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Holomorphic Functions on Strict Inductive Limits of Banach Spaces

SEÁN DINEEN and LUIZA A. MORAES

ABSTRACT. In this article we show that a number of apparently different properties coincide on the set of holomorphic functions on a strict inductive limit (all inductive limits are assumed to be countable and proper) of Banach spaces and that they are all satisfied only in the trivial case of a strict inductive limit of finite dimensional spaces. Thus the linear properties of a strict inductive limit of Banach spaces rarely translate themselves into holomorphic properties.

§0

The main ingredients in our proof are the extension to strict inductive limits of a property of the τ_δ semi-norms which was previously known for direct sums of Banach spaces [4], a characterization of Schwartz spaces using limited sets due to Lindström [8] and a close examination of a special hypoanalytic function on a strict inductive limit of Banach spaces constructed in [7].

We refer to [6] for details concerning holomorphic functions on locally convex spaces. Information regarding holomorphic functions on direct sums of Banach spaces, $\mathcal{DF}$ spaces and on inductive limits of Fréchet spaces may be found in [1, 3, 4, 6, 7, 9]. We thank J. M. Ansemil for some helpful comments.

§1

In this section we recall some definitions and prove a number of results that we shall use later.

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If E is a locally convex space over $\mathbf{C}$ we let $\mathscr{H}_G(E)$ denote the set of $\mathbf{C}$ valued functions on E whose restriction to each finite dimensional subspace of E is a holomorphic function (of several complex variables). If $f \in \mathscr{H}_G(E)$ is continuous on compact sets we say that f is hypoanalytic and the space of all such functions is denoted by $\mathscr{H}_{HY}(E)$. If $f \in \mathscr{H}_G(E)$ is continuous then f is called holomorphic and we let $\mathscr{H}(E)$ denote the space of all holomorphic functions on E. We let $\mathscr{P}(^n E)$ denote the space of all continuous n-homogeneous polynomials on E.

The compact open topology on $\mathscr{H}(E)$ and $\mathscr{H}_{HY}(E)$ is denoted by τ_0. A semi-norm p on $\mathscr{H}(E)$ is said to be τ_ω continuous if there exists a compact subset K of E such that for each open subset V of E, $K \subset V$, there exists $C(V) > 0$ such that

$$p(f) \leq C(V) \sup_{x \in V} |f(x)|$$

for all $f \in \mathscr{H}(E)$.

We introduce the τ_δ topology (see [6, proposition 3.27]) in a somewhat unusual fashion but one which is more suitable for the purposes of this article.

The τ_δ topology on $\mathscr{H}(E)$ is the topology generated by all semi-norms which satisfy the following two conditions

$$p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) \tag{1.1}$$

for every f in $\mathscr{H}(E)$,

$$p|\mathscr{P}(^n E) \tag{1.2}$$

is τ_ω continuous for each positive integer n.

If τ is a locally convex topology on $\mathscr{H}(E)$, then $(\mathscr{H}(E), \tau)$ is T.S. τ-complete if for any sequence $(P_n)_{n=0}^{\infty}$, $P_n \in \mathscr{P}(^n E)$ all n, such that $\sum_{n=0}^{\infty} p(P_n) < \infty$ for every τ-continuous seminorm p on $\mathscr{H}(E)$ we have $\sum_{n=0}^{\infty} P_n \in \mathscr{H}(E)$ ([6, definition 3.32]).

A subset B of E is said to be *bounding* if $\|f\|_B := \sup_{x \in B} |f(x)| < \infty$ for every $f \in \mathscr{H}(E)$ and $L \subset E$ is said to be *limited* if for every equicontinuous weak* null sequence in E', $(\phi_n)_n$, we have $\|\phi_n\|_L \to 0$ as $n \to \infty$.

Proposition 1. *([8]) A locally convex space E is a Schwartz space if and only if it is quasi-normable and the bounded subsets of E are limited.*

Proposition 2. *If E is a barrelled locally convex space then every weak* null sequence in E' converges uniformly to zero on the bounding subsets of E. In particular the bounding subsets of E are limited.*

Proof. Let B denote a bounding subset of E and let $(\phi_n)_{n=1}^{\infty}$ denote a weak* null sequence in E'. Let $f=\sum_{n=1}^{\infty} \phi_n^n$. Since the sequence $(\phi_n)_n$ is weak* null the function f is defined on all of E and hence belongs to $\mathscr{H}_G(E)$. Since the sequence $(\phi_n)_n$ is pointwise bounded in the barrelled space E, it is equicontinuous. Thus f belongs to $\mathscr{H}(E)$ by [2, proposition 1]. By corollary 4.19 (b) of [6] this implies

$$\lim_{n\to\infty} (\|\phi_n^n\|_B)^{\frac{1}{n}} = \lim_{n\to\infty} \|\phi_n\|_B = 0$$

and hence B is a limited subset of E. This completes the proof.

