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Comparison Results for a Class of Variational Inequalities

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ABSTRACT. In this paper we study a variational inequality related to a linear differential operator of elliptic type. We give a pointwise bound for the rearrangement of the solution u, and an estimate for the L^2-norm of the gradient of u.

1. INTRODUCTION

Let Ω be an open bounded set of $\mathbb{R}^n$ and let

$$Au = -(a_{ij}(x)u_{x_i})_{x_j} + (b_i(x)u)_{x_i} + c(x)u$$

be a differential operator whose coefficients satisfy:

$$(1.1)\qquad \begin{cases} a_{ij},\ b_i,\ c \in L^\infty(\Omega) \\ a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, & \text{for a.e. } x\in\Omega,\ \ \xi\in R^n \\ \Sigma b_i^2 \leq B^2. \end{cases}$$

Let $u \in H_0^1(\Omega)$ be a solution of the related variational inequality

$$(1.2)\qquad a(u, U-u) \geq \int_\Omega f(x)(U-u)\,dx \qquad \forall\, U\in H_0^1(\Omega),\ U,\ u\geq 0$$

where

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$$a(\varphi, \psi)=\int_\Omega (a_{ij}(x)\varphi_{x_i}\psi_{x_j}-b_i(x)\varphi\psi_{x_i}+c(x)\varphi\psi)dx,$$

and $f\in L^2(\Omega)$.

Schwarz symmetrization allows us to obtain sharp estimates for a solution u of (1.2) comparing u with the solution $v\in H_0^1(\Omega^\#)$ of the «symmetrized» variational inequality

$$(1.3)\qquad \begin{aligned} a^\#(v, V-v) = \int_{\Omega^\#} v_{x_i}(V-v)_{x_i}+B\frac{x_i}{|x|}v(V-v)_{x_i}-c_-^\#(x)v(V-v)dx\geq \\ \geq \int_{\Omega^\#} f^\#(x)(V-v)dx \qquad \forall\, V\in H_0^1(\Omega^\#), \qquad V, v\geq 0 \end{aligned}$$

where $\Omega^\#$ is the n-dimensional ball centered at the origin with measure $|\Omega|$, $f^\#(x)$, $c_-^\#(x)$ are the spherically symmetric decreasing rearrangements of $f(x)$ and $c_-(x)=\text{Max}(-c(x), 0)$, respectively (see § 2).

More precisely in this paper we show (see theorem 2.2) that the pointwise estimate

$$(1.4)\qquad u^\#(x)\leq v(x)$$

holds for all $x\in\Omega^\#$. Such an estimate gives an optimal upper bound for the L^p-norm of u and for the measure of the set $\{x:u(x)>0\}$ or, that is the same, an optimal lower bound for the coincidence set of u.

As consequence of (1.4) we derive an estimate for $a(u,u)$ and for the L^2-norm of ∇u in terms of $a^\#(v, v)$ and the L^2-norm of ∇v respectively. Finally the previous results are applied to the obstacle problem when the obstacle is in $H_0^1(\Omega)$.

The first results in this direction, in the case $b_i=0$, $c\geq 0$, are due to Bandle-Mossino [BM] and Maderna-Salsa [MS]. In particular in [BM] the case of a non-linear operator is considered. Subsequently Alvino-Matarasso-Trombetti [AMT] considered a variational inequality of general form taking into account also the influence of the term $c_+(x)=\max(c(x), 0)$, but with constraints on the coefficients different from (1.1). As far as variational parabolic inequalities concern we recall [DM].

The method used was introduced by Talenti [Ta1] for an elliptic equation without lower-order terms and was extended to elliptic equations of more general form by several authors (see i.e. [AT], [ALT2], [Ba], [FP], [GT], [Ta2], [TV]).

