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Smooth and Analytic Solutions for Analytic Linear Systems

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ABSTRACT. We give some approximation theorems in the Whitney topology for a general class of analytic fibre bundles. This leads to a classification theorem which generalizes the classical ones.

INTRODUCTION

Approximation theorems have been a fundamental tool to prove relevant results in real geometry, as, for instance, Nash conjecture ([T5]) and classification theorems for real analytic bundles ([T1], [T2], [T3]). They assume a particularly expressive form in the case of vector bundles.

In this paper we give approximation theorems for sections of a more general class of vector bundles over a coherent real analytic space: namely vector bundles that, in general, are not locally trivial.

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In fact the results are obtained for coherent sheaves and we come back to bundles via duality theory (§6).

First we define the Whitney topology on the set of sections of a coherent sheaf $\mathcal{F}$ (§1); then (§2 and §3) that is dense in the set of sections of the sheaf $\mathcal{F} \otimes \mathcal{E}_X$ is the sheaf of germs of smooth functions.

As an application we get approximation for smooth solutions of analytic linear systems: more precisely we prove that if an analytic linear system $\sum a_{hk}(x)y_k = g_h(x)$, defined on an open set $U \subset \mathbb{R}^n$, admits a C^∞ solution φ, then in any neighbourhood B_φ of φ in the Whitney topology of $C^\infty(U)^q$ there exists an analytic solution of the system.

Approximation theorems can be stated also for sheaf homomorphisms. We prove that the set of isomorphisms between two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ is open in $\mathcal{H}om\ (\mathcal{F},\mathcal{G})$, so, again by duality, we get a Grauert-like theorem for generalized vector bundles.

Finally in §5 we consider the same problems in the algebraic context and we obtain similar results with some obvious modifications.

1. THE WHITNEY TOPOLOGY FOR SECTIONS OF A SHEAF

Let X be a paracompact, locally compact space and $\mathcal{F} = \{\mathcal{F}_U, r_V^U\}$ be a sheaf of real vector spaces; so, for any $A \subset X$, the set $\Gamma(A,\mathcal{F})$ has a structure of real vector space and the restriction maps are linear.

Definition 1.1. *A local system of seminorms $\mathcal{L}$ in $\mathcal{F}$ is given by the following data:*

(1) A locally finite open covering $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ of X by relatively compact open sets.

(2) For any compact set $K \subset U_\lambda$, for any open neighbourhood U of K and any natural number p, a seminorm $|| \ ||^p_{K,\lambda}$ (depending on λ) defined on $\Gamma(U,\mathcal{F})$ with the following properties:

a) If $\gamma_1 \in \Gamma(U_1,\mathcal{F})$, $\gamma_2 \in \Gamma(U_2,\mathcal{F})$ and $r_U^{U_1}\gamma_1 = r_U^{U_2}\gamma_2$ for an open neighbourhood U of K, then for any p

$$||\gamma_1||^p_{K,\lambda} = ||\gamma_2||^p_{K,\lambda} = ||r_U^{U_1}\gamma_1||^p_{K,\lambda}$$

b) If $K \subset U_\lambda \cap U_{\lambda'}$, for each integer p there exist two positive numbers α and β such that for each $\gamma \in \Gamma(U,\mathcal{F})$

$$\alpha||\gamma||^p_{K,\lambda} \leq ||\gamma||^p_{K,\lambda'} \leq \beta||\gamma||^p_{K,\lambda}$$

c) If $K = \bigcup_{i=1}^{n} K_i$ is a decomposition of K as finite union of compact sets, then for each $\gamma \in \Gamma(U,\mathcal{F})$ and each p

$$||\gamma||^p_{K,\lambda} = \sup_{i=1,\ldots,n} ||\gamma||^p_{K_i,\lambda}$$

In particular if $K \subset K' \subset U_\lambda$ then $||\gamma||^p_{K,\lambda} \leq ||\gamma||^p_{K',\lambda}$

d) If $U \supset \bar{U}_\lambda$ then $\sup_{\substack{K \subset U_\lambda \\ K \text{ compact}}} ||\gamma||^p_{K,\lambda} < \infty$, for any $\gamma \in \Gamma(U,\mathcal{F})$.

Let now K be any compact set in X and suppose $K \cap U_\lambda = \emptyset$ if λ is different from $\lambda_1, \ldots, \lambda_q$.

Definition 1.2. *For any $\gamma \in \Gamma(U,\mathcal{F})$ with $K \subset U$ we define*

$$||\gamma||^p_K = \sup_{i=1,\ldots,q} \sup_{\substack{H \subset U_{\lambda_i} \cap K \\ H \text{ compact}}} ||\gamma||^p_{H,\lambda_i}$$

Property d) of Definition 1.1 ensures that $||\gamma||^p_K < \infty$.

Definition 1.3. *Let $U \subset X$ be an open set. The weak topology defined by the local system of seminorms $\mathcal{L}$ for $\Gamma(U,\mathcal{F})$ is the topology having the family*

$$\mathcal{U}^{\mathcal{L}}_{K,p,\varepsilon} = \{\gamma \in \Gamma(U,\mathcal{F}) \mid ||\gamma||^p_K < \varepsilon,\ K \text{ compact set},\ K \subset U\}$$

as a fundamental system of neighbourhoods of 0.

Remark 1.4. The restriction maps r^U_V are continuous with respect to the weak topology.

Now we are ready to define the Withney topology, as usual, as a limit of the weak topology.

