REVISTA MATEMATICA de la
Universidad Complutense de Madrid
Volumen 7, número 2: 1994
http://dx.doi.org/10.5209/rev_REMA.1994.v7.n2.17749

Existence of Periodic Solutions for a Class of Nonlinear Evolution Equations

RADU CAŞCAVAL and IOAN I. VRABIE

ABSTRACT. In the present paper we prove an existence result concerning T-periodic solutions to a class of nonlinear evolution equations of the form

$$u'(t) + Au(t) \ni f(t, u(t)),\ t \in \mathbf{R},$$

where A is an m-accretive operator acting in a real Hilbert space H and such that $-A$ is the generator of a compact semigroup, while $f: \mathbf{R} \times \overline{D(A)} \to H$ is continuous and T-periodic with respect to the first argument.

1. INTRODUCTION

Our goal in the present paper is to prove an existence result concerning T-periodic solutions to a class of nonlinear evolution equations of the form

$$u'(t) + Au(t) \ni f(t, u(t)),\ t \in \mathbf{R} \tag{1.1}$$

1991 Mathematics Subject Classification: 34C25, 34G20, 35K55
Editorial Complutense. Madrid, 1994.

In all that follows $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ is a real Hilbert space, $B(0,r)$ is the closed ball with radius $r > 0$ and centered at $0, A: D(A) \subseteq H \to 2^H$ is an m-accretive operator and $f: \mathbf{R}\times\overline{D(A)} \to H$ is a continuous function which is T-periodic with respect to its first argument. Our main result is Theorem 1.1 below.

Theorem 1.1. *Assume that $A: D(A) \subseteq H \to 2^H$ is an m-accretive operator and $-A$ generates a compact semigroup. Assume further that $f: \mathbf{R}\times\overline{D(A)} \to H$ is a continuous function which is T-periodic with respect to its first argument, and which is bounded on bounded subsets in $\mathbf{R}\times\overline{D(A)}$. If there exists $r > 0$ such that $B(0,r)\cap D(A)$ is nonempty, and for each $x \in D(A)$ with $\|x\| = r$, each $y \in Ax$ and each $t \in [0,T]$*

$$\langle x, y - f(t,x)\rangle \geq 0, \tag{1.2}$$

then the problem (1.1) has at least one T-periodic integral solution.

Results on this kind of problems have been obtained previously under various assumptions on A and on f by many authors and we mention here only [4,6,10,13,16]. For further details on the subject see [16]. For the case in which f does not depend on u see [1,9,11,12,14].

One of the most usual method for proving an existence result for T-periodic solutions to (1.1) is to show that the corresponding Poincaré map, i.e. the map which assigns to each $x \in \overline{D(A)}$ the values at T of all integral solutions of (1.1) satisfying $u(0) = x$, has at least one fixed point. Since in our case this map is neither single-valued nor even convex-valued (we recall that f is only continuous) this method is no longer applicable directly. In order to avoid this difficulty, in a recent paper by the second author [16], the existence of T-periodic solutions of (1.1) is obtained by looking for fixed points for a suitably defined mapping which is always single-valued and continuous. The approach introduced there permits to prove the existence of T-periodic solutions to (1.1) without approximating the perturbing term f by smooth functions in order to guarantee the uniqueness of the integral solutions to the associated Cauchy problem.

Very recently, Hirano [10] improves the main result in [16] (which is valid in general Banach spaces) in the specific case in which A is the subdifferential of a l.s.c. convex and proper function acting on a real Hilbert space H. More precisely, Hirano [10] shows that if A is a subdifferential and $-A$ generates a compact semigroup, while $f: \mathbf{R} \times H \to H$ is a Carathéodory function, T-periodic with respect to its first argument and there exist $M_1, M_2, a, b > 0$ such that

$$||f(t,x)|| \leq M_1||x|| + M_2 \tag{1.3}$$

for all $t \in \mathbf{R}$ and $x \in H$, and

$$\langle x, y - f(x,t) \rangle \geq a||x||^2 - b \tag{1.4}$$

for all $t \in \mathbf{R}$, $x \in D(A)$ and $y \in Ax$, then the problem (1.1) has at least one T-periodic integral (in fact strong) solution.