2. MAIN RESULTS

If Ω is an open bounded set of $\mathbb{R}^n$, we will denote by $|\Omega|$ its measure and by $\Omega^{\#}$ the ball of $\mathbb{R}^n$ centered at the origin whose measure is $|\Omega|$. Moreover if φ is a measurable function,

$$\mu(t)=|\{x\in\Omega:\varphi(x)>t\}|, \qquad t\in\mathbb{R},$$

is the distribution function of φ and

$$\varphi^*(s)=\sup\{t\in\mathbb{R}:\mu(t)>s\}, \qquad s\in[0,\ |\Omega|[,$$

is its decreasing rearrangement. If C_n is the measure of the n-dimensional unit ball,

$$\varphi^{\#}(x)=\varphi^*(C_n|x|^n), \qquad x\in\Omega^{\#},$$

is the spherically symmetric decreasing rearrangement of $\varphi(x)$. For an exhaustive treatment of rearrangements we refer for example to [ALT1], [Ba], [CR], [HLP], [Ka], [Mo]. Here we just recall the well known Hardy inequality:

$$\int_{\Omega} u(x)v(x)dx \leq \int_0^{|\Omega|} u^*(s)v^*(s)\mathrm{ds}$$

where $u(x)$, $v(x)$ are measurable functions.

The firts step to get the comparison result is to obtain a differential inequality involving the decreasing rearrangement of u. In the following we will consider $f_+, f_- \not\equiv 0$. In the other cases proofs are simpler.

Theorem 2.1. *Let u be a solution of the variational inequality (1.2) with conditions (1.1), then*

$$\begin{aligned} -u^{*\prime}(s) \leq {} & \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s (f^*(\sigma)+c_{\pm}^*(\sigma)u^*(\sigma))d\sigma \, + \\ & + \frac{Bs^{-1+1/n}}{nC_n^{1/n}} u^*(s) \text{ a.e. in } [0,\ |u>0|]. \end{aligned} \tag{2.1}$$

Proof. Following [BM] we choose as test function U in (1.2) $u \pm \Phi_h$ where

$$\Phi_h(x)=\begin{cases} h & \text{if } u>t+h \\ (u(x)-t) & \text{if } t<u(x)\leq t+h \\ 0 & \text{otherwise} \end{cases}$$

with $h>0$ and $t\in[0, \sup u[$.

We have

$$\frac{1}{h}a(u,\Phi_h)=\frac{1}{h}\int_\Omega f\Phi_h dx$$

and by ellipticity condition, letting h go to 0, we obtain

$$-\frac{d}{dt}\int_{u>t}|\nabla u|^2dx\leq -\frac{d}{dt}\int_{u>t}b_i(x)u_{x_i}u\,dx+\int_{u>t}(f(x)-c(x)u)dx.$$

From now on we give a sketch of the proof because the tools are standard. We have (see [ALT2])

$$-\frac{d}{dt}\int_{u>t}b_i(x)u_{x_i}u\,dx\leq Bt\left(-\frac{d}{dt}\int_{u>t}|\nabla u|dx\right)$$

and by Hardy inequality

$$\int_{u>t}(f(x)-c(x)u(x))dx\leq\int_0^{\mu(t)}(f^*(\sigma)+c_-^*(\sigma)u^*(\sigma))d\sigma.$$

Then

$$\begin{aligned}&-\frac{d}{dt}\int_{u>t}|\nabla u|^2dx-Bt\left(-\frac{d}{dt}\int_{u>t}|\nabla u|dx\right)\leq \\ (2.2)\qquad &\leq\int_0^{\mu(t)}(f^*(\sigma)+c_-^*(\sigma)u^*(\sigma))d\sigma.\end{aligned}$$

Isoperimetric inequality [DG], Fleming-Rishel formula [FR] and Schwarz inequality give (see [ALT2], [Ta2])

$$nC_n^{1/n}\mu(t)^{1-1/n}\leq-\frac{d}{dt}\int_{u>t}|\nabla u|\leq$$

(2.3)
$$\leq [-\mu'(t)]^{1/2}\left(-\frac{d}{dt}\int_{u>t}|\nabla u|^2\right)^{1/2}.$$

Now we observe that the right hand side of (2.2) is non negative. As a matter of fact, letting t go to 0, from (2.2) we get

$$\int_0^{\mu(0)}(f^*(\sigma)+c_-^*(\sigma)u^*(\sigma))d\sigma\geq 0$$

and then

(2.4)
$$\int_0^{\mu(t)}(f^*(\sigma)+c_-^*(\sigma)u^*(\sigma))d\sigma\geq 0 \qquad \forall\, t\in[0,\sup u]$$

because $f^*(\sigma)+c_-^*(\sigma)u^*(\sigma)$ is a decreasing function.

