

Multiplicative functionals on function algebras

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ABSTRACT. Let X be a completely regular Hausdorff space and $C(X)$ the algebra of all continuous \mathbb{K} -valued functions on X ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}). If $A \subseteq C(X)$ is a subalgebra, in [4] can be found conditions on A under which each character of A , i.e., each non-zero \mathbb{K} -linear multiplicative functional $\phi: A \rightarrow \mathbb{K}$, is given by a point evaluation at some point of X .

In this paper we present a «Michael» type theorem for the particular case in which X is a real Banach space. As consequence it is showed that if E is a separable Banach space or E is the topological dual space of a separable Banach space and A is the algebra of all real analytic or the algebra of all real C^m -functions, $m = 0, 1, \dots, \infty$, on E , then every character ϕ of A is a point evaluation at some point of E .

Let E be a real Banach space with topological dual E' and let $C(E)$ be the algebra of all continuous \mathbb{R} -valued functions on E . Let $l^1(\mathbb{N}) = \{\alpha = (\alpha_n) \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} |\alpha_n| < \infty\}$.

Theorem 1. Assume that there exists $(\phi_n)_{n=1}^{\infty} \subset E'$, $\|\phi_n\| \leq 1$ for every $n \in \mathbb{N}$, such that (ϕ_n) separates points of E . Let $A \subseteq C(E)$ be a subalgebra with $1 \in A$. Assume:

(i) If $f \in A$, $f(x) \neq 0$ for all $x \in E$, then $1/f \in A$.

(ii) $E' \subset A$ and for every $\alpha = (\alpha_n) \in l^1(\mathbb{N})$, the function $\sum_{n=1}^{\infty} \alpha_n \cdot \phi_n^2$ belongs to A .

Then every character $\phi: A \rightarrow \mathbb{R}$, such that $\phi(\phi_n) = \phi_n(a)$ for every $n \in \mathbb{N}$ and some $a \in E$, is the point evaluation at a .

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Proof. Let $\alpha = (\alpha_n) \in l^1(\mathbb{N})$ with $\alpha_n > 0$ for all $n \in \mathbb{N}$. Condition (ii) implies that the functions:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n^2(x-a) \text{ and } g(x) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n} \phi_n^2(x-a) \text{ belong to } A$$

For each $N \in \mathbb{N}$, let $x \in E$ such that $\phi(f) = f(x)$; $\phi(g) = g(x)$ and $\phi(\phi_i) = \phi_i(x)$, $i = 1, \dots, N$ (a such x exists after condition (i)). For this $x \in E$, we have

$$\phi(f) = \sum_{n=1}^{\infty} \alpha_n \phi_n^2(x-a) ; \quad \phi(g) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n} \phi_n^2(x-a)$$

Therefore $0 \leq N\phi(g) \leq \phi(f)$ and it follows that $\phi(g) = 0$.

If $h \in A$ is given, let $y \in E$ such that $\phi(h) = h(y)$ and $\phi(g) = g(y)$. Since $\phi(g) = g(y) = 0$, it follows that $\phi_n(y) = \phi_n(a)$ for all $n \in \mathbb{N}$, i.e., $y = a$ and $\phi(h) = h(a)$.

Remark 1. The hypothesis on the real Banach space E in Theorem 1 is equivalent to say that E' is $\sigma(E'; E)$ -separable. Therefore it holds when E is a separable Banach space and when E is the topological dual space of a separable Banach space.

Consequences

Let $A(E)$ be, respectively $C^m(E)$ ($m = 0, 1, \dots, \infty$), the subalgebra of $C(E)$ of all real analytic functions (see [2]), respectively of all C^m -functions in the Fréchet sense, on E .

Corollary 1. *If E is finite dimensional and $A = A(E)$ or $A = C^m(E)$, then every character $\phi: A \rightarrow \mathbb{R}$ is a point evaluation at some point of E .*

Proof. This follows from Theorem 1 if we consider (ϕ_n) as the canonical projections.