His method of proof which has its roots in the calculus of variations in essentially based on conditions (1.3) and (1.4), and rests heavily on the fact that A is a subdifferential. As we can easily see, our result is applicable to a strictly broader class of problems of the form (1.1) inasmuch as we do not assume that A is a subdifferential and the conditions (1.3) and (1.4) are replaced by the less restrictive ones: f is bounded on bounded subsets in $\mathbf{R} \times \overline{D(A)}$, and respectively by (1.2). It is easy to see that if there exist $a, b, \alpha > 0$ such that

$$\langle x, y - f(t,x) \rangle \geq a||x||^\alpha - b$$

for each $t \in \mathbf{R}$, $x \in D(A)$ and $y \in Ax$, then (1.2) holds for every $r \geq (b/a)^{1/\alpha}$ for which $B(0,r) \cap D(A)$ is nonempty.

We conclude this section by noticing that the compactness assumption on the semigroup generated by $-A$ is essential. More precisely, it is not possible to obtain a variant of Theorem 1.1 in the case in which the semigroup generated by $-A$ is not compact, even if we assume that f is a compact operator satisfying (1.2). Indeed, a very simple and instructive example due to Deimling [8, Exercise 6, p. 85] shows that there exists a compact mapping $f: l^2 \to l^2$ satisfying $\langle x, f(x) \rangle < 0$ for each $x \in l^2$ with $||x|| = r$, and such that the problem $u'(t) = f(u(t))$, $t \in \mathbf{R}$, has no T-periodic solution.

2. PRELIMINARIES

Although we assume familiarity with the theory of nonlinear evolution equations governed by m-accretive operators, we recall for easy reference some basic concepts and results in the field which we shall use frequently in the sequel. For further details on this subject see [3,5,7,17].

An operator $A : D(A) \subseteq H \to 2^H$ is called *m-accretive* if for each $x_i \in D(A)$, $y_i \in Ax_i$, $i = 1,2$,

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0,$$

and for each $\lambda > 0$, $I + \lambda A$ is surjective.

Consider the Cauchy problem

$$u'(t) + Au(t) \ni g(t), \ 0 \leq t \leq T, \tag{2.1}$$

$$u(0) = u_0,$$

where A is an m-accretive operator, $u_0 \in \overline{D(A)}$, and $g \in L^1([0,T]; H)$. By an *integral solution* of (2.1) we mean a continuous function $u : [0,T] \to \overline{D(A)}$, with $u(0) = u_0$, and satisfying

$$||u(t) - x||^2 \leq ||u(s) - x||^2 + 2\int_s^t \langle u(\tau) - x, g(\tau) - y \rangle d\tau \tag{2.2}$$

for each $x \in D(A)$, $y \in Ax$ and $0 \leq s \leq t \leq T$.

It is known that for each $(u_0, g) \in \overline{D(A)} \times L^1([0,T]; H)$ the problem (2.1) has a unique integral solution $u = \Im(u_0, g)$. Moreover, if $u = \Im(u_0, g)$ and $v = \Im(v_0, h)$, then

$$||u(t) - v(t)|| \leq ||u(s) - v(s)|| + \int_s^t ||g(\tau) - h(\tau)|| d\tau \tag{2.3}$$

for each $o \leq s \leq t \leq T$. See [3] or [5].

Let $S(t) : \overline{D(A)} \to \overline{D(A)}$, $t \geq 0$ be the semigroup of nonexpansive operators generated by $-A$, i.e. $S(t)u_0 = \Im(u_0, 0)(t)$ for each $u_0 \in \overline{D(A)}$ and $t \geq 0$.

The semigroup $S(t):\ \overline{D(A)} \to \overline{D(A)},\ t \geq 0$ is called *compact* if $S(t)$ is a compact operator for each $t > 0$.

We recall that a family $G \subseteq L^1([0,T];H)$ is called *uniformly integrable* if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every measurable subset E in $[0,T]$ whose Lebesgue measure is less than $\delta(\varepsilon)$ we have

$$\int_E \|g(t)\|dt < \varepsilon$$

uniformly for $g \in G$.

A remarkable property of compact semigroups is given below.

Theorem 2.1. *If $-A$ generates a compact semigroup, then, for each bounded subset B of $\overline{D(A)}$ and each uniformly integrable family G in $L^1([0,T];H)$, the set $\Im(B \times G)$ is relatively compact in $C([\delta,T];H)$ for each $\delta \in (0,T)$. If, in addition, B is relatively compact in H, then $\Im(B \times G)$ is relatively compact even in $C([0,T];H)$.*

The proof of this slight extension of a result due to Baras [2] and to Vrabie [15] is quite similar to that of [17, Theorem 2.3.2, p. 64] and so we omit it.

