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The Space of Countably Simple Bounded Functions with Values in a DF-Space

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ABSTRACT. We study the posibility of lifting some properties, as being a (barrelled, quasi-barrelled, bornological or ultrabornological) DF, gDF or quasi-normable space, from a locally convex space E to thê space $S_{\aleph_0}(\mu, E)$, of countably-valued and bounded (classes of μ-a.e. equal) functions from a measure space (Ω, Σ, μ) into E.

Let (Ω, Σ, μ) be a measure space and $E \neq \{0\}$ be a Hausdorff locally convex space. Denote by $S_{\aleph_0}(\Sigma, E)$ the space of all functions $\varphi: \Omega \longrightarrow E$ that can be written as

$$\varphi(\cdot) = \sum_{n=1}^{\infty} \chi_{S_n}(\cdot) x_n, \qquad (*)$$

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where (x_n) is a bounded sequence from E and (S_n) is a sequence of non-empty and pairwise disjoint subsets of Σ covering Ω. A function of this form is called a countably simple bounded function. If we endow $S_{\aleph_0}(\Sigma, E)$ with the uniform convergence topology and identify functions that are equal μ-a.e., we obtain the quotient space $S_{\aleph_0}(\mu, E)$. This paper is devoted to the possibility of lifting some properties from E to $S_{\aleph_0}(\mu, E)$, mainly in the case when E is a DF-space. For the space $CB(X, E)$ of continuous bounded functions, similar results to our Theorems 1-4 here were obtained by Bierstedt, Bonet and Schmets in [2], and for the space $L^\infty(\mu, E)$ of essentially bounded measurable functions by Fernández and Florencio in [6]. We refer the reader to the monographs of Jarchow [8] and Pérez Carreras and Bonet [10] for the terminology used in this paper.

To fix some notation, denote by $\mathcal{Q}(E)$ the family of all continuous seminorms defining the topology of E, by $\mathcal{U}(E)$ the family of all absolutely convex and closed zero-neighbourhoods of E and by $\mathcal{B}(E)$ the family of all absolutely convex and closed bounded subsets of E. Then, a fundamental system of seminorms for the topology defined on $S_{\aleph_0}(\Sigma, E)$ is given by the mappings

$$\tilde{q}\Big(\sum_{n=1}^{\infty} \chi_{S_n} \cdot x_n\Big) = \sup\{q(x_n) : n \geq 1\},$$

where q runs through the set $\mathcal{Q}(E)$.

When μ is the cardinal measure, both spaces $S_{\aleph_0}(\Sigma, E)$ and $S_{\aleph_0}(\mu, E)$ coincide. In particular, if μ is the cardinal measure on the set $\mathbf{N}$ of all positive integers, we have that $S_{\aleph_0}(\mu, E) = \ell^\infty(E)$, the space of all bounded sequences from E. Thus, the case studied here can be considered a generalization of this space of vector-valued sequences. We shall see, however, that when μ is atomless the behaviour can be very different (see Theorem 5 below).

We start by proving that $S_{\aleph_0}(\mu, E)$ is a Hausdorff quotient of $S_{\aleph_0}(\Sigma, E)$.

Proposition 1. *Let $\mathcal{N}_\mu$ be the subspace of all functions in $S_{\aleph_0}(\Sigma, E)$ that are equal μ-a.e. to the zero function. Then $\mathcal{N}_\mu$ is closed.*

Proof. Let φ be a countably simple bounded function such that $\varphi \notin \mathcal{N}_\mu$. Then, we can write φ as in $(*)$, requiring in addition that $\mu(S_1) > 0$ and $x_1 \neq 0$. Thus, there is a continuous seminorm $q \in \mathcal{Q}(E)$ such that $q(x_1) > 0$. Consider the following open neighbourhood of φ,

$$V := \{\psi \in S_{\aleph_0}(\Sigma, E) : \; \tilde{q}(\psi - \varphi) < q(x_1)/2\}.$$

We only have to show that $V \cap \mathcal{N}_\mu$ is empty. Take $\psi \in V$. According to $(*)$, ψ can be written as $\psi = \sum_{m=1}^{\infty} \chi_{T_m} y_m$. Consider the element of Σ defined by

$$T := \bigcup_{m \in I} T_m, \qquad \text{where } I := \{m \in \mathbf{N} : \; y_m \neq 0\}.$$

