Pub. Mat. UAB
N° 25 Juny 1981

Two Coincidence Theorems

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Rebut el 16 de Març del 1981

I. INTRODUCTION

This note deals with the existence and unicity of coincidence points $f(x) = g(x)$ for two maps, like in Cerdà [1], related by a contractive type relation similar to the properties required in some generalisations of the Banach fixed point theorem.

In a first theorem we consider maps f, g from a topological space to a complete metric space, following the condition

(i) For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(f(x),f(y)) < \varepsilon + \delta \Rightarrow d(g(x),g(y)) < \varepsilon$$

like in Meir-Keeler [2]. We shall prove that the coincidence set S is non-empty and both functions are constant on S.

We give an example of mappings keeping contractive relation (i) but not the one in [1].

In a second theorem based on an article by Chi Song Wong [3], mappings f, g are defined on a uniform space $(X,\mathcal{U})$ and we consider the uniformity basis $\mathcal{U}_\varphi$ defined by

$$\mathcal{U}_\varphi = \{\varphi^{-1}(U) \times \varphi^{-1}(U) \cup \Delta : U \in \mathcal{U}\}$$

being $\varphi(x)=(f(x),g(x))$ and Δ the diagonal of XxX, and we shall prove that the coincidence set S has a unique point if f is uniformly continuous from

$(X,\mathcal{U}_\varphi)$ into $(X,\mathcal{U})$ and keeps some other conditions.

2. THEOREM I

THEOREM I. Let X be a non-empty topological space and Y a non- empty metric space. Let f and g be two mappings of X into Y keeping (i), and

(ii) f be proper and continuous

(iii) $g(X) \subset f(X)$

(iv) $\overline{g(X)}$ complete.

Then S is non-empty and f(S) has a unique point.

Proof. If $f(x) \neq f(y)$, (i) implies $d(g(x),g(y)) < d(f(x),f(y))$ (1)

and then f(S) has at great a point.

Now we shall prove that S is non-empty. Let $x_o \in X$, (iii) implies $f^{-1}\{g(x_o)\} \neq \emptyset$. Pick $x_1 \in f^{-1}\{g(x_o)\}$. Repeating the same operation we obtain a sequence x_n of X keeping $f(x_{n+1}) = g(x_n)$ for each $n \in \mathbb{N}$.

Let $c_n = d(f(x_n),f(x_{n+1}))$; if there exists n_o which $c_{n_o} = 0$, then $f(x_{n_o}) = f(x_{n_o+1}) = g(x_{n_o})$; therefore, $x_{n_o} \in S$ and the theorem is proved. On the contrary, we will have $c_n > 0$ for each n, and, at the first time, we are going to prove $c_n \searrow 0$. Since $0 < c_n$, (i) implies $c_n < d(g(x_n),g(x_{n+1})) = c_{n+1}$; the sequence is decreasing and positive. Let $c = \lim_n c_n$. If $c > 0$, (i) implies that there exists a positive number δ such that, taking c_n that keeps $c_n < c+\delta$ we have

$$c \leq c_n < c + \delta \Rightarrow c_{n+1} < c \qquad \text{and it contradicts}$$

$c_n \searrow c$; therefore, $c_n \searrow 0$ as we would like to see.

Now, we are going to prove that $f(x_n)$ is a Cauchy sequence. The proof is by contradiction. If $f(x_n)$ wasn't a Cauchy sequence there would exist $\varepsilon > 0$ so that Cauchy relation wouldn't keep for 2ε. For a such ε, there exists $\delta > 0$ keeping (i). For a such δ pick s so that

$$c_s < \delta/3 . \qquad (2)$$

Pick $k,m > s$ fulfilling $d(f(x_k), f(x_m)) > 2\varepsilon$ and $k < m$. If $n \in [k,m]$,

$$|d(f(x_k), f(x_{n+1})) - d(f(x_k), f(x_n))| < \delta/3. \qquad (3)$$

Consider $A = \{n \in [k,m] : d(f(x_k), f(x_n)) \geq \varepsilon + \delta\}$, obviously $m \in A$, and let $i \in [k,m]$ be such that $i+1 = \min A$. We have

$$d(f(x_k),f(x_{i+1})) - d(f(x_k),f(x_i)) \geq \varepsilon + \delta - d(f(x_k), f(x_i)) \qquad (4)$$

and, (3) and (4) imply $d(f(x_k), f(x_i)) > \varepsilon + 2\delta/3$. (5)

However, $d(f(x_k), f(x_i)) \leq d(f(x_k), f(x_{k+1})) + d(f(x_{k+1}), f(x_{i+1})) +$

$$+ d(f(x_{i+1}), f(x_i)) < \varepsilon + 2\delta/3$$

because of (2), $i \notin A$, (5) and (i). It contradicts (5). This contradiction proves that $f(x_n)$ must be a Cauchy sequence.

Because of (iv), $f(x_n) \longrightarrow y \in \overline{g(X)}$. Let $B = \{y\} \cup \{f(x_n)\}_{n \in \mathbb{N}}$ which is a compact of Y. (ii) implies that $f^{-1}(B)$ is compact; therefore, there exists a partial sequence of x_n converging to $x \in f^{-1}(B)$, and $f(x) = y$ by the continuity of f. The continuity of g, given by (i), implies that the image by g of this partial converges to $g(x)$; moreover, the handling of the sequence x_n implies $g(x) = f(x)$ and S is non-empty as we would like to prove. The theorem is proved.