3. PROOF OF THE MAIN RESULT

For the sake of convenience and clarity we divide the proof of Theorem 1.1 into three lemmas.

First, let us consider the Cauchy problem

$$u'(t) + Au(t) \ni f(t,u(t)),\ 0 \leq t \leq T \tag{3.1}$$

$$u(0) = u_0.$$

In the next lemmas we will assume that f satisfies a slightly stronger condition than (1.2). Namely, we will assume that

(C) there exist $r > 0$ and $\rho > 0$ such that $B(0,r) \cap D(A)$ is nonempty and for each $x \in D(A)$ with $\|x\| = r$, each $y \in Ax$ and $t \in [0,T]$

$$\langle x, y - f(t,x)\rangle \geq \rho.$$

Lemma 3.1. *Assume that $-A$ generates a compact semigroup and $f : [0,T] \times \overline{D(A)} \to H$ is continuous, bounded on bounded subsets in $[0,T] \times \overline{D(A)}$ and satisfies (C). Then, for each $u_0 \in \overline{D(A)}$ with $\|u_0\| \leq r$, the problem (3.1) has at least one integral solution $u : [0,T] \to \overline{D(A)}$ satisfying*

$$\|u(t)\| \leq r \text{ for all } t \in [0,T]. \tag{3.2}$$

Lemma 3.2. *Assume that $-A$ generates a compact semigroup and $f : [0,T] \times \overline{D(A)} \to H$ is continuous, bounded on bounded subsets in $[0,T] \times \overline{D(A)}$ and satisfies (C). Assume, in addition, that for each $u_0 \in B(0,r) \cap \overline{D(A)}$ the problem (3.1) has a unique integral solution $u = \mathcal{E}(u_0)$ defined on the whole $[0,T]$. Then, the mapping $u_0 \mapsto \mathcal{E}(u_0)$ is continuous from $B(0,r) \cap \overline{D(A)}$ into $C([0,T];H)$.*

Lemma 3.3. *Assume that $-A$ generates a compact semigroup and $f : [0,T] \times \overline{D(A)} \to H$ is continuous, bounded on bounded subsets in $[0,T] \times \overline{D(A)}$ and satisfies (C). Assume, in addition, that for each $u_0 \in B(0,r) \cap \overline{D(A)}$ the problem (3.1) has a unique integral solution $u = \mathcal{E}(u_0)$ defined on the whole $[0,T]$. Then, the problem*

$$u'(t) + Au(t) \ni f(t,u(t)), \; 0 \leq t \leq T \tag{3.3}$$

$$u(0) = u(T)$$

has at least one integral solution.

Proof of Lemma 3.1. The fact that for each $u_0 \in \overline{D(A)}$ the problem (3.1) has at least one noncontinuable integral solution $u : [0,T_m) \to \overline{D(A)}$ follows from [17, Theorem 3.8.2, p.180]. Then, let $u_0 \in \overline{D(A)}$ with $\|u_0\| \leq r$ amd let $u : [0,T_m) \to \overline{D(A)}$ be such a noncontinuable integral solution of (3.1). For each $\varepsilon > 0$ let $f_\varepsilon : [0,T_m) \to H$ be a C^1-function which approximates $t \mapsto f(t,u(t))$ uniformly on compact subsets in $[0,T_m)$, and let $u_{0\varepsilon} \in D(A)$ satisfying $\|u_{0\varepsilon}\| \leq r$, and $\|u_0 - u_{0\varepsilon}\| \leq \varepsilon$. We note that such an element $u_{0\varepsilon}$ always exists since $B(0,r) \cap D(A)$ is nonempty and $\overline{D(A)}$ is convex - see [3, Proposition 2.6, p.77].

Now, let us consider the approximate problem

$$u'_\varepsilon(t) + Au_\varepsilon(t) \ni f_\varepsilon(t),\ 0 \le t < T_m \tag{3.4}$$

$$u_\varepsilon(0) = u_{0\varepsilon}$$

In view of [5, Proposition 3.3, p.68] this problem has a unique strong solution $u_\varepsilon : [0,T_m) \to \overline{D(A)}$ satisfying $u_\varepsilon(t) \in D(A)$ for each $t \in [0,T_m)$ and such that u_ε is differentiable from the right at each $t \in [0,T_m)$.