Since $\psi \neq 0$, T is non-empty. Let us see that $\mu(S_1 \setminus T) = 0$. Indeed, if we suppose that $\mu(S_1 \setminus T) > 0$, then $S_1 \setminus T$ is non-empty. In this set, $\psi - \varphi$ takes the value $-x_1$. Therefore $q(x_1)/2 > \tilde{q}(\psi - \varphi) \geq q(x_1)$ and this is a contradiction. Since $\mu(S_1) > 0$, there must be an index $m_0 \in I$ such that $\mu(S_1 \cap T_{m_0}) > 0$, thus $\mu(T_{m_0}) > 0$. Since $y_{m_0} \neq 0$, we conclude $\psi \notin \mathcal{N}_\mu$. ∎

To avoid trivial cases, we assume the following condition (C):

(C) There is a sequence (Δ_n) of pairwise disjoint sets in Σ with $\mu(\Delta_n) > 0$, for all $n \in \mathbf{N}$.

We follow a common habit and do not distinguish by notation between a map and its μ-equivalence class. In particular, using $(*)$ above and condition (C), for a non-zero element $\varphi \in S_{\aleph_0}(\mu, E)$ we can always choose a representative of the form $\sum_{n=1}^{\infty} \chi_{S_n} x_n$, where (x_n) is a bounded sequence from E and (S_n) is a pairwise disjoint sequence of subsets of Σ with positive measure convering Ω. The set $R(\varphi) := \{x_n : \; n \in \mathbf{N}\}$ is clearly well-defined, we call it the essential range of the function φ. The quotient topology of $S_{\aleph_0}(\mu, E)$ can be defined by the family of seminorms q_∞ given by

$$\begin{aligned} q_\infty(\varphi) : &= \inf\{\tilde{q}(\psi) : \psi = \varphi \; (\mu - \text{a.e.})\} \\ &= \sup\{q(x) : x \in R(\varphi)\} \qquad (q \in \mathcal{Q}(E)). \end{aligned}$$

We give now some technical results that describe the behaviour of those subsets of $S_{\aleph_0}(\mu, E)$ that are lifted from sets in E in a natural way.

If A is a subset of E, we say that a function $\varphi \in S_{\aleph_0}(\mu, E)$ takes its values essentially in A if $R(\varphi) \subset A$. We denote by $L(A)$ the subset of all functions in $S_{\aleph_0}(\mu, E)$ that take their values essentially in A. The set $L(A)$ inherits certain properties from A. We list some of them that will be useful and can be easily checked:

(1) For all subsets A, B of E we have that $L(A) \subset L(B)$ if and only if $A \subset B$.

(2) $L(\cap A_n) = \cap L(A_n)$, for every sequence (A_n) of subsets of E.

(3) $L(\alpha A) = \alpha L(A)$ for all $A \subseteq E$ and scalars α.

(4) If either A or B is bounded, then $L(A) + L(B) = L(A + B)$.

(5) $L(A)$ is absolutely convex if so is A.

(6) $U \subset E$ is a zero-neighbourhood in E if and only if $L(U)$ is a zero-neighbourhood in $S_{\aleph_0}(\mu, E)$. The system $\{L(U) : U \in \mathcal{U}(E)\}$ is a basis of zero-neighbourhoods for the topology of $S_{\aleph_0}(\mu, E)$.

(7) A is a bounded subset of E if and only if $L(A)$ is a bounded subset of $S_{\aleph_0}(\mu, E)$. Moreover, for every bounded subset C of $S_{\aleph_0}(\mu, E)$, there exists $A \in \mathcal{B}(E)$ such that $C \subset L(A)$; just take A to be the closed absolutely convex hull:

$$A = \overline{\text{acx}}\left(\bigcup_{\varphi \in C} R(\varphi)\right).$$

From the list above, it is clear that many properties —like the existence of a fundamental sequence of bounded subsets, the metrizability of the bounded subsets or the countable boundedness property —are equivalent for the spaces E and $S_{\aleph_0}(\mu, E)$.