Consider a local system $\mathcal{L}$ of seminorms on the sheaf $\mathcal{F}$. Let $U \subset X$ be an open set.
Take:

(1) An exhaustive sequence of compact sets

$$\mathcal{K} = \{K_i\}_{i\in\mathbb{N}} \quad K_i \subset \overset{\circ}{K}_{i+1} \quad \bigcup K_i = U,$$

(2) A sequence $\mathcal{M} = \{m_i\}_{i\in\mathbb{N}}$ of natural numbers,

(3) A sequence $\mathcal{E} = \{\varepsilon_i\}_{i\in\mathbb{N}}$ of positive numbers.

Then:

Definition 1.5. *A fundamental system of neighbourhoods of* $0 \in \Gamma(U,\mathcal{F})$ *for the Whitney topology on* $\Gamma(U,\mathcal{F})$ *is given by the sets*

$$\mathcal{U}^{\mathcal{L}}_{\mathcal{K},\mathcal{M},\varepsilon} = \{\gamma \in \Gamma(U,\mathcal{F}) \mid \forall n \sup_{p\le m_n} ||\gamma||^p_{K_n - \overset{\circ}{K}_{n-1}} < \varepsilon_n\}$$

Remarks 1.6.

1. The weak topology can be given by a countable family of seminorms, namely $||\ ||^{p_n}_{K_n}$ for any exhaustive sequence of compact sets. Hence the weak topology is induced by a metric. This is not true for the Whitney topology because the family $\mathcal{U}^{\mathcal{U}}_{\mathcal{K},\mathcal{M},\varepsilon}$ is not countable and does not have any countable cofinal subfamily.
2. If X is compact then the weak and the Whitney topologies coincide.

Definition 1.7. *Two local systems of seminorms over* X

$$\mathcal{L} = \{\{U_\lambda\}, ||\ ||^p_{K,\lambda}\} \text{ and } \mathcal{L}' = \{\{U'_{\lambda'}\}, ||\ ||^p_{K,\lambda'}\}$$

are said to be equivalent if for each compact $K \subset X$ *and any* p *there exist two positive numbers* α, β *such that*

$$\alpha(||\gamma||^p_K)_{\mathcal{L}} \le (||\gamma||^p_K)_{\mathcal{L}'} \le \beta(||\gamma||^p_K)_{\mathcal{L}}$$

for each $\gamma \in \Gamma(U,\mathcal{F})$, *with* $K \subset U$.

Lemma 1.8. *If $\mathcal{L}$ and $\mathcal{L}'$ are equivalent, they induce the same weak topology and the same Whitney topology on $\Gamma(U,\mathcal{F})$, for any open set $U \subset X$.*

Proof. It is clear by the definitions. ∎

In the following we shall omit the restriction maps when there is no risk of confusion.

Examples.

(1) If $U \subset \mathbb{R}^n$ is an open set, we have the classical seminorms for functions in $C^\infty(U)$ or in $C^\omega(U)$

$$||f(x_1,\ldots,x_n)||_K^p = \sup\left[\sup_K |f(x)|, \sup_{\substack{K \\ j_1+\ldots+j_n=j\leq p}} \left|\frac{\partial^j f(x_1,\ldots,x_n)}{\partial x_1^{j_1},\ldots,\partial x_n^{j_n}}\right|\right]$$

which give to $C^\infty(U)$ and $C^\omega(U)$ the usual compact open topology (or weak topology) and Whitney (or strong) topology.

(2) Let $(X,\mathcal{O}_X)$ be a reduced real coherent analytic space. We can find a locally finite open covering $\{U_\lambda\}$ of X such that for each λ there exists an isomorphism $j_\lambda : U_\lambda \to X_\lambda$, where X_λ is a closed real analytic subset of an open set Ω_λ in $\mathbb{R}^{n_\lambda}$. The isomorphism j_λ induces a surjective map $\pi_\lambda : C^\omega(\Omega_\lambda) \to \Gamma(U_\lambda,\mathcal{O}_X)$ which is the composition of j_λ^{-1} with the quotient map. So for each $K \subset U_\lambda$ and each $f \in \Gamma(U_\lambda,\mathcal{O}_X)$ we can define

$$||f||_K^p = \inf_{g\in\pi_\lambda^{-1}(f)} ||g||_{j_\lambda(K)}^p$$

By this local system of seminorms we can define the weak and the strong topology on $\Gamma(U,\mathcal{O}_X)$ for any open set $U \subset X$.

If X is not coherent we can extend any analytic function on a local model to a C^∞ function on Ω_λ and then use C^∞ seminorms.

(3) Let $(X,\mathcal{O}_X)$ be a reduced complex analytic space and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then we can find an open covering $\{U_\lambda\}$ of X by holomorphically convex open sets and for each λ a resolution of $\mathcal{F}$ on U_λ

$$\mathcal{O}^p_{U_\lambda} \to \mathcal{O}^q_{U_\lambda} \to \mathcal{F}_{U_\lambda} \to 0.$$

It induces a surjective map

$$\beta : \ \Gamma(U, \mathcal{O}^q) \to \Gamma(U, \mathcal{F}) \to 0$$

for each open Stein set $U \subset U_\lambda$. Hence for any compact set $K \subset U$ we can define

$$||\gamma||_K = \inf_{\substack{\tilde{\gamma}=(\gamma_1,\ldots,\gamma_q)\\ \beta(\tilde{\gamma})=\gamma}} \left\{ \sup_K (|\gamma_1| + \ldots + |\gamma_q|) \right\}$$

With this local system of seminorms the weak topology gives to $\Gamma(U, \mathcal{F})$ a structure of Frechét space (see [G.R] Chap. VII).

(4) Let now $(X, \mathcal{O}_X)