Since $\lim_{\varepsilon\downarrow 0} f_\varepsilon(t) = f(t,u(t))$ uniformly on compact subsets in $[0,T_m)$ and $\lim_{\varepsilon\downarrow 0} u_{0\varepsilon} = u_0$, from (2.3) it follows that

$$\lim_{\varepsilon\downarrow 0} u_\varepsilon(t) = u(t) \tag{3.5}$$

uniformly on compact subsets in $[0,T_m)$.

At this point let us observe that either there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0,\varepsilon_0)$ and each $t \in [0,T_m)$ we have

$$||u_\varepsilon(t)|| \le r, \tag{3.6}$$

or there exist a sequence $(\varepsilon_n)_{n\in N}$ converging to 0 and a sequence $(t_n)_{n\in N}$ in $[0,T_m)$ - denoted for simplicity by (ε) and (t_ε) - such that $||u_\varepsilon(t_\varepsilon)|| > r$.

If (3.6) holds, then by (3.5) we easily conclude that

$$||u(t)|| \le r \tag{3.7}$$

for each $t \in [0,T_m)$, and thus, in view of [17, Theorem 3.8.2, p.180], u can be continued to the right of T_m if $T_m < T$. Hence, in this case $T_m = T$, u is defined on $[0,T]$ and the proof of Lemma 3.1 is complete.

Therefore, let us assume that (3.6) does not hold. Clearly, for each $\varepsilon \in (\varepsilon)$ there exist $s_\varepsilon \in [0,T_m)$ and $\lambda_\varepsilon > 0$ such that $[s_\varepsilon, s_\varepsilon + \lambda_\varepsilon) \subseteq [0,T_m)$,

$$||u_\varepsilon(s)|| \le r \text{ for } s \in [0,s_\varepsilon] \tag{3.8}$$

and

$$\|u_\varepsilon(s)\| > r \text{ for } s \in (s_\varepsilon, s_\varepsilon + \lambda_\varepsilon). \tag{3.9}$$

If $\limsup_{s\downarrow 0} s_\varepsilon = T_m$, from (3.5) and (3.8) we deduce (3.7) and this completes the proof. So, let us assume by contradiction that $\limsup_{\varepsilon\downarrow 0} s_\varepsilon < T_m$. Then, there exists $T_0 \in [0, T_m)$ such that $s_\varepsilon \in [0, T_0]$ for each $\varepsilon \in (\varepsilon)$.

Now, taking the inner product in both sides of (3.4) by $u_\varepsilon(t)$ for $t = s_\varepsilon$, we get

$$\frac{1}{2}\frac{d^+}{dt}\|u_\varepsilon(s_\varepsilon)\|^2 + \langle u_\varepsilon(s_\varepsilon), v_\varepsilon(s_\varepsilon) - f_\varepsilon(s_\varepsilon)\rangle = 0,$$

where $v_\varepsilon(s_\varepsilon) \in Au_\varepsilon(s_\varepsilon)$. A simple computation along with (C) yields

$$\frac{1}{2}\frac{d^+}{dt}\|u_\varepsilon(s_\varepsilon)\|^2 < -\rho + \|u_\varepsilon(s_\varepsilon)\| \cdot \|f(s_\varepsilon, u_\varepsilon(s_\varepsilon)) - f_\varepsilon(s_\varepsilon))\|$$

for each $\varepsilon \in (\varepsilon)$. Recalling that $\lim_{\varepsilon\downarrow 0} f_\varepsilon(t) = f(t, u(t))$ and $\lim_{\varepsilon\downarrow 0} u_\varepsilon(t) = u(t)$ uniformly on $[0, T_0]$ and that $s_\varepsilon \in [0, T_0]$ for $\varepsilon \in (\varepsilon)$, from the last inequality, we conclude that for a sufficiently small $\varepsilon \in (\varepsilon)$ we have

$$\frac{1}{2}\frac{d^+}{dt}\|u_\varepsilon(s_\varepsilon)\|^2 < 0.$$

Hence there exists $\delta_\varepsilon > 0$ such that $[s_\varepsilon, s_\varepsilon + \delta_\varepsilon) \subseteq [0, T_m)$ and

$$\|u_\varepsilon(s)\| < r \text{ for } s \in (s_\varepsilon, s_\varepsilon + \delta_\varepsilon),$$

relation which obviously contradicts (3.9). Thus, the supposition that $\limsup_{\varepsilon\downarrow 0} s_\varepsilon < T_m$ is false, and this completes the proof of Lemma 3.1. ■