Theorem 1. *$S_{\aleph_0}(\mu, E)$ is a DF-space (resp. a gDF-space) if and only if E is a DF-space (resp. a gDF-space).*

Proof. ($\Leftarrow$) Assume that E is a DF-space and let (B_n) be an increasing fundamental sequence of absolutely convex bounded subsets in E. As we pointed out above, $(L(B_n))$ is a fundamental sequence of absolutely convex bounded subsets in $S_{\aleph_0}(\mu, E)$. Now, we have to prove that $S_{\aleph_0}(\mu, E)$ is countably-quasi-barrelled. Let (W_n) be a sequence of absolutely convex zero-neighbourhoods in $S_{\aleph_0}(\mu, E)$ such that $W = \cap_n W_n$ is bornivorous. We have to see that W is also a zero-neighbourhood. For every $n \in \mathbf{N}$, take $r_n > 0$ with $r_n L(B_n) \subset 2^{-(n+1)}W$. Then we have

$$\bigcup_{n\geq 1}(r_1 L(B_1) + r_2 L(B_2) + \cdots + r_n L(B_n)) \subset \frac{1}{2}W.$$

Since (W_n) are zero-neighbourhoods in $S_{\aleph_0}(\mu, E)$, there exists a sequence (V_n) in $\mathcal{U}(E)$ such that $L(V_n) \subseteq \frac{1}{2}W_n$, for all $n \in \mathbf{N}$. Consider, for $n = 1, 2, \ldots$, the absolutely convex zero-neighbourhoods in E given by

$$U_n := r_1 B_1 + r_2 B_2 + \cdots + r_n B_n + V_n.$$

It is clear that $U = \cap_n U_n$ is bornivorous in the DF-space E. Therefore U is a zero-neighbourhood in E. Finally, we will show that $L(U) \subset W$.

$$\begin{aligned} L(U) &= L\left(\bigcap_{n=1}^{\infty} U_n\right) = \bigcap_{n=1}^{\infty} L(U_n) = \bigcap_{n=1}^{\infty} L\left(\sum_{k=1}^{n} r_k B_k + V_n\right) \\ &= \bigcap_{n=1}^{\infty}\left(\sum_{k=1}^{n} r_k L(B_k) + L(V_n)\right) \subseteq \bigcap_{n=1}^{\infty}\left(\frac{1}{2}W + \frac{1}{2}W_n\right) \\ &\subseteq \bigcap_{n=1}^{\infty} W_n = W. \end{aligned}$$

($\Rightarrow$) Now, assume that $S_{\aleph_0}(\mu, E)$ is a DF-space. If (C_n) is a fundamental sequence of absolutely convex bounded subsets of $S_{\aleph_0}(\mu, E)$, we can find a sequence of absolutely convex bounded subsets (B_n) in E such that $C_n \subset L(B_n)$ for all $n \in \mathbf{N}$. It is easy to see that (B_n) is an fundamental sequence of bounded sets in E. To show that E is countably-quasi-barrelled, let (U_n) be a sequence of absolutely convex zero-neighbourhoods in E such that $U = \cap_n U_n$ is bornivorous. Then $(L(U_n))$ is a sequence of zero-neighbourhoods in $S_{\aleph_0}(\mu, E)$ such that $L(U) = \cap_n L(U_n)$ is bornivorous. Since $S_{\aleph_0}(\mu, E)$ is a DF-space, it follows that $L(U)$ is a zero-neighbourhood in $S_{\aleph_0}(\mu, E)$, hence U is a zero-neighbourhood in E. This finishes the proof.

