ADAMS, The classification problem for capacities associated with the Besov and Triebel-Lizorkin spaces, *PWN Polish Sci. Publ.* **22**, Banach Center Publ., Warsaw, Poland (1989), 9–24.

[A3] D. R. ADAMS, A note on Choquet integrals with respect to Hausdorff capacity, in "*Function Spaces and Applications*," Lund 1986, Lecture Notes in Math. **1302**, Springer-Verlag, 1988, pp. 115–124.

[A4] D. R. ADAMS, Sets and functions of finite capacity, *Indiana Univ. Math. J.* **27** (1978), 611–627.

[A5] D. R. ADAMS, Lectures on L^p-potential theory, Umeå Univ. Reports **2** (1981).

[A6] D. R. ADAMS, Capacity and the obstacle problem, *Appl. Math. Optim.* **8** (1981), 39–57.

[A7] D. R. ADAMS, L^p-capacitary integrals with some applications, *Proc. Sympos. Pure Math.* **35** (1979), 359–367.

[A8] D. R. ADAMS, Weakly elliptic systems with obstacle constraints, Part I, in "*Partial Differential Equations with Minimal Smoothness and Applications*," Math. and its Appl. **42**, Springer-Verlag, 1992, pp. 1–14.

[A9] D. R. ADAMS, Weighted capacity and the Choquet integral, *Proc. Amer. Math. Soc.* **102** (1988), 879–887.

[A10] D. R. ADAMS, Weighted nonlinear potential theory, *Trans. Amer. Math. Soc.* **297** (1986), 73–94.

[A11] D. R. ADAMS, Weakly elliptic systems with obstacle constraints, Part II, *J. Geom. Anal.* (to appear).

[AN] D. R. ADAMS AND E. NIEMINEN, Choquet integrals for obstacle problems, Preprint, University of Kentucky (1995).

[APe] D. R. ADAMS AND M. PIERRE, Capacity strong type estimates in semilinear problems, *Ann. Inst. Fourier (Grenoble)* **41** (1991), 117–135.

[AL] D. R. ADAMS AND H. LOPES, Nonlinear weakly elliptic 2×2 systems of variational inequalities with unilateral obstacle constraints, *Electron. J. Differential Equations* **1997(18)** (1997), 1–20.

[AV] D. R. Adams and R. Vandenhouten, Stability for polyharmonic obstacle problems with varying obstacles, Preprint, University of Kentucky (1998).

[Ai] H. Aikawa, Quasiadditivity of Riesz capacity, *Math. Scand.* **69** (1991), 15–30.

[AE] H. Aikawa and M. Essen, "*Potential Theory-Selected topics*," Lecture Notes in Math. **1633**, Springer-Verlag, 1996.

[An] B. Anger, Representations of capacities, *Math. Ann.* **229** (1977), 245–258.

[AP] H. Attouch and C. Picard, Inequations variationnelles avec obstacle et espaces functionnels en theorie duality potential, *Appl. Anal.* **12** (1981), 287–306.

[BZ] T. Bagby and W. P. Ziemer, Pointwise differentiability and absolute continuity, *Trans. Amer. Math. Soc.* **191** (1974), 129–148.

[BM] L. Baccardo and F. Murat, Homogenization of nonlinear unilateral problems. Composite media and homogenization theory, *Progr. Nonlinear Differential Equations Appl.* (Birkhauser) **5** (1991), 81–105.

[BI] B. Bojarski and T. Iwaniec, Analytical foundations of the theory of quasi conformal mappings in $\mathbb{R}^n$, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **8** (1983), 257–324.

[C] G. Choquet, Theory of capacities, *Ann. Inst. Fourier (Grenoble)* **5** (1953–54), 131–295.

[CK] L. Caffarelli and D. Kinderlehrer, Potential methods in variational inequalities, *J. Anal. Math.* **37** (1980), 285–295.