Proposition 1. *For every character $\phi: A(E) \rightarrow \mathbb{R}$, the restriction $\phi|_E$ is $\sigma(E'; E)$ -sequentially continuous.*

Proof. Assume that $(x'_n) \subset E'$ converges to zero for the $\sigma(E'; E)$ -topology. If $\phi(x'_n) \not\rightarrow 0$, there are $\alpha > 0$ and (x'_{n_p}) , subsequence of (x'_n) , such that

$$\left[\phi \left[\frac{x'_{n_p}}{\sqrt{\alpha}} \right] \right]^2 > 1$$

for every $p \in \mathbb{N}$. Since $(x'_{n_p}) \rightarrow 0$ ($p \rightarrow \infty$) for the $\sigma(E'; E)$ -topology,

the function

$$f(x) = \sum_{p=1}^{\infty} \left[\frac{x'_{np}(x)}{\sqrt{\alpha}} \right]^{2p}$$

is well defined and $f \in A(E)$. (See ([2], Th. 6)). For each $N \in \mathbb{N}$,

$$\phi(f) \geq \phi \left[\sum_{p=1}^N \left[\frac{x'_{np}}{\sqrt{\alpha}} \right]^{2p} \right] = \sum_{p=1}^N \left[\phi \left[\frac{x'_{np}}{\sqrt{\alpha}} \right] \right]^{2p}$$

Therefore $\sum_{p=1}^{\infty} \left[\phi \left[\frac{x'_{np}}{\sqrt{\alpha}} \right] \right]^{2p} < \infty$ and then $\left[\phi \left[\frac{x'_{np}}{\sqrt{\alpha}} \right] \right]^{2p} \rightarrow 0$ ($p \rightarrow \infty$), which is a

contradiction because $\left[\phi \left[\frac{x'_{np}}{\sqrt{\alpha}} \right] \right]^2 > 1$ for all $p \in \mathbb{N}$.

Corollary 2. *Let E be a separable Banach space and $\phi: A(E) \rightarrow \mathbb{R}$ a character. Then $\phi|_E$ is a point evaluation at some point of E .*

Proof. This is immediate from Prop. 1, since by ([5], Ch. IV; Th. 6.2 and Corollary 3) for $\phi|_E$ to be $\sigma(E'; E)$ -continuous it suffices to show that $\phi|_E$ is $\sigma(E'; E)$ -sequentially continuous.

Corollary 3. *Let E be a separable Banach space and $\phi: A(E) \rightarrow \mathbb{R}$ a character. Then ϕ is a point evaluation at some point of E .*

Proof. This is immediate from Theorem 1, Remark 1 and Corollary 2.

Let F be a separable Banach space and $(y_n)_{n=1}^{\infty}$ a dense subset in $\{y \in F : \|y\| \leq 1\}$. Let $E = F'$. Let $\phi_n: E \rightarrow \mathbb{R}$ be defined as $\phi_n(x) = x(y_n)$. Then $\phi_n \in E'$, $\|\phi_n\| \leq 1$ and $(\phi_n)_{n=1}^{\infty}$ separates points of E . The mapping $y \rightarrow \phi_y$, defined as $\phi_y(x) = x(y)$, allow us identify F with a subspace of $E' = F''$. Thus, if $\phi: A(E) \rightarrow \mathbb{R}$ is a character, Prop. 1 implies that $\phi|_F$ is $\|\cdot\|$ -continuous, therefore $\phi|_F \in F' = E$. Then, it follows that there exists $a \in E$ such that $\phi(\phi_n) = \phi_n(a)$ for all $n \in \mathbb{N}$. Now the following Corollary is clear after Theorem 1.

Corollary 4. *Let E be a topological dual space of a separable Banach space and $\phi: A(E) \rightarrow \mathbb{R}$ a character. Then ϕ is a point evaluation at some point of E .*

Corollary 5. *Assume that E is a separable Banach space or E is the topological dual space of a separable Banach space. Then every character $\phi: C^m(E) \rightarrow \mathbb{R}$, $m=0, 1, \dots, \infty$, is a point evaluation at some point of E .*

Proof. $\phi_{|A(E)}$ is a point evaluation by Corollary 3 and Corollary 4. Thus, ϕ satisfies conditions of Theorem 1 with $A = C^m(E)$.

Remark 2. The Corollary 5, for the particular case E a separable Banach space and $m = \infty$, can be found in [1]. Also, for E with C^m -partitions of unity and $m < \infty$, see [3].

References

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