Hence by (2.3) and (2.4) we have

$$\int_0^{\mu(t)}(f^*(\sigma)+c_-^*(\sigma)u^*(\sigma))d\sigma\leq$$

$$\leq\frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}}[-\mu'(t)]^{1/2}\left(-\frac{d}{dt}\int_{u>t}|\nabla u|^2\right)^{1/2}\int_0^{\mu(t)}(f^*(\sigma)+$$
$$+c_-^*(\sigma)u^*(\sigma))d\sigma.$$

Then by (2.2) and (2.3)

(2.5)
$$\left(-\frac{d}{dt}\int_{u>t}|\nabla u|^2dx\right)^{1/2}\leq$$
$$\leq Bt[-\mu'(t)]^{1/2}+[-\mu'(t)]^{1/2}\,\frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}}\int_0^{\mu(t)}(f^*(\sigma)+c_-^*(\sigma)u^*(\sigma))d\sigma.$$

Using again (2.3), we obtain

$$\frac{nC_n^{1/n}\mu(t)^{1-1/n}}{[-\mu'(t)]^{1/2}}\leq Bt[-\mu'(t)]^{1/2}+$$
$$+[-\mu'(t)]^{1/2}\frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}}\int_0^{\mu(t)}(f^*(\sigma)+c_-^*(\sigma)u^*(\sigma))d\sigma.$$

Making a change of variables we get (2.1). □

We consider the variational inequality (1.3) whose data are spherically symmetric. If its solution v is spherically symmetric and decreasing, then repeating the above proof, it is easy to verify that all the inequalities used to get (2.1) are equalities. As in [AMT] a way to establish the existence of such a solution is given by the following proposition:

Proposition 2.1. *Let us consider the differential operator*

$$A^{\#}v=-\Delta v-B\left(v\frac{x_i}{|x|}\right)_{x_i}-c_-^{\#}(x)v.$$

Let us suppose $c_-^{\#}=0$ or let any one of the following conditions be satisfied:

(i) there exists a non negative function $H\not\equiv 0$ such that the Dirichlet problem

$$\begin{cases}A^{\#}Z=H & \text{in } \Omega^{\#}\\ Z\in H_0^1(\Omega^{\#})\end{cases}$$

has a non negative solution Z;

(ii) the first eigenvalue of the problem

$$\begin{cases}A^{\#}\varphi=\lambda p(x)\varphi & \text{in } \Omega^{\#}\\ \varphi\in H_0^1(\Omega^{\#})\end{cases} \tag{2.6}$$

where $p\in L^{n/2}(\Omega^{\#})$, $p(x)\not\equiv 0$ is a non negative function, is positive;

(iii) there exists $\alpha>0$ such that

$$\int_{\Omega^{\#}}e^{B|x|}\left(|\nabla\Phi|^2-B\frac{(n-1)}{|x|}\Phi^2-c_-^{\#}\Phi^2\right)dx\geq$$

$$\geq\alpha\int_{\Omega^{\#}}e^{B|x|}|\nabla\Phi|^2dx\quad\forall\,\Phi\in H_0^1(\Omega^{\#}).$$

Then (1.3) has a unique solution $v=v^{\#}$. Moreover for $A^{\#}$ a maximum principle holds, that is

$$\text{(2.7)} \qquad A^{\#}V \geq 0 \; in \; \Omega^{\#}, \qquad V \geq 0 \; on \; \partial\Omega^{\#} \Rightarrow V \geq 0 \; in \; \Omega^{\#}.$$

The proof of this proposition will be given in the appendix.

The above arguments yield to state the following

Proposition 2.2. *If one of the conditions of proposition 2.1 is verified then the solution $v = v^{\#}$ of (1.3) satisfies*

$$\text{(2.8)} \qquad \begin{aligned} -v^{*\prime}(s) = & \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s (f^*(\sigma) + c_-^*(\sigma) v^*(\sigma)) d\sigma + \\ & + \frac{B s^{-1+1/n}}{n C_n^{1/n}} v^*(s) \qquad a.e. \; in \; [0, |v>0|]. \end{aligned}$$

At this point we are able to prove the comparison result.