To prove that $S_{\aleph_0}(\mu, E)$ is a gDF-space if and only if E is a gDF-space use the fact that for all sequences (B_n) in $\mathcal{B}(E)$ and (U_n) in $\mathcal{U}(E)$ we have that

$$L\left(\bigcap_{n\geq 1}(B_n + U_n)\right) = \bigcap_{n\geq 1}(L(B_n) + L(U_n)),$$

together with condition [8, 12.3.1]. We leave the details to the reader. ■

Theorem 2. *$S_{\aleph_0}(\mu, E)$ is quasi-normable if and only if E is quasi-normable.*

Proof. ($\Leftarrow$) Given any zero-neighbourhood W in $S_{\aleph_0}(\mu, E)$, there is an absolutely convex zero-neighbourhood U in E with $L(U) \subset W$. By hypothesis, we can find $V \in \mathcal{U}(E)$ such that for every $\varepsilon > 0$ there exists $B \in \mathcal{B}(E)$ with $V \subset B + \varepsilon U$. Then

$$L(V) \subset L(B + \varepsilon U) = L(B) + \varepsilon L(U) \subset L(B) + \varepsilon W,$$

so $S_{\aleph_0}(\mu, E)$ is quasi-normable.

($\Rightarrow$) On the other hand, given $U \in \mathcal{U}(E)$, since $L(U)$ is a zero-neighbourhood in $S_{\aleph_0}(\mu, E)$, by hypothesis there exists $V \in \mathcal{U}(E)$ such that for every $\varepsilon > 0$ there is $B \in \mathcal{B}(E)$ with $L(V) \subset L(B) + \varepsilon L(U) = L(B + \varepsilon U)$. It follows that $V \subset B + \varepsilon U$, and the proof is finished. ■

We now study when the space $S_{\aleph_0}(\mu, E)$ is quasi-barrelled or bornological for E a DF-space. In the characterization of the quasi-barrelled and bornological spaces $\ell^\infty(E)$ given by Bierstedt and Bonet in [1, Thm. 5 and Cor. 8] the *dual density condition* and the *strong dual density condition*, introduceed and studied by them in the same paper, play an essential role. We shall see that these conditions are also essential in our more general case.

These conditions read technically as follows (see [1, Prop. 1.4(b)]): A DF-space E with a fundamental sequence (B_n) of bounded subsets, verifies the dual density condition (resp. strong dual density condition) if and only if for every decreasing sequence $(\lambda_n)_{n\geq 1}$ of positive real numbers, there exists $U \in \mathcal{U}(E)$ such that for every $n \geq 1$, we can find $m \geq n$ and $\varepsilon_n > 0$ with

$$B_n \cap \varepsilon_n U \subseteq \overline{\operatorname{acx}}\Big(\bigcup_{k=1}^{m} \lambda_k B_k\Big) \qquad \left(\text{resp. } B_n \cap \varepsilon_n U \subseteq \operatorname{acx}\Big(\bigcup_{k=1}^{m} \lambda_k B_k\Big)\right).$$

We also know that a DF-space satisfies the dual density condition if and only if its bounded subsets are metrizable [1, Thm. 1.5].

Theorem 3. *Let (Ω, Σ, μ) be a measure space and E be a DF-space. Then, the following assertions are equivalent:*

(1) E satisfies the dual density condition or, equivalently, each bounded subset of E is metrizable.

(2) $S_{\aleph_0}(\mu, E)$ is quasi-barrelled.

Proof. (1) $\Rightarrow$ (2) By Theorem 1, $S_{\aleph_0}(\mu, E)$ is a DF-space whose bounded subsets are metrizable by properties (6) and (7) above. The implication follows from a well-known result on DF-spaces [9, §29 3.(12)].

(2) $\Rightarrow$ (1) Consider an increasing fundamental sequence (B_n) of closed absolutely convex bounded subsets of E, and suppose that (1) does not hold. By reading the dual density condition in the technical form given above, we can see that there exists a decreasing sequence (λ_n) of strictly positive numbers such that for every $U \in \mathcal{U}(E)$, we can find $n \geq 1$ with the property that for every $m \geq n$ and every $\varepsilon > 0$, in particular $\varepsilon = 1$, we have

$$B_n \cap U \not\subseteq C_m := \overline{\operatorname{acx}}(\lambda_1 B_1 \cup \lambda_2 B_2 \cup \cdots \cup \lambda_m B_m).$$

This gives us an increasing sequence (C_m) of closed absolutely convex bounded subsets of the DF-space E such that:

(i) $(\lambda_m^{-1} C_m)$ is a fundamental sequence of bounded sets in E.

(ii) For each $U \in \mathcal{U}(E)$, there is $k \geq 1$ with $B_k \cap U \not\subset C_m$ for all $m \geq 1$.

