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Oscillation of Neutral Differential Equations with "Maxima"

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ABSTRACT. In the paper ordinary neutral differential equations with "maxima" are considered. Sufficient conditions for oscillation of all solutions are obtained.

1. INTRODUCTION

In the paper the following equation is considered

$$[x(t)+p(t)x(t-\tau)]' + q(t)\max_{[t-\sigma,t]} x(s) = 0 \qquad (1)$$

Though differential equations with maxima are often met in the applications, for instance in the theory of automatic control [3], [4], the qualitative theory of these equations is relatively little developed. The existence of periodic solutions of the equations with maxima is considered in [5] and [6]. The asymptotic stability of the solutions of these

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equations is investigated in [7]. The only paper in which the oscillatory properties of equations with maxima are considered is [1].

The main goal of the present paper is to obtain sufficient conditions for oscillation of all solutions of equation (1).

2. AUXILIARY ASSERTIONS

Definition 1. *The function f is said to eventually enjoy the property K if there exists t_0 such that for $t \geq t_0$ the function f enjoys the property K.*

Define the function $z(t)$ as follows:

$$z(t) = x(t) + p(t)x(t - \tau) \tag{2}$$

Definition 2. *The function x defined for all sufficiently large values of t is said to be an eventual solution of (1) if for all t large enough x is a continuous function, z is a continously differentiable function and x satisfies eventually equation (1).*

Remark 1. In the paper solutions for which $x(t) \equiv 0$ eventually are not considered.

Definition 3. *The eventual solution $x(t)$ of (1) is said to oscillate if the set of its zeros is unbounded above. Otherwise, the solution is said to be nonoscillating.*

By Definition 3 the nonoscillating solutions of (1) are characterized as being eventually positive or eventually negative.

We shall say that conditions (H) are met if the following conditions hold:

H1. $p(t) \in C([t_0, \infty), \mathbb{R})$

H2. $q(t) \in C([t_0, \infty), (0, \infty))$

H3. $\tau, \sigma \in \mathbb{R},\ \tau > 0,\ \sigma > 0$

H4. $\int_{t_0}^{\infty} q(t)dt = \infty$

Lemma 1. ([2], p. 46) *For the equation*

$$x'(t) + p(t)x(t-\tau) = 0, \; t \geq t_0$$

let the following conditions hold:

(i) $p(t) \in C([t_0, \infty), \mathbb{R}_+)$

(ii) $\tau > 0$

(iii) $\liminf\limits_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}$

Then the following assertions are valid:

a) the inequality $\dot{x}(t) + p(t)x(t-\tau) \leq 0$ *has no eventually positive solution;*

b) the inequality $x'(t) + p(t)x(t-\tau) \geq 0$ *has no eventually negative solution.*

Lemma 2. ([2], p. 46). *Let the conditions of Lemma 1 hold.*

Then the following assertions are valid:

a) the inequality $x'(t) - p(t)x(t+\tau) \geq 0$ *has no eventually positive solution;*

b) the inequality $x'(t) - p(t)x(t+\tau) \leq 0$ *has no eventually negative solution;*

c) the equation $x'(t) - p(t)x(t+\tau) = 0$ *has only oscillating solutions.*

Lemma 3. *Let conditions (H) hold. Then the following assertions are valid:*

(i) If $p(t) \leq -1$ *and* $x(t)$ *is an eventually positive solution of (1), then the function* $z(t)$ *is eventually decreasing and* $z(t) < 0$ *evetually.*

(ii) If $p(t) \leq -1$ *and* $x(t)$ *is an eventually negative solution of (1), then* $z(t)$ *is an eventually increasing function and* $z(t) > 0$ *eventually.*

(iii) If $-1 \leq p(t) \leq 0$ *and* $x(t) > 0$ *eventually, then* $z(t) > 0$ *eventually.*

(iv) If $-1 \leq p(t) \leq 0$ *and* $x(t) < 0$ *eventually, then* $z(t) < 0$ *eventually.*

Proof.

(i) From the definition of $z(t)$ there follows the equality

$$z'(t) + q(t) \max_{[t-\sigma,t]} x(s) = 0 \tag{3}$$

Since $x(t) > 0$ eventually, then $z'(t) < 0$ and $z(t)$ is an eventually decreasing function. Suppose that $z(t) > 0$ eventually.