Theorem 2.2. *If one of the conditions of proposition 2.1 is verified and u, v are solutions respectively of (1.2), (1.3) with the assumptions (1.1), then*

$$u^*(s) \leq v^*(s) \qquad \text{in } [0, |\Omega|].$$

Proof. Starting from (2.1) and (2.8) we show that, setting $w = u^* - v^*$, we will show that

$$\text{(2.9)} \quad -w'(s) \leq B \frac{s^{-1+1/n}}{n C_n^{1/n}} w + \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s c_-^*(\sigma) w(\sigma) d\sigma \quad \text{in } [0, |u>0|].$$

Obviously (2.9) holds for all $s \in [0, \min\{|u>0|, |v>0|\}]$, then it is trivial if $|u>0| \leq |v>0|$. We suppose $|u>0| > |v>0|$. By regularity results (see [BS]) the solution of (1.3) is in $H^2(\Omega^{\#})$, then $v^* \in C^1(]0, |\Omega|])$ and $v^{*\prime}(|v>0|) = 0$.

Hence by (2.8)

$$\text{(2.10)} \qquad \int_0^{|v>0|} f^*(\sigma) d\sigma + \int_0^{|v>0|} c_-^*(\sigma) v^*(\sigma) d\sigma = 0.$$

By (2.1), taking into account that $w(s)=u^*(s)$ for $|v>0|\leq s\leq|u>0|$ and (2.10), we get

$$-w'(s)\leq B\frac{s^{-1+1/n}}{nC_n^{1/n}}w+\frac{s^{-2+2/n}}{n^2C_n^{2/n}}\int_0^s c_-^*(\sigma)w(\sigma)d\sigma+\frac{s^{-2+2/n}}{n^2\,C_n^{2/n}}\int_{|v>0|}^s f^*(\sigma)d\sigma$$

for $|v>0|\leq s\leq|u>0|$.

Then we obtain (2.9) observing that by (2.10)

$$\int_0^{|v>0|} f^*(\sigma)d\sigma\leq 0,$$

which implies $f^*(s)<0$ in $[|v>0|,|\Omega|]$ because f^* is a decreasing function. To get the thesis we first suppose $c_-^*>0$. Setting $V=\int_0^s c_-^*(\sigma)w(\sigma)d\sigma$, (2.9) can be written as

$$(2.11)\quad \begin{cases} -\left(\exp\left(\dfrac{Bs^{1/n}}{C_n^{1/n}}\right)\dfrac{V'}{c_-^*}\right)'-\dfrac{s^{-2+2/n}}{n^2C_n^{2/n}}\exp\left(\dfrac{Bs^{1/n}}{C_n^{1/n}}\right)V\leq 0 \text{ in } [0,|u>0|]\\ V(0)=0\\ V'(|u>0|)\leq 0.\end{cases}$$

We will show that $V\leq 0$. If we suppose ab absurdo $V_+\not\equiv 0$, then there exists $\bar{s}$ such that $V(\bar{s})>0, V'(\bar{s})=0$ and $V_+\not\equiv 0$ in $[0,\bar{s}]$. We denote by $\bar{B}$ the ball centered at the origin whose measure is $\bar{s}$ and we consider the eigenvalue problem

$$(2.12)\qquad \begin{cases} A^{\#}\varphi=\lambda c_-^{\#}(x)\varphi & \text{in } \bar{B}\\ \varphi\in H_0^1(\bar{B}).\end{cases}$$

If we observe that

$$A^{\#}\varphi=\lambda c_-^{\#}(x)\varphi\Leftrightarrow-(e^{B|x|}\varphi_{x_i})_{x_i}-e^{B|x|}B\frac{(n-1)}{|x|}\varphi-e^{B|x|}\;c_-^{\#}(x)\varphi=e^{B|x|}\lambda c_-^{\#}(x)\varphi$$

then the first eigenvalue of (2.12) can be characterized by

$$\lambda_1=\min_{\Phi\in H_0^1(\bar{B})}\frac{\int_{\bar{B}}e^{B|x|}(|\nabla\Phi|^2-B\,\dfrac{(n-1)}{|x|}\Phi^2-c_-^{\#}(x)\Phi^2)dx}{\int_{\bar{B}}e^{B|x|}c_-^{\#}(x)\Phi^2dx}.$$