We now adapt a technique due to S. Dierolf (see [5, Prop. 4.5]) to get a contradiction. Since every bounded set in $S_{\aleph_0}(\mu, E)$ is absorbed by some $L(C_n)$, the set

$$W = \bigcup_{n \geq 1} L(C_n)$$

is bornivorous in $S_{\aleph_0}(\mu, E)$. Then, the closure $\overline{W}$ of W in $S_{\aleph_0}(\mu, E)$ is a bornivorous barrel in $S_{\aleph_0}(\mu, E)$, and by hypothesis, it is a zero-neighbourhood in $S_{\aleph_0}(\mu, E)$. Then, there exists $U \in \mathcal{U}(E)$ such that

$$L(U) \subseteq \frac{1}{2}\overline{W}.$$

Now, by (ii) there is a bounded sequence (x_n) in U such that $x_n \notin C_n$. Since (C_n) are closed subsets of E, there is a sequence $(V_n) \subset \mathcal{U}(E)$ such that

$$x_n \notin C_n + V_n, \ n \in \mathbf{N}. \tag{$**$}$$

By (i), and since the sequence (C_n) is increasing, the set

$$V = \bigcap_{k \geq 1} \left(\frac{1}{2} C_k + V_k \right)$$

is bornivorous in the DF-space E. Hence, V is a zero-neighbourhood in E and $L(V)$ is a zero-neighbourhood in $S_{\aleph_0}(\mu, E)$. We have that

$$\begin{aligned} L(U) &\subset \frac{1}{2}\overline{W} \subset \frac{1}{2}W + L(V) = \bigcup_{n \geq 1} \frac{1}{2} L(C_n) + L(V) \\ &= \bigcup_{n \geq 1} \frac{1}{2} L(C_n) + \bigcap_{n \geq 1} \left(\frac{1}{2} L(C_n) + L(V_n) \right) \\ &\subset \bigcup_{n \geq 1} (L(C_n) + L(V_n)) = \bigcup_{n \geq 1} L(C_n + V_n). \end{aligned} \tag{$***$}$$

Clearly, the function $\varphi : \Omega \to E$ defined by $\varphi = \sum_{n\geq 1} x_n\chi_{\Delta_n}$, where (Δ_n) is the sequence given by condition (C), is in $S_{\aleph_0}(\mu, E)$. Moreover, $\varphi \in L(U)$ because the bounded sequence (x_n) is contained in U. By $(***)$, there is $n \in \mathbf{N}$ such that $\varphi \in L(C_n + V_n)$. Since $\mu(\Delta_n) > 0$, we have that $x_n \in C_n + V_n$, which is in contradiction with $(**)$. ∎

Theorem 4. *Let (Ω, Σ, μ) be a measure space and E a DF-space. Then, the following assertions are equivalent:*

(1) E satisfies the strong dual density condition.

(2) $S_{\aleph_0}(\mu, E)$ is bornological.

Proof. Let (B_n) be an increasing fundamental sequence of absolutely convex closed bounded subsets of E. To prove that (2) $\Rightarrow$ (1), suppose that (1) does not hold. Reading the strong dual density condition in the technical form given above, we have that there exists a decreasing sequence (λ_n) of strictly positive numbers such that for each $U \in \mathcal{U}(E)$, we can find $n \geq 1$ with the property that for every $m \geq n$ and $\varepsilon > 0$, in particular $\varepsilon = 1$, we have

$$B_n \cap U \not\subset C_m := \mathrm{acx}(\lambda_1 B_1 \cup \lambda_2 B_2 \cup \cdots \cup \lambda_m B_m).$$

This gives us an increasing sequence (C_m) of absolutely convex bounded subsets of E. Take $W = \cup_n L(C_n)$. This set W is absolutely convex and bornivorous in $S_{\aleph_0}(\mu, E)$. Since $S_{\aleph_0}(\mu, E)$ is bornological, then W is a zero-neighbourhood, and we can find $U \in \mathcal{U}(E)$ such that $L(U) \subset W$.