The following result is a reformulation of [7, proposition 3]. We sketch the proof for the readers' convenience

Proposition 3. *Let $E=\varinjlim_{n} E_n$ denote a strict inductive limit of Banach spaces and suppose dimension $(E_n)=\infty$ for some integer n. Then there exists a g in $\mathscr{H}_{HY}(E)$ with the following properties;*

$$g \notin \mathscr{H}(E) \tag{1.3}$$

$$\text{if} \quad Z_m = \left\{ n; \frac{\hat{d}^n g(0)}{n!}\bigg|_{E_m} \neq 0 \right\}$$
$$\text{then} \sum_{n\in Z_m} \frac{\hat{d}^n g(0)}{n!} \in \mathscr{H}(E) \tag{1.4}$$

for every positive integer m.

Proof. Since $\dim(E_n)=\infty$ for some n, E is not a Schwartz space and hence proposition 1 implies that E contains a bounded subset which is not limited. By proposition 2 the bounding subsets and the limited subsets of E coincide and hence E contains a bounded set, which we may suppose is the unit ball B_1 of E_1, which is not bounding. Hence there exists $f\in\mathscr{H}(E)$,

$f=\sum_{n=1}^{\infty} P_n$, such that $\|P_{n_j}\|_{B_1} \geq 1$ for all j where $(n_j)_j$ is a strictly increasing sequence of positive integers satisfying $n_{j+1} > n_j + j$ for all j.

Let $\phi_n \in E'$, $n \geq 2$, satisfy $\phi_n | E_{n-1} = 0$ and $\phi_n(x_n) = 1$ for some $x_n \in E_n$ with $\|x_n\| = 1$ and let $\theta_k(z) = kz$ for all $z \in E$.

We consider the function

$$g=\sum_{k=2}^{\infty}\left\{\left(\sum_{j=k}^{\infty} P_{n_j}\right)\circ \theta_k\right\}\phi_k^k.$$

Since $j \geq k$ the condition $n_{j+1} > n_j + j$ implies that $\sum_{k=2,\, j\geq k}^{\infty} (P_{n_j}\circ\theta_k)\,\phi_k^k$ is the Taylor series expansion of g.

Since

$$g\,|\,E_n=\sum_{k=2}^{m+1}\left\{\sum_{j=k}^{\infty} P_{n_j}\circ\theta_k\right\}\phi_k^k\Big|_{E_n}$$

it follows that $g \in \mathscr{H}_{HY}(E)$ and, moreover,

$$\sum_{n\in Z_m}\frac{\hat{d}^n g(0)}{n!} \in \mathscr{H}(E) \quad \text{for all } m.$$

We refer to [7] for the details which show that $g \notin \mathscr{H}(E)$.

Our next result was proved for countable direct sums of Banach spaces in [4] (see also [6, proposition 4.40]).

Proposition 4. *If $E=\varinjlim_n E_n$ is a strict inductive limit of Banach spaces and p is a τ_δ continuous seminorm on $\mathscr{H}(E)$ then there exists a positive integer m such that $f \in \mathscr{H}(E)$ and $f\,|\,E_m = 0$ imply $p(f)=0$.*

Proof. We may suppose without loss of generality that the semi-norm p satisfies (1.1) and (1.2). If the result is not true then for every positive integer n there exists a continuous homogeneous polynomial P_n such that $P_n | E_n = 0$ and $p(P_n) \neq 0$. We now show that the sequence $(P_n)_{n=1}^{\infty}$ is locally bounded.

Let K denote a compact subset of E. Then K is contained and compact in some E_k. For each positive integer j let B_j denote the unit ball in E_j.

Now choose $M>0$ and $\lambda_1, \dots, \lambda_k$ positive numbers such that

$$\|P_j\|_{K+\sum_{r=1}^{k} \lambda_r B_r} \leq M \quad \text{for } j=1, \dots, k$$

Using a binomial expansion and the fact that $P_{k+1}|E_{k+1}=0$ we can find $\lambda_{k+1}>0$ such that

$$\|P_j\|_{K+\sum_{r=1}^{k+1}\lambda_r B_r}\leq M+\frac{1}{2^{k+1}} \text{ for } j=1 ...., k+1.$$

By induction we can find a sequence of positive numbers $(\lambda_r)_{r=1}^{\infty}$ such that

$$\|P_j\|_{K+\sum_{r=1}^{\infty}\lambda_r B_r}\leq M+1 \quad \text{for all } j.$$

Since E is bornological it follows that $\sum_{r=1}^{\infty}\lambda_r B_r$ is a neighbourhood of zero in E. Hence $\{P_j\}_j$ is locally bounded.

If $Q_j=\dfrac{j\,P_j}{p(P_j)}$ then $Q_{j+1}|_{E_j}=0$ and the above argument shows that $\{Q_j\}_j$ is also a locally bounded and hence a τ_δ bounded sequence in $\mathscr{H}(E)$ (see [6, lemma 2.43]).

Since $p(Q_j)=j$ this is impossible and proves our result.