Since $\tilde{B}\subset\Omega^{\#}$, by (ii) of proposition 2.1 we find $\lambda_1>0$. On the other hand if φ_1 is the eigenfunction corresponding to λ_1, we have

$$-\varphi_1'(s)-B\frac{s^{-1+1/n}}{nC_n^{1/n}}\varphi_1(s)-\frac{s^{-2+2/n}}{n^2C_n^{2/n}}\int_0^s c_-^*(\sigma)\varphi_1(\sigma)d\sigma=$$
$$=\lambda_1\frac{s^{-2+2/n}}{n^2C_n^{2/n}}\int_0^s c_+^*(\sigma)\varphi_1(\sigma)d\sigma \qquad \text{a.e. in } [0,\bar{s}].$$

Hence setting $\Phi=\int_0^s c_-^*(\sigma)\varphi_1(\sigma)d\sigma$, it is easy to verify that λ_1 is the same as the first eigenvalue of the problem

$$(2.13)\qquad \begin{cases} -\left(\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)\frac{\Phi'}{c_-^*}\right)'-\frac{s^{-2+2/n}}{n^2C_n^{2/n}}\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)\Phi= \\ =\lambda_1\frac{s^{-2+2/n}}{n^2C_n^{2/n}}\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)\Phi \\ \Phi(0)=\Phi'(\bar{s})=0. \end{cases}$$

By (2.13), using the variational characterization of λ_1 and (2.11), we get

$$\lambda_1\leq\frac{\int_0^{\bar{s}}\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)\frac{V_+'^2}{c_-^*}-\frac{s^{-2+2/n}}{n^2C_n^{2/n}}\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)V_+^2ds}{\int_0^{\bar{s}}\frac{s^{-2+2/n}}{n^2C_n^{2/n}}\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)V_+^2ds}\leq 0$$

which gives the absurd.

Using again (2.11) we have

$$\begin{cases} -\left(\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right)\frac{V'}{c_-^*}\right)'\leq 0 \\ V(0)=0 \\ V'(|u>0|)\leq 0. \end{cases}$$

Integrating between s and $|u>0|$, we obtain $V'(s)\leq 0$, that is $w(s)\leq 0$. The result in the case $c_-^*\geq 0$ can be proved by approximation techniques. □

Making use of the techniques of the proof of theorem 2.1 it is possible to estimate the L^2-norm of ∇u. We have the following

Theorem 2.3. *If one of the conditions of proposition 2.1 is verified and u, v are solution respectively of (1.2), (1.3) with the assumptions (1.1), then*

$$\int_\Omega |\nabla u|^2 dx \leq \int_{\Omega^*} |\nabla v|^2 dx.$$

Proof. Starting from (2.5) and squaring both sides we get

$$-\frac{d}{dt}\int_{u>t} |\nabla u|^2 dx \leq$$
$$\leq -\mu'(t)\left[Bt + \frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}}\int_0^{\mu(t)} (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma))d\sigma\right]^2.$$

Integrating between 0 and sup u and making a change of variables we obtain

$$\int_\Omega |\nabla u|^2 dx \leq$$
$$\leq \int_0^{|u>0|}\left[Bu^*(s) + \frac{s^{-1+1/n}}{nC_n^{1/n}}\int_0^s (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma))d\sigma\right]^2 ds.$$

By Theorem 2.2 we have $u^*(s) \leq v^*(s)$, which implies also $|u>0| \leq |v>0|$, therefore we conclude observing that

$$\int_{\Omega^*} |\nabla v|^2 dx =$$
$$= \int_0^{|v>0|}\left[Bv^*(s) + \frac{s^{-1+1/n}}{nC_n^{1/n}}\int_0^s f^*(\sigma) + c_-^*(\sigma)v^*(\sigma)d\sigma\right]^2 ds. \qquad \square$$

Using the previous results we can get also an estimate for $a(u,u)$.

Proposition 2.3. *Under the same assumptions of theorem 2.2, the following estimates hold*

(α) $a(u,u)+\int_\Omega c_-(x)u^2dx\leq a^\#(v,v)+\int_{\Omega^\#} c_-^\#(x)v^2dx;$

(β) $a(u,u)\leq a^\#(v,v)$ *if* $b_i(x)=0$.