Since $B_n \cap U \not\subset C_m$, we can take $x_m \in (B_n \cap U) \backslash C_m$ for all $m \geq 1$. If we set $\varphi = \sum_{m=1}^{\infty} x_m \chi_{\Delta_m}$, where (Δ_m) is the sequence from condition (C), then $\varphi \in S_{\aleph_0}(\mu, E)$. Moreover, $\varphi \in L(U) \subset W = \cup_n L(C_n)$. Therefore, there exists $n_0 \in N$ such that $\varphi \in L(C_{n_0})$. Since $\mu(\Delta_{n_0}) > 0$, then $x_{n_0} \in C_{n_0}$ and this is a contradiction with the selection of the x_n's.

(1) $\Rightarrow$ (2) Since $(L(B_n))$ is an increasing fundamental sequence of absolutely convex closed bounded subsets in $S_{\aleph_0}(\mu, E)$ and this is a DF-space, we only have to show that if W is an absolutely convex bornivorous subset of $S_{\aleph_0}(\mu, E)$, then $W \cap L(B_n)$ is a zero neighbourhood

in $L(B_n)$ for every $n \in \mathbf{N}$, when $L(B_n)$ is endowed with the topology inherited from $S_{\aleph_0}(\mu, E)$.

Since W is bornivorous, there exists a decreasing sequence of positive real numbers (λ_n) such that $\lambda_n L(B_n) \subset W$ for every $n \in \mathbf{N}$.

Bearing in mind the definition of the strong dual density condition, there exists $U \in \mathcal{U}(E)$ such that from all $n \in \mathbf{N}$, we can find $m \geq n$ and $\varepsilon_n > 0$ with

$$B_n \cap \varepsilon_n U \subset \operatorname{acx}\left(\frac{\lambda_1}{2} B_1 \cup \frac{\lambda_2}{2^2} B_2 \cup \cdots \cup \frac{\lambda_m}{2^m} B_m \right).$$

Therefore,

$$\begin{aligned} L(B_n) \cap \varepsilon_n L(U) = L(B_n \cap \varepsilon_n U) &\subset L\left(\operatorname{acx}\left(\bigcup_{k=1}^{m} \frac{\lambda_k}{2^k} B_k \right)\right) \\ &\subset L\left(\sum_{k=1}^{m} \frac{\lambda_k}{2^k} B_k \right) = \sum_{k=1}^{m} \frac{\lambda_k}{2^k} L(B_k) \subset W. \end{aligned}$$

Finally, $L(B_n) \cap \varepsilon_n L(U) \subset W \cap L(B_n)$ for every $n \in \mathbf{N}$, so the proof is finished. ∎

To study when $S_{\aleph_0}(\mu, E)$ is barrelled or ultrabornological, we shall use the abstract results given in [3] (barrelledness) and in [4] (ultrabornology) for a locally convex space endowed with a suitable Boolean algebra of projections. A family $P_\Sigma = \{P_S : S \in \Sigma\}$ of continuous linear *projections in* E *is called an* (Ω, Σ, μ)*-Boolean algebra of projections if* the following conditions are satisfied:

(i) P_Ω is the identity on E.

(ii) $P_S = 0$ whenever $S \in \Sigma$ and $\mu(S) = 0$.

(iii) $P_{S \cap T} = P_S \cdot P_T$ for all $S, T \in \Sigma$.

(iv) $P_{S \cup T} = P_S + P_T$ for all disjoint $S, T \in \Sigma$.

The results mentioned above can be stated as follows. (Similar results as in [3] and [4] for some spaces of (scalar or vector-valued) continuous functions defined on an interval $[a,b] \subset \mathbf{R}$, have been obtained independently and about the same time by Gilioli [7].)

Theorem A. ([3, Cor. 1 and 2] and [4, Cor. 1 and 2].) *Let (Ω,Σ,μ) be a σ-finite measure space. Let E be a Hausdorff locally convex space and P_Σ be an (Ω,Σ,μ)-Boolean algebra of projections. Assume that P_Σ is equicontinuous and that the following conditions holds:*

($\bullet$) *If (Ω_n) is a decreasing sequence in Σ with $\mu(\cap_n \Omega_n) = 0$, (x_n) is a bounded sequence in E such that every x_n is supported in Ω_n (i.e. $P_{\Omega_n}(x_n) = x_n$), and (α_n) is a sequence in ℓ^1, then the series $\sum_n \alpha_n x_n$ converges in E.*

Then we have:

(1) *If E is quasi-barrelled and $P_S(E)$ is barrelled for each atom $S \in \Sigma$, then E is barrelled.*

(2) *If E is quasi-barrelled and μ is atomless, then E is barrelled.*

(3) *If E is bornological and $P_S(E)$ is ultrabornological for each atom $S \in \Sigma$, then E is ultrabornological.*

(4) *If E is bornological and μ is atomless, then E is ultrabornological.*

To use Theorem A in our case, we state the following lemma.