COROLLARY. If we change f proper by g proper between the hypothesis of the theorem I, this one will remain true.

EXAMPLE. Let $X = [0,1] \cup \{3n,3n+1\}_{n \in \mathbb{N}}$ and $Y = \mathbb{R}^2$, both with euclidean distance, and let

$$f(x) = (x/2, \sqrt{3}/2\ x)$$

$$g(x) = \begin{cases} (x/4, \sqrt{3}/4\ x) & \text{if } x \in [0,1] \\ (0,0) & \text{if } x = 3n \\ (1/2 - 1/2n+4, \sqrt{3}/2 - \sqrt{3}/2n+4) & \text{if } x = 3n+1 \end{cases}$$

There isn't any mapping φ such that $d(g(x),g(y)) \leq \varphi[d(f(x),f(y))]$ keeping $\varphi(\varepsilon) < \varepsilon$ for each $\varepsilon > 0$ because $\varphi(1)$ couldn't be minor than 1 since $d(f(3n),f(3n+1)) = 1$ and $d(g(3n),g(3n+1)) = 1 - 1/n+2$, and φ should keep $1 - 1/n+2 \leq \varphi(1)$ for each n. Conversely, we can prove without any difficulty that these mappings keep the hypothesis of theorem I.

3. THEOREM II

THEOREM II. Let $(X,\mathcal{U})$ be a non-empty Hausdorff complete uniform space, f and g be two functions from X into X. If

(i) f is uniformly continuous of $(X,\mathcal{U}_\varphi)$ into $(X,\mathcal{U})$

(ii) $\forall U \in \mathcal{U}\quad \exists V \in \mathcal{U}$ such that $(f(x),f(y)) \in V \Rightarrow (x,y) \in U$

(iii) $\varphi^{-1}(U)$ is non-empty and closed for each closed symmetric member U of $\mathcal{U}$.

Then f and g have a unique coincidence point.

Proof. Pick the filter $\mathcal{F} = \{\varphi^{-1}(U) : U \in \mathcal{B}\}$, where $\mathcal{B}$ is the set formed by all the closed symmetric members of $\mathcal{U}$.

We will see $\mathcal{F}$ is a Cauchy filter. Let $U \in \mathcal{U}$, pick $V \in \mathcal{U}$ such that $(f(x),f(y)) \in V \Rightarrow (x,y) \in U$ since (ii). By (i) we can find $W \in \mathcal{B}$ such that $(x,y) \in \varphi^{-1}(W) \times \varphi^{-1}(W) \cup \Delta \Rightarrow (f(x),f(y)) \in V$. Taking $\varphi^{-1}(W)$ we have $\varphi^{-1}(W) \in \mathcal{F}$ and $\varphi^{-1}(W) \times \varphi^{-1}(W) \subset U$; therefore $\mathcal{F}$ is a Cauchy filter.

Since $\mathcal{F}$ is a Cauchy filter, X complete Hausdorff and (iii), we have

$$\bigcap_{W\in\mathcal{B}} \varphi^{-1}(W) = \{x_0\} ;$$

moreaver, taking images by φ it results that $\Delta = \bigcap_{W\in\mathcal{B}} W \ni (f(x_0),g(x_0))$ implies $f(x_0) = g(x_0)$. We shall now prove the unicity. If y_0 is a coincidence point $\varphi(y_0) \in \Delta \Rightarrow \varphi(y_0) \in \bigcap_{W\in\mathcal{B}} W \Rightarrow y_0 \in \bigcap_{W\in\mathcal{B}} \varphi^{-1}(W) = \{x_0\} \Rightarrow y_0 = x_0$ and the theorem is proved.

COROLLARY. If X is a complete metric space, the theorem says:

(i) $\forall\varepsilon > 0 \quad \exists\delta > 0$: $d(f(x),g(x)) + d(f(y),g(y)) < \delta \Rightarrow d(f(x),f(y)) < \varepsilon$

(ii) $\forall\varepsilon > 0 \quad \exists\delta' > 0$: $d(f(x),f(y)) < \delta' \Rightarrow d(x,y) < \varepsilon$

(iii) $\forall\varepsilon > 0$ $\{x : x\in X$ and $d(f(x),g(x)) \leq \varepsilon\}$ is non-empty and closed.

Then, f and g have a unique coincidence point.

Proof. It is enough to observe that a vecinity of $\mathcal{U}_\varphi$ has the form $U_\varphi^r = \{(x,y) : x = y$ or $[d(f(x),g(x)) < r$ and $d(f(y),g(y)) < r]\}$ and if we consider $B_\varphi^r = \{(x,y) : x = y$ or $d(f(x),g(x)) + d(f(y),g(y)) < r\}$ we will have $B_\varphi^r \subset U_\varphi^r \subset B_\varphi^{2r}$.

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Aquest treball fou presentat a les Jornades MHL de Santander (1979).