Proof. If in (1.2) we choose as test function $U=0$ and $U=2u$ we obtain

$$a(u,u)=\int_\Omega f(x)u\,dx.$$

By Hardy inequality we have

$$a(u,u)+\int_\Omega c_-(x)u^2dx\leq \int_0^{|\Omega|}(f^*(s)u^*+c_-^*(s)u^{*2})ds,$$

and integration by parts gives

$$\int_0^{|\Omega|}(f^*(s)+c_-^*(s)u^*)u^*ds=\int_0^{|\Omega|}\int_0^s(f^*(t)+c_-^*(t)u^*)dt(-du^*(s)).$$

By (2.4) we have

$$\int_0^s(f^*(t)+c_-^*(t)u^*)dt\geq 0,$$

moreover (2.1), (2.8) and theorem 2.1 imply

$$-\frac{du^*}{ds}\leq -\frac{dv^*}{dv}.$$

Then

$$\int_0^{|\Omega|}(f^*(s)u^*+c_-^*(s)u^{*2})ds\leq \int_0^{|\Omega|}\int_0^s(f^*(t)+c_-^*(t)v^*)dt(-dv^*(s))=$$

$$=\int_0^{|\Omega|}(f^*(s)+c_-^*(s)v^*)v^*ds=a^\#(v,v)+\int_{\Omega^\#}c_-^*(x)v^2dx,$$

that is (α) or, which is the same, (β) when $c_-(x)=0$.

If $b_i(x)=0$, then the bilinear form $a^\#(u,v)$ is symmetric and hence we get

$$a(u,u)=\int_\Omega f(x)u\,dx\le\int_{\Omega^\#} f^\#(x)u^\# dx=a^\#(v,u^\#)\le \tag{2.14}$$
$$\le\sqrt{a^\#(v,v)}\,\sqrt{a^\#(u^\#,u^\#)}.$$

Ellipticity condition, Polya Szego and Hardy inequalities give

$$a^\#(u^\#,u^\#)=\int_{\Omega^\#}|\nabla u^\#|^2dx-\int_{\Omega^\#} c_-^\#(x)u^{\#2}dx\le \tag{2.15}$$
$$\le\int_\Omega|\nabla u|^2dx-\int_\Omega c_-(x)u^2dx\le a(u,u).$$

By (2.14) and (2.15) (β) follows. □

Remark. We can apply the previous results to the obstacle problem

$$a(u,\varphi-u)\ge\int_\Omega f(\varphi-u) \qquad \forall\ \varphi\in H_0^1(\Omega)\ \varphi,u\ge\psi \tag{2.16}$$

where the obstacle $\psi\in H_0^1(\Omega)$ and

$$A\psi=-(a_{ij}(x)\psi_{x_i})_{x_j}+(b_i(x)\psi)_{x_i}+c(x)\psi \qquad \text{in } L^2(\Omega).$$

Setting $\bar u=u-\psi$ *and* $g=f-A\psi$ we have that $\bar u$ satisfies

$$a(\bar u,\varphi-\bar u)\ge\int_\Omega g(\varphi-\bar u) \qquad \varphi\in H_0^1(\Omega)\ \varphi,\ \bar u\ge 0.$$

Hence we can compare $\bar u$ with the solution of the symmetrized problem

$$a^\#(v,V-v)\ge\int_{\Omega^\#} g^\#(x)(V-v)\,dx \qquad \forall\ V\in H_0^1(\Omega^\#)\ V,v\ge 0.$$

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APPENDIX

Proof of Proposition 2.1. Following [AMT] at first we prove that conditions (i), (ii), (iii) are equivalent. In order to do that let us observe that

(A.1) $A^{\#}Z=H \Leftrightarrow -(e^{B|x|}Z_{x_i})_{x_i} - e^{B|x|}B\dfrac{(n-1)}{|x|}Z - e^{B|x|}\,c_-^{\#}(x)Z = e^{B|x|}H$

then the first eigenvalue of (2.6) can be characterized by the Rayleigh principle

$$\text{(A.2)}\quad \lambda_1 = \min_{\Phi\in H_0^1(\Omega^{\#})} \frac{\displaystyle\int_{\Omega^{\#}} e^{B|x|}\left(|\nabla\Phi|^2 - B\frac{(n-1)}{|x|}\Phi^2 - c_-^{\#}(x)\Phi^2\right)dx}{\displaystyle\int_{\Omega^{\#}} e^{B|x|}\,p(x)\Phi^2\,dx}.$$

(i) $\Rightarrow$ (ii)

By (A.1)

$$\lambda_1\int_{\Omega^{\#}} e^{B|x|}p(x)\Phi_1 Z\,dx = \int_{\Omega^{\#}} e^{B|x|}H\Phi_1\,dx,$$

where Φ_1 is the first eigenfunction of (2.6). Then, since Φ_1 has constant sign in $\Omega^{\#}$, we have $\lambda_1>0$.