Lemma. *For every subset $S \in \Sigma$, denote*

$$P_S : \varphi \in S_{\aleph_0}(\mu, E) \to P_S(\varphi) = \chi_S \cdot \varphi \in S_{\aleph_0}(\mu, E).$$

Then we have:

(1) *The set $\{P_S : S \in \Sigma\}$ is an equicontinuous Boolean algebra of projections on $S_{\aleph_0}(\mu, E)$.*

(2) *If S is an atom, then $P_S(S_{\aleph_0}(\mu, E))$ is isomorphic to E.*

(3) *$S_{\aleph_0}(\mu, E)$ satisfies the condition (•) in Theorem A, i.e. if (Ω_n) is a decreasing sequence of subsets of Σ such that $\mu(\cap_n \Omega_n) = 0$ and (φ_n) is a bounded sequence in $S_{\aleph_0}(\mu, E)$ such that φ_n is supported in Ω_n, for each $n \in \mathbf{N}$ and if (α_n) is a sequence from ℓ_1, then the series $\sum_n \alpha_n \varphi_n$ converges in $S_{\aleph_0}(\mu, E)$.*

Proof. (1) The algebraic part is easy. To prove the equicontinuity, simply note that $P_S(L(U)) \subset L(U)$, for every $U \in \mathcal{U}(E)$ and $S \in \Sigma$.

(2) It is enough to prove that any function of $S_{\aleph_0}(\mu, E)$ is constant on S. In this case, the isomorphism is the natural. Suppose that $\varphi \in S_{\aleph_0}(\mu, E)$ is not constant on S. Then, there exist $x_1, x_2 \in E$ with $x_1 \neq x_2$, such that the subsets

$$S_1 = \{\omega \in S : \varphi(\omega) = x_1\} \quad \text{and} \quad S_2 = \{\omega \in S : \varphi(\omega) = x_2\}$$

are disjoint and they have positive measure. Since S is an atom, we have that $\mu(S_1) = \mu(S_2) = \mu(S)$ and we get a contradiction.

(3) Since (Ω_n) is decreasing and each φ_n is supported in Ω_n, it follows that the series $\sum_n \alpha_n \varphi_n$ converges pointwise μ-a.e. to a function φ because outside every Ω_n there is only a finite number of non-zero terms and $\mu(\cap_n \Omega_n) = 0$. Finally, note that φ is countably simple and bounded because

$$R(\varphi) \subset \bigcup_{n=1}^{\infty} \left\{ \sum_{k=1}^{n} \alpha_k x_k, \ x_k \in R(\varphi_k), \ k \in \mathbf{N} \right\}$$

is bounded and countable, and that φ is also the limit in $S_{\aleph_0}(\mu, E)$ of $\sum_n \alpha_n \varphi_n$. ■

From this lemma and Theorem A, we have the following.

Theorem 5. *Let (Ω, Σ, μ) be a σ-finite measure space and E a DF-space.*

(a) *If the measure μ has atoms, then*

(1) *$S_{\aleph_0}(\mu, E)$ is barrelled if and only if E is barrelled and each bounded subset of E is metrizable.*

(2) $S_{\aleph_0}(\mu, E)$ *is ultrabornological if and only if E is ultrabornological and E satisfies the strong dual density condition.*

(b) *If the measure* μ *is atomless, then*

(1) $S_{\aleph_0}(\mu, E)$ *is barrelled if and only if each bounded subset of E is metrizable.*

(2) $S_{\aleph_0}(\mu, E)$ *is ultrabornological if and only if E satisfies the strong dual density condition.*

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