Proof of Lemma 3.2. Let $(u_{0n})_{n\in N}$ be a sequence in $B(0, r) \cap \overline{D(A)}$ with $\lim_{n\to\infty} u_{0n} = u_0$. Let us denote by u_n the unique integral solution of (3.1) with initial datum u_{0n}. From (3.2) we deduce that the sequence $(u_n)_{n\in N}$ is bounded in $C([0, T]; H)$. Consequently, the family $\{f(\cdot, u_n(\cdot)); n \in \mathbf{N}\}$ is bounded in $C([0, T]; H)$ and therefore uniformly integrable in $L^1([0, T]; H)$. Using Theorem 2.1 we deduce that $\{u_n; n \in$

$\mathbf{N}\}$ is relatively compact in $C([0,T];H)$. Therefore, to conclude the proof, it suffices to show that the only limit point of $(u_n)_{n\in N}$ is the unique integral solution u of (3.1) corresponding to the initial datum u_0. To this aim let us consider a subsequence of $(u_n)_{n\in N}$ - denoted for simplicity also by $(u_n)_{n\in N}$ - which converges in $C([0,T];H)$ to some function u. Since, for each $n \in \mathbf{N}$, u_n satisfies (2.2), it follows that u satisfies (2.2) too, and thus it is an integral solution of (3.1) corresponding to the initial datum u_0. Since this solution is unique, the proof of Lemma 3.2 is complete. ■

Proof of Lemma 3.3. Let $B_A(0,r) = B(0,r) \cap \overline{D(A)}$ - which is nonempty, closed, bounded and convex - and let $Q: B_A(0,r) \to H$ be the Poincaré map, i.e. $Q(u_0) = u(T)$ for each $u_0 \in B_A(0,r)$, where u is the unique integral solution of the problem (3.1) corresponding to the initial datum u_0. Obviously, Lemma 3.1 shows that Q maps $B_A(0,r)$ into itself, while Theorem 2.1 implies that Q has a relatively compact range in H. Since by Lemma 3.2 Q is continuous from $B_A(0,r)$ into itself, Q satisfies the hypotheses of Schauder's Fixed Point Theorem, and thus it has at least one fixed point $u_0 \in B_A(0,r)$. Now it is clear that the integral solution of (3.1) with the initial datum u_0 is in fact an integral solution of the problem (3.3), and this completes the proof of Lemma 3.3. ■

Proof of Theorem 1.1. Since $f: \mathbf{R} \times \overline{D(A)} \to H$ is continuous, for each $\varepsilon > 0$ there exists a locally Lipschitz function $f_\varepsilon: \mathbf{R} \times \overline{D(A)} \to H$ such that

$$||f(t,u) - \varepsilon \cdot u - f_\varepsilon(t,u)|| < (\varepsilon \cdot r)/2 \tag{3.10}$$

for each $(t,u) \in \mathbf{R} \times \overline{D(A)}$. See [8, Theorem 7.2, p.44 and Exercice 6, p. 53].

We may easily verify that, for each $\varepsilon > 0$, f_ε satisfies (C) with the same r as f does and with $\rho = (\varepsilon \cdot r^2)/2$.

Now let us consider the problem (3.3) with f replaced with f_ε. From Lemma 3.3 we know that for each $\varepsilon > 0$ the problem (3.3) has at least one integral solution $u_\varepsilon: [0,T] \to \overline{D(A)}$. In view of Lemma 3.1 we have $||u_\varepsilon(t)|| \leq r$ for each $t \in [0,T]$. This inequality along with (3.10) shows that $\{f_\varepsilon(\cdot, u_\varepsilon(\cdot)); \varepsilon > 0\}$ is bounded in $C([0,T];H)$. From Theorem 2.1 we then deduce that $\{u_\varepsilon; \varepsilon > 0\}$ is relatively compact in $C([\delta,T];H)$

for each $\delta \in (0,T)$. Inasmuch as $u_\varepsilon(0) = u_\varepsilon(T)$, and $\{u_\varepsilon(T); \varepsilon > 0\}$ is relatively compact in H, we conclude that $\{u_\varepsilon(0); \varepsilon > 0\}$ is relatively compact too. Using once again Theorem 2.1 we deduce that $\{u_\varepsilon; \varepsilon > 0\}$ is relatively compact in $C([0,T]; H)$. Thus, for each sequence $\varepsilon \downarrow 0$, at least on a subsequence, $(u_\varepsilon)_{\varepsilon>0}$ converges to a function u which obviously is an integral solution of (3.3). Since f is T-periodic with respect to its first argument this solution can be continued on the whole $\mathbf{R}$ as a T-periodic integral solution of (1.1), and this completes the proof of Theorem 1.1. ■