From (2) there follow the inequalities

$$x(t) > -p(t)x(t-\tau) \geq x(t-\tau)$$

From the inequality $x(t) > x(t-\tau)$ and the fact that $x(t)$ is an eventually positive function it follows that there exists a constant $m > 0$ such that $x(t) > m$ eventually and $\max\limits_{[t-\sigma,t]} x(s) > m$ eventually. From (3) we obtain the estimate

$$z'(t) = -q(t) \max_{[t-\sigma,t]} x(s) < -mq(t)$$

Integrate the last inequality from t_1 to t, where t_1 is a sufficiently large number, and obtain

$$z(t) < z(t_1) - m \int_{t_1}^{t} q(s)ds \tag{4}$$

Passing to the limit in (4), from condition H4 it follows that $\lim\limits_{t\to\infty} z(t) = -\infty$ which contradicts the assumption that $z(t) > 0$ eventually.

(ii) From (3) and the inequality $x(t) < 0$ eventually it follows that $z'(t)$ is an eventually positive function and $z(t)$ is an eventually increasing function. Suppose that $z(t) < 0$ eventually. From (2) and from the condition $p(t) \leq -1$ we deduce the inequality $x(t) < x(t-\tau)$. Hence there exists a negative constant m such that $x(t) < m$ eventually and $\max\limits_{[t-\sigma,t]} x(s) < m$ eventually. From (3) it follows that $z'(t) \geq -q(t)m$.

Further on, as in the proof of (i) it is shown that $\lim_{t\to\infty} z(t) = \infty$ which contradicts the assumption that $z(t) < 0$ eventually. Hence $z(t) > 0$ eventually.

(iii) Suppose that $z(t) < 0$ eventually. Then from (2) and from the condition $-1 \le p(t) \le 0$ we obtain the inequality

$$x(t) < x(t-\tau) \tag{5}$$

Since $x(t) > 0$ eventually, then from the above inequality it follows that $x(t)$ is a bounded function for $t \in [t_0, \infty)$, hence $z(t)$ is a bounded function too. It is immediately verified that $z'(t) < 0$ eventually and $z(t)$ is an eventually decreasing negative function. Since $z(t)$ is a bounded function, then there exists the finite limit $\lim_{t\to\infty} z(t) = l$ $(l < 0)$. Let $c = \liminf_{t\to\infty} x(t)$. Suppose that $c > 0$. Eventually the inequality $x(t) \ge \frac{c}{2}$ is valid, hence $\max_{[t-\sigma,t]} x(s) \ge \frac{c}{2}$ eventually. Then as in (i) it is proved that $\lim_{t\to\infty} z(t) = -\infty$ which contradicts the fact that $z(t)$ is a bounded function.

Thus we proved that $\liminf_{t\to\infty} x(t) = 0$. There exists a sequence $\{t_n\}_1^\infty$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} x(t_n - \tau) = 0$. From (5) it follows that $\lim_{n\to\infty} x(t_n) = 0$. Passing to the limit in the equality $z(t_n) = x(t_n) + p(t_n)x(t_n - \tau)$ as $n \to \infty$, we obtain that $\lim_{n\to\infty} z(t_n) = 0$, which contradicts the fact that $l = \lim_{t\to\infty} z(t) < 0$. Hence $z(t) > 0$ eventually.

The proof of (iv) is analogous to the proof of (iii). ■

Remark 2. We shall emphasize that equation (1) is nonlinear and in general the fact that $x(t)$ is a solution of (1) does not imply that $-x(t)$ is also a solution of (1). That is why in the proof of Lemma 3 (and hence in the proof of the main theorems) the cases $x(t) > 0$ and $x(t) < 0$ eventually are considered separately.

3. MAIN RESULTS

Theorem 1. *Let conditions (H) hold and* $p(t) \equiv 1$.

Then each solution of equation (1) oscillates.

Theorem 1 is an immediate corollary of Lemma 3.

Theorem 2. *Let conditions (H) hold,* $p(t) \leq -1$ *and* $\tau > \sigma$. *Moreover, let the following condition hold*

$$\liminf_{t\to\infty} \int_{t-\tau+\sigma}^{t} \frac{q(s)}{\max\limits_{u\in[s-\sigma,s]} \{-p(u+\tau)\}}\, ds > \frac{1}{e} \tag{6}$$

Then each solution of equation (1) oscillates.

Proof. Suppose that equation (1) has a nonoscillating solution $x(t)$. Let $x(t) < 0$ eventually. From Lemma 3 (ii) it follows that $z(t) > 0$ eventually. From (2) there follow the inequalities

$$\begin{aligned} z(t) &< p(t)x(t-\tau) \\ x(t) &< \frac{z(t+\tau)}{p(t+\tau)} \\ \max_{[t-\sigma,t]} x(s) &\leq \max_{[t-\sigma,t]} \frac{z(s+\tau)}{p(s+\tau)} \end{aligned} \tag{7}$$