[CF] L. Caffarelli and A. Friedman, The obstacle problem for the biharmonic operator, *Ann. Scuola Norm. Sup. Pisa* **6** (1979), 151–184.

[CFR] C. Calderon, G. Fabes and N. Riviere, Maximal smoothing operators, *Indiana Univ. Math. J.* **23** (1974), 889–898.

[CM] D. Cioranescu and F. Murat, Un terme etrange venu d'ailleurs, Nonlinear Partial Differential Equations and their Applications, *College de France Seminar* (Pitman) **II** (1982), 98–138;, **III** (1983), 154–178.

[CZ] A. Calderon and A. Zygmund, Local properties of solutions of elliptic partial differential equation, *Studia Math.* **20** (1961), 171–225.

[Ca1] L. Carleson, "*Selected Problems on Exceptional Sets*," Van Nostrand, Princeton, 1967.

[Ca2] L. Carleson, An interpolation problem for bounded analytic functions, *Amer. J. Math.* **80** (1958), 921–930.

[FJK] G. Fabes, D. Jerison and C. Kenig, The Wiener test for degenerate elliptic equations, *Ann. Inst. Fourier (Grenoble)* **32** (1982), 151–182.

[FLP] D. Feyel and A. de La Pradelle, Capacites Gaussiennes, *Ann. Inst. Fourier (Grenoble)* **41** (1991), 49–76.

[FP] C. Fernstrom and J. Polking, Bounded point evaluations and approximations in L^p by solutions of elliptic partial differential equations, *J. Funct. Anal.* **28** (1978), 1–20.

[Fe] R. Fefferman, A theory of entropy in Fourier analysis, *Adv. Math.* **30** (1978), 171–201.

[F] J. Freshe, On systems of second order variational inequalities, *Israel J. Math.* **15** (1973), 421–429.

[Fu] B. Fuglede, Capacity as a sublinear functional generalizing an integral, *Mat. Fys. Medd. Danske Vid. Selsk.* **38(7)** (1971), 44 p.

[GF] J. Garcia-Cuerva and J. L. Rubio de Francia, "*Weighted Norm Inequalities and Related Topics*," Math. Studies **116**, North Holland, 1985.

[H] L. L. Helms, "*Introduction to Potential Theory*," Wiley Series Pure and Appl. Math **22**, 1969.

[Ha] K. Hansson, Imbedding theorems of Sobolev type in potential theory, *Math. Scand.* **45** (1979), 77–102.

[He1] L. I. Hedberg, On certain convolution inequalities, *Proc. Amer. Math. Soc.* **36** (1972), 505–510.

[He2] L. I. Hedberg, Bounded point evaluations and capacity, *J. Funct. Anal.* **10** (1972), 269–280.

[HKM] J. Heinonen, T. Kilpeläinen and O. Martio, "*Nonlinear Potential Theory of Degenerate Elliptic Equations*," Oxford Univ. Press, Oxford, 1993.

[Ho] L. Hormander, L^p estimate for (pluri-) subharmonic functions, *Math. Scand.* **20** (1967), 65–78.

[HW] S. Hildebrant and K. O. Widman, Variational inequalities for vector valued functions, *J. fur Math.* **309** (1979), 191–220.

[K] S. Krantz, Fractional integration on Hardy spaces, *Studia Math.* **63** (1982), 87–94.

[KM] T. Kilpeläinen and J. Maly, The Wiener test and potential estimates for quasilinear elliptic equations, *Acta Math.* **172** (1994), 137–161.

[KS] D. Kinderlehrer and G. Stampacchia, "*Introduction to Variational Inequalities and their Applications*," Academic Press, 1980.

[LM] G. Li and O. Martio, Stability in Obstacle problems, *Math. Scand.* **75** (1994), 87–100.

[LSW] W. Littman, G. Stampacchia and H. Wienberger, Regular points for elliptic equations with discontinuous coefficients, *Ann. Scuola Norm. Sup. Pisa* **17** (1963), 45–79.