Proposition 5. ([*6, example 1.24*]) *If* $E=\varinjlim_n E_n$ *is a inductive limit of Banach spaces then* C*-valued homogeneous hypocontinuous polynomials on* E *are continuous.*

We refer to [1], [3, proposition 4.1] and [6, example 1.38] for information regarding the next proposition.

Proposition 6. *If* $E=\varinjlim_n E_n$ *is an inductive limit of Banach spaces then, for each positive integer* n, τ_ω *on* $\mathscr{P}(^nE)$ *is the topology of uniform convergence on the bounded subsets of* E *and* $(\mathscr{P}(^nE), \tau_0)$ *and* $(\mathscr{P}(^nE), \tau_\omega)$ *are complete locally convex spaces.*

§2

The following theorem is our main result.

Theorem 7. *If* $E=\varinjlim_n E_n$ *is a strict inductive limit of Banach spaces then the following are equivalent:*

(a) $\mathscr{H}(E)=\mathscr{H}_{HY}(E)$,

(b) the bounded subsets of E *are bounding,*

(c) *the bounded subsets of E are limited,*

(d) $E \approx \mathbf{C}^{(N)}$,

(e) $(\mathscr{H}(E), \tau_0)$ *is a Fréchet space,*

(f) $E = \varinjlim_n E_n$ *in the category of locally convex spaces and holomorphic mappings,*

(g) $(\mathscr{H}(E), \tau_0)$ *is complete (resp. quasicomplete, sequentially complete, T. S. τ_0 complete),*

(h) *the τ_0 bounded subsets of $\mathscr{H}(E)$ are locally bounded,*

(i) $(\mathscr{H}(E), \tau_\delta)$ *is complete (resp. quasicomplete, sequentially complete, T. S. τ_δ complete),*

(j) *τ_δ bounded subsets of $\mathscr{H}(E)$ are locally bounded.*

Proof. The implication (a) $\Rightarrow$ (b) is proved in [7] (and also the implication (b) $\Rightarrow$ (a) if E is separable). By proposition 2 we have (b) $\Rightarrow$ (c).

Since E is a $\mathscr{DF}$ space it is quasinormable and hence if (c) is satisfied then proposition 1 implies that E is a Schwartz space. Hence each E_n is finite dimensional and (c) $\Rightarrow$ (d). By [5] condition (d) implies all the other conditions except (f). Since $\mathbf{C}^{(N)}$ is a Schwartz space (d) $\Rightarrow$ (f). Hence conditions (a),..., (d) are all equivalent. By proposition 6 and [6, proposition 3.36] all the conditions in (g) are equivalent, all the conditions in (i) are equivalent and (e) $\Rightarrow$ (g) $\Rightarrow$ (i). If the τ bounded subsets of $\mathscr{H}(E)$ are locally bounded, $\tau = \tau_0$ or τ_δ, then we have $\mathscr{H}(E)$ T. S. τ-complete, and hence (h) $\Rightarrow$ (g) and (j) $\Rightarrow$ (i). Hence, to complete the proof, it suffices to show (f) $\Rightarrow$ (a) and $\sim$(d) $\Rightarrow$ $\sim$(i). Suppose (f) is satisfied. Let $f \in \mathscr{H}_{HY}(E)$. For each positive integer n, $f|_{E_n} \in \mathscr{H}_{HY}(E_n) = \mathscr{H}(E_n)$, since E_n is a Banach space and the topology induced by E on E_n is its original Banach space topology. By (f) we have $f \in \mathscr{H}(E)$ and (f) $\Rightarrow$ (a).

Now suppose that (d) is not satisfied. By proposition 3 there exists an $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}$ in $\mathscr{H}_{HY}(E)$ satisfying (1.3) and (1.4).

Let p denote a τ_δ continuous semi-norm on $\mathscr{H}(E)$ satisfying (1.1) and (1.2).

Let m denote an integer associated with p as in proposition 4 and let Z_m have the meaning given to it in (1.4).

Since $\sum_{n \in Z_m} \frac{\hat{d}^n f(0)}{n!} \in \mathscr{H}(E)$ we have $\sum_{n \in Z_m} p\left(\frac{\hat{d}^n f(0)}{n!}\right) < \infty$.

If $n \notin Z_m$ then $\left.\frac{\hat{d}^n f(0)}{n!}\right|_{E_m} = 0$ and $p\left(\frac{\hat{d}^n f(0)}{n!}\right) = 0$ by proposition 4.

Hence $\sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) < \infty$ and $\left\{\sum_{n=0}^{m} \frac{\hat{d}^n f(0)}{n!}\right\}_{m=1}^{\infty}$ is a Cauchy sequence in $(\mathscr{H}(E), \tau_\delta)$.

Since $\sum_{n=0}^{m} \frac{\hat{d}^n f(0)}{n!} \notin \mathscr{H}(E)$ this Cauchy sequence does not converge and (i) is not satisfied. This completes the proof.

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Department of Mathematics,
University College Dublin,
Belfield, Dublin 4,
Ireland.

Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530
21945 Rio de Janeiro
Brazil

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