(ii) $\Rightarrow$ (iii)

Choosing in (ii) $p(x)\geq B\dfrac{(n-1)}{|x|}+c_-^{\#}(x)$, and using (A.2) we get

$$\int_{\Omega^{\#}} e^{B|x|}\left(|\nabla\Phi|^2 - B\frac{(n-1)}{|x|}\Phi^2 - c_-^{\#}\Phi^2\right)dx - \alpha\int_{\Omega^{\#}} e^{B|x|}|\nabla\Phi|^2dx =$$

$$=(1-\alpha)\int_{\Omega^{\#}} e^{B|x|}|\nabla\Phi|^2dx - \int_{\Omega^{\#}} e^{B|x|}\left(B\frac{(n-1)}{|x|}\Phi^2 + c_-^{\#}\Phi^2\right)dx \geq$$

$$\geq \lambda_1(1-\alpha)\int_{\Omega^{\#}} e^{B|x|}p(x)\Phi^2 - \alpha\int_{\Omega^{\#}} e^{B|x|}\left(B\frac{(n-1)}{|x|}\Phi^2 + c_-^{\#}\Phi^2\right)dx \geq$$

$$\geq (\lambda_1 - \alpha\lambda_1 - \alpha)\int_{\Omega^{\#}} e^{B|x|}\left(B\frac{(n-1)}{|x|}\Phi^2 + c_-^{\#}\Phi^2\right)dx.$$

Taking $0<\alpha<\dfrac{\lambda_1}{\lambda_1+1}$, (iii) follows.

(iii) $\Rightarrow$ (i)

By (iii) the problem

$$\text{(A.3)}\quad \begin{cases} -(e^{B|x|}Z_{x_i})_{x_i} - e^{B|x|}\, B\dfrac{(n-1)}{|x|}Z - e^{B|x|}c_-^\#(x)Z = e^{B|x|}H & \text{in } \Omega^\# \\ Z\in H_0^1(\Omega^\#) \end{cases}$$

has a unique solution. Moreover this solution is positive. As a matter of fact setting $Z_- = \max(-Z,0)$ by (A.3)

$$\alpha\int_{\Omega^\#} e^{B|x|}|\nabla Z_-|^2 dx \leq \int_{\Omega^\#} e^{B|x|}\left(|\nabla Z_-|^2 - B\frac{(n-1)}{|x|}Z_-^2 - c_-^\# Z_-^2\right)dx =$$

$$= -\int_{\Omega^\#} e^{B|x|}HZ_- dx \leq 0$$

and then $Z_- = 0$.

With similar arguments, by (iii), we obtain (2.7).

Now we observe that (1.3) is equivalent to the variational inequality

$$\int_{\Omega^\#} e^{B|x|}\left[v_{x_i}(\Phi - v)_{x_i} - B\frac{(n-1)}{|x|}v(\Phi - v) - c_-^\#(x)v(\Phi - v)\right]dx \geq$$

$$\geq \int_{\Omega^\#} e^{B|x|}f^\#(x)(\Phi - v)dx \qquad \forall\, \Phi\in H_0^1(\Omega^\#)\ \ \Phi, v\geq 0$$

which, by (iii), has a unique solution v.