[Mn] P. Malliavin, Applications and foundations of quasi sure analysis, *Progr. Probab.* **36** (1995), 199–203.

[Ma1] V. G. Maz'ya, On certain integral inequalities for functions of many variables, *J. Soviet Math.* **1** (1973), 205–234.

[Ma2] V. G. Maz'ya, The negative spectrum of the n-dimensional Schrödinger operator, *Soviet Math.* **3** (1962), 808–810.

[Ma3] V. G. Maz'ya, On the theory of the n-dimensional Schrödinger operator, *Izv. Akad. Nauk. SSSR ser Mat.* **28** (1964), 1145–1172.

[Me] M. S. Melnikov, Metric properties of analytic α-capacity and approximation of analytic functions with a Hölder condition by rational functions, *Math. USSR Sbornik* **8** (1969), 115–124.

[M1] Y. Mizuta, Fine differentiability of Riesz potentials, *Hiroshima Math. J.* **8** (1978), 505–514.

[M2] Y. Mizuta, "*Potential Theory in Euclidean Spaces*," Gakuto International Series, Mathematical Sciences and Applications **6**, Gakkotosho, Tokyo, Japan, 1996.

[M] P. Mikkonen, On the Wolff potential and quasilinear elliptic equations involving measures, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **104** (1996), 1–71.

[MZ] J. Maly and W. P. Ziemer, "*Fine Regularity of Solutions of Elliptic Partial Differential Equations*," Amer. Math. Society Math. Monographs **51**, 1997.

[Mo] U. Mosco, Convergence of convex sets and solutions of variational inequalities, *Adv. Math.* **3** (1969), 510–585.

[Ne1] Netrusov, Metric estimates for the capacities of sets in Besov spaces, *Trudy Mat. Inst. Steklov.* **190** (1989), 159–185 (Russian); Engl. trans.: *Proc. Steklov Inst. Math.* **190** (1992), 167–192.

[Ne2] Netrusov, Estimates of capacities associated with Besov spaces, *Azpiski Nauchn. Semin. LOMI* **201, xx92)**, 124-156 (Russian); Engl. trans.: *J. Soviet Math.*.

[OV] J. Orobitg and J. Verdera, Choquet integrals, Hausdorff content and the Hardy-Littlewood maximal operator, *Bull. London*

Math. Soc. **30** (1998), 145–150.

[RS] T. Runst and W. Sickel, "*Sobolev Spaces of Fractional Order Nemytskij Operators, and Nonlinear Partial Differential Equations*," de Gruyter, 1996.

[R] J.-F. Rodrigues, "*Obstacle Problems in Mathematical Physics*," Math. Studies **134**, North-Holland, 1987.

[Sa] E. Sawyer, Weighted norm inequalities for fractional maximal operators, *Canad. Math. Soc. Conference Proc.* **1** (1981), 283–309.

[S] B. Schild, On the coincidence set in biharmonic variational inequalities with thin obstacles, *Ann. Scuola Norm. Sup. Pisa* **13** (1986), 559–616.

[St1] E. Stein, "*Singular Integrals and Differentiability of Functions*," Princeton U. Press, 1970.

[St2] E. Stein, "*Harmonic Analysis: Real-variable methods orthogonality, and oscillatory integrals*," Princeton U. Press, 1993.

[SW] E. Stein and G. Wiess, "*Introduction to Fourier Analysis on Euclidean Spaces*," Princeton U. Press, 1971.

[TW] M. Taibleson and G. Weiss, The molecular characterization of certain Hardy spaces, *Astérisque* **77** (1980), 67–149.

[To] G. M. Troianello, "*Elliptic Differential Equations and Obstacle Problems*," Plenum Press, 1987.

[T] B. O. Turesson, Nonlinear Potential Theory and Weighted Sobolev Spaces, in Lecture Notes in Math., Springer-Verlag (to appear).

[Z] W. P. Ziemer, "*Weakly Differentiable Functions*," Graduate Texts in Math. **120**, Springer-Verlag, 1989.

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