4. AN EXAMPLE

The aim of this section is to illustrate the effectiveness of the abstract existence result we have proved by showing how this applies to nonlinear partial differential equations of parabolic type. Thus, let us consider the nonlinear heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta_p u &= g(t,x,u) && \text{for } (t,x) \in \mathbf{R} \times \Omega \\ u &= 0 && \text{for } (t,x) \in \mathbf{R} \times \partial\Omega \\ u(t,x) &= u(t+T,x) && \text{for } (t,x) \in \mathbf{R} \times \Omega \end{aligned} \tag{4.1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $p \geq 2$, $g : \mathbf{R} \times \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ is a continuous function and Δ_p is the pseudo-Laplace operator, i.e.

$$\Delta_p u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right).$$

Our aim here is to show that, under some rather general assumptions on g, the problem (4.1) can be rewritten as an abstract evolution equation of the form (1.1) satisfying the hypotheses of Theorem 1.1.

We begin by recalling that an equivalent norm on $W_0^{1,p}(\Omega)$ is given by the $L^p(\Omega)$-norm of the gradient. Therefore there exists $c > 0$ such that

$$c\left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p dx \right) \leq \int_\Omega |\nabla u(x)|^p dx \tag{4.2}$$

for every $u \in W_0^{1,p}(\Omega)$.

We are now prepared to prove:

Theorem 4.1. *Assume that* $g: \mathbf{R} \times \overline{\Omega} \times \mathbf{R} \to \mathbf{R}$ *is continuous, T-periodic with respect to its first argument and there exists* $a > 0$ *and* $b > 0$ *such that*

$$|g(t,x,u)| \leq a|u| + b \text{ for all } (t,x,u) \in \mathbf{R} \times \overline{\Omega} \times \mathbf{R}. \tag{4.3}$$

Assume in addition that there exist $\alpha \in (0,c)$ *(where* $c > 0$ *satisfies (4.2)) and* $\beta > 0$ *such that*

$$u \cdot g(t,x,u) \leq \alpha|u|^p + \beta \text{ for all } (t,x,u) \in \mathbf{R} \times \overline{\Omega} \times \mathbf{R}. \tag{4.4}$$

Then there exists at least one solution $u: \mathbf{R} \to L^2(\Omega)$ *of the problem (4.1) satisfying*

$$u \in C([0,T]; L^2(\Omega)) \cap L^\infty([0,T]; W_0^{1,p}(\Omega)) \tag{4.5}$$

$$\frac{\partial u}{\partial t} \in L^2([0,T]; L^2(\Omega)). \tag{4.6}$$

Proof. Take $H = L^2(\Omega)$ with the usual inner product and let us define the operator $A: D(A) \subseteq H \to H$ by

$$Au = -\Delta_p u$$

for each $u \in D(A)$, where $D(A) = \{u \in W_0^{1,p}(\Omega); \Delta_p u \in L^2(\Omega)\}$, and $f: \mathbf{R} \times H \to H$ by

$$f(t,u)(x) = g(t,x,u(x))$$

for each $u \in L^2(\Omega)$, $t \in \mathbf{R}$ and a.e. for $x \in \Omega$.

Now it is clear that (4.1) can be rewritten as a problem of the form (1.1) with A and f as before. It is known that $-A$ generates a compact semigroup [17, Proposition 2.3.2, p.59] and, in view of (4.3), that f is everywhere defined, continuous and T-periodic with respect to its first argument. Thus, in order to appeal to Theorem 1.1, we have to show that A and f verify (1.2). To this aim let us denote by $|\Omega|$ the Lebesgue measure of Ω, choose

$$r \geq \beta^{1/p}(c-\alpha)^{-1/p}|\Omega|^{1/2}$$

and let us observe that, in view of (4.4), (4.2) and of the choice of r, it follows

$$\langle Au - f(t,u), u\rangle = \int_\Omega (|\nabla u(x)|^p - g(t,x,u(x))u(x))dx$$

$$\geq \int_\Omega (|\nabla u(x)|^p - \alpha|u(x)|^p - \beta)dx$$

$$\geq c\int_\Omega |\nabla u(x)|^p dx + (c-\alpha)\int_\Omega |u(x)|^p dx - \beta\cdot|\Omega|$$

$$\geq c\int_\Omega |\nabla u(x)|^p dx + (c-\alpha)|\Omega|^{(2-p)/2}\left(\int_\Omega |u(x)|^2 dx\right)^{p/2} - \beta\cdot|\Omega|.$$