Since $z(t)$ is an eventually increasing function, then for sufficiently large t the following estimate is valid

$$z(t+\tau-\sigma) \leq z(s+\tau)\ , \ \ s \in [t-\sigma, t]$$

Then

$$\max_{[t-\sigma,t]} \frac{z(t+\tau-\sigma)}{-p(s+\tau)} \geq \max_{[t-\sigma,t]} \frac{z(s+\tau)}{p(s+\tau)}$$

$$\frac{-z(t+\tau-\sigma)}{\max\limits_{[t-\sigma,t]}\{-p(s+\tau)\}} \geq \max_{[t-\sigma,t]} \frac{z(s+\tau)}{p(s+\tau)}$$

From the last inequality and from (7) we deduce the inequality

$$\max_{[t-\sigma,t]} x(s) \leq \frac{-z(t+\tau-\sigma)}{\max\limits_{[t-\sigma,t]}\{-p(s+\tau)\}} \tag{8}$$

From (3) and from (8) it follows that the eventually positive function $z(t)$ satisfies the inequality

$$z'(t) - \frac{q(t)}{\max\limits_{[t-\sigma,t]}\{-p(s+\tau)\}}\, z(t+(\tau-\sigma)) \geq 0 \tag{9}$$

But from (6) and from Lemma 2 it follows that inequality (9) has no eventually positive solutions. The contradiction obtained shows that equation (1) has no eventually negative solutions. We shall show that it has no eventually positive solutions either. Suppose that this is not true. Let $x(t) > 0$ eventually. From Lemma 3 (i) it follows that $z(t) < 0$ eventually. As above, we obtain the estimate

$$\max_{[t-\sigma,t]} x(s) \geq \frac{-z(t+(\tau-\sigma))}{\max\limits_{[t-\sigma,t]}\{-p(s+\tau)\}} \tag{10}$$

From (3) and from (10) it follows that the eventually negative function $z(t)$ satisfies the inequality

$$z'(t) - \frac{q(t)}{\max\limits_{[t-\sigma,t]}\{-p(s+\tau)\}}\, z(t+(\tau-\sigma)) \leq 0 \tag{11}$$

But from (6) and from Lemma 2 it follows that inequality (11) has no eventually negative solutions. Hence (1) has no eventually positive solutions. Since in view of what was proved above it has no eventually negative solutions, then each solution of equation (1) oscillates. ∎

Theorem 3. *Let conditions (H) hold,* $-1 \le p(t) \le 0$, $\sigma \ge t$ *and let*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} q(s) \max_{[s-\sigma,s]} \{-p(u)\} ds > \frac{1}{e} \tag{12}$$

Then each solution $x(t)$ *of equation (1) oscillates.*

Proof. Suppose that (1) has a nonoscillating solution $x(t)$ and let $x(t) > 0$ eventually. From Lemma 3 it follows that $z(t)$ is an eventually decreasing positive function. Then

$$z(t) < x(t) \quad \text{and} \quad \max_{[t-\sigma,t]} z(s) < \max_{[t-\sigma,t]} x(s)$$

From (3) and from the last inequality we obtain that

$$z'(t) + q(t) \max_{[t-\sigma,t]} z(s) \le 0$$

Since $z(t)$ is an eventually decreasing function, then

$$\max_{[t-\sigma,t]} z(s) = z(t-\sigma)$$

Consequently, the eventually positive function $z(t)$ satisfies the inequality

$$z'(t) + q(t)z(t-\sigma) < 0 \tag{13}$$

Since $\max_{[t-\sigma,t]} \{-p(u)\} \le 1$, then from (12) and from Lemma 1 it follows that inequality (13) has no eventually positive solutions. The contradiction obtained shows that equation (1) has no eventually positive solutions.

We shall show that it has no eventually negative solutions either. Suppose that this is not true. Let $x(t) < 0$ eventually. From Lemma 3 it follows that $z(t)$ is an eventually negative increasing function. From the inequality $z(t) < 0$ eventually and from the definition of $z(t)$ there follow the inequalities

$$x(t) < -p(t)x(t-\tau) < -p(t)z(t-\tau)$$

$$\max_{[t-\sigma,t]} x(s) < \max_{[t-\sigma,t]}\{-p(s)z(s-\tau)\} \leq \max_{[t-\sigma,t]}\{-p(s)\}z(t-\tau)$$

From (3) and from the last inequality we obtain that the eventually negative function $z(t)$ satisfies the inequality

$$z'(t) + q(t)\max_{[t-\sigma,t]}\{-p(s)\}z(t-\tau) > 0 \tag{14}$$

From (12) and from Lemma 1 it follows that inequality (14) has no eventually negative solutions. Hence (1) has no eventually negative solutions and since in view of what was proved above the equation (1) has no eventually positive solutions either, then each solution of (1) oscillates. ∎

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