To prove that v is decreasing for the sake of simplicity we suppose v sufficiently smooth. Setting $r=|x|, R=\left(\dfrac{|v>0|}{C_n}\right)^{1/n}$, $V=v_r$ and deriving with respect to r the equation

$$A^\# v = f^\# \qquad \text{in } \{x : v(x)>0\}$$

we obtain

$$-V_{rr} - \frac{(n-1)}{r}V_r - BV_r + \left(\frac{(n-1)}{r^2} - B\frac{(n-1)}{r} - c_-^\#(x)\right)V =$$

$$= f_r^\# - B\frac{(n-1)}{r^2}v + (c_-^\#)_r v \qquad \text{in } [0,R].$$

Since $f_r^\# - B\dfrac{(n-1)}{r^2}v \leq 0$, we have

$$A^{\#}V+\frac{(n-1)}{|x|^2}V\leq 0$$

and $V(R)\leq 0$.

We conclude observing that the operator $A^{\#}+\frac{(n-1)}{|x|^2}$ has the property (2.7). □

References

[ALT1] ALVINO, A.-LIONS, P. L.-TROMBETTI, G.: *On optimization problems with prescribed rearrangements*, Nonlinear Analysis, T.M.A., **13** (1989), 185-220.

[ALT2] ALVINO, A.-LIONS, P. L.-TROMBETTI, G.: *Comparison results for elliptic and parabolic equations via Schwarz symmetrization*, Ann. Inst. Henri Poincarè, (2) **7** (1990), 37-65.

[AMT] ALVINO, A.-MATARASSO, S.-TROMBETTI, G.: *Variational Inequalities and Rearrangements*, Rend. Acc. Naz. Linc., (1992).

[AT] ALVINO, A.-TROMBETTI, G.: *Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri*, Ricerche Mat., **27** (1978), 413-428.

[Ba] BANDLE, C.: *Isoperimetric Inequalities and Applications*, Monographs and Studies in Math., No. 7, Pitman, London, 1980.

[BM] BANDLE, C.-MOSSINO, J.: *Rearrangement in Variational Inequalities*, Ann. Mat. Pura e Appl, **138** (1984), 1-14.

[BS] BREZIS, H. R.-STAMPACCHIA, G.: *Sur la régularité de la solution d'inéquations elliptiques*, Bull. Soc. Math. France, **96** (1968), 153-180.

[CR] CHONG, K. M.—RICE, N. M.: *Equimeasurable rearrangements of functions*, Queen's papers in pure and applied mathematics, No. 28, Queen's University, Ontario, 1971.

[DG] DE GIORGI, E.: *Su una teoria generale della misura (r − 1)-dimensionale in uno spazio ad r dimensioni*, Ann. Mat. Pura e Appl., **36** (1954), 191-213.

[DM] DÍAZ, J. I.-MOSSINO, J.: *Isoperimetric inequalities in the parabolic obstacle problem*, to appear in Journal de Math. Pures et App.

[FP] FERONE, V.-POSTERARO, M. R.: *Symmetrization results for elliptic equations with lower order terms*, Atti Sem. Mat. Fis. Modena, **39** (1991), 47-61.

[FR] FLEMING, W.-RISHEL, R.: *An integral formula for total gradient variation*, Arch. Math., **11** (1960), 218-222.

[GT] GIARRUSSO, E.-TROMBETTI, G.: *Estimates for solutions of elliptic equations in a limit case*, Bull. Austral. Math. Soc., **36** (1987), 425-434.

[HLP] HARDY, G. H.-LITTLEWOOD, J. E.-POLYA, G.: *Inequalities*, Cambridge University Press, 1964.

[Ka] KAWHOL, B.: *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Math., No. 1150, Springer, Berlin-New York, 1985.

[MS] MADERNA, C.-SALSA, S.: *Some Special Properties of Solutions to Obstacle Problems*, Rend. Sem. Mat. Univ. Padova, **71** (1984), 121-129.

[Mo] MOSSINO, J.: *Inégalités isopérimétriques et applications en physique*, Collection Travaux en Cours, Hermman, Paris, 1984.

[Ta1] TALENTI, G.: *Elliptic equations and reattangements*, Ann. Scuola Norm. Sup. Pisa, (4) **3** (1976), 697-718.

[Ta2] TALENTI, G.: *Linear Elliptic P.D.E.'s: Level Sets, Rearrangements and a priori Estimates of Solutions*, Boll. U.M.I., (6) 4-B (1985), 917-949.

[TV] TROMBETTI, G.-VÁZQUEZ, J. L.: *A symmetrization result for elliptic equations with lower order terms*, Ann. Fac. Sci. de Toulose, **7** (1985), 137-150.

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