Consequently, if $u \in \overline{D(A)}$ and

$$\left(\int_\Omega |u(x)|^2 dx\right)^{1/2} = r,$$

we have

$$\langle Au - f(t,u), u\rangle \geq c\int_\Omega |\nabla u(x)|^p dx \geq 0.$$

Thus A and f satisfy all the hypotheses of Theorem 1.1. Since (4.5) and (4.6) follows from [3, Proposition 2.4, p.204], the proof of Theorem 4.1 is complete. ■

Remark 4.1. With some obvious modifications the proof of Theorem 4.1 can be addapted to handle the case in which (4.3) and (4.4) hold for some $b \in L^\infty([0,T]; L^2(\Omega))$ and $\beta \in L^\infty([0,T]; L^1(\Omega))$.

References

[1] Badii, M. and Diaz, J.I.: *Periodic solutions of a degenerate parabolic problem arising in unsaturated flow through a porous medium*, to appear.

[2] Baras, P.: *Compacité de l'opérateur $f \to u$ solution d'une équation non linéaire, $(du)/(dt) + Au \ni f, C, R$.* Acad. Sci. Paris, 289 (1978), 1113-1116.

[3] Barbu, V.: *Nonlinear semigroups and differential equations in Banach spaces*, Editura Academiei Bucuresti, Noordhoff, 1976.

[4] Becker, R.I.: *Periodic solutions of semilinear equations of evolution of compact type*, J. Math. Anal. Appl., 82 (1981), 33-48.

[5] Brézis, H.: *Opérateurs maximaux monotones et semi-groupes de contractions dans un space de Hilbert*, North-Holland, 1973.

[6] Browder, F.E.: *Existence of periodic solutions for nonlinear equations of evolution,* Proc. Nat. Acad. Sci. U.S.A. 53 (1965), 1100-1103.

[7] Crandall, M.G.: *Nonlinear semigroups and evolution governed by accretive operators*, Proc. Sympos. Pure Math., vol. 45 Part 1 Amer. Math. Soc., Providence R.I., F.E. Browder editor, 1986, 305-337.

[8] Deimling, K.: *Nonlinear Functional Analysis*, Springer Verlag

Berlin-Heidelberg-New York-Tokyo, 1980.

[9] Haraux, A. and Kenmochi,N.: *Asymptotic behaviour of solutions to some degenerate problems*, Preprint, Analyse Numérique, Paris IV (1988), 1-23.

[10] Hirano, N.: *Existence of periodic solutions for nonlinear evolution equations in Hibert spaces*, Proc. Amer. Math. Soc., 120 (1994), 185-192.

[11] Kenmochi, N. and Kubo, M.: *Periodic solutions to a class of nonlinear variational inequalities with time dependent constraints*, Funkcialaj Ekvacioj, 30 (1987) 333-349.

[12] Kenmochi, N. and Kubo, M.: *Periodic solutions of parabolic-elliptic obstacle problems*, J. Diff. Eqs., 88 (1990).

[13] Prüss, J.: *Periodic solutions of semi-linear evolution equations*, Nonlin. Anal. T.M.A., 3 (1979), 601-612.

[14] Seidman, T.: *Periodic solutions of a nonlinear parabolic equation*, J. Diff. Eqs., 19 (1975) 242-257.

[15] Vrabie, I.I.: *The nonlinear version of Pazy's local existence theorem*, Israel J. Math., 32 (1979) 221-235.

[16] Vrabie, I.I.: *Periodic solutions for nonlinear evolution equations*, Proc. Amer. Math. Soc., 109 (1990) 653-661.

[17] Vrabie, I.I.: *Compactness methods for nonlinear evolutions*, Longman Scientific and Technical, 1987.

Dept. of Mathematics,
University "Al. I. Cuza" of Iasi,
Iasi 6600,
ROMANIA

Recibido: 28 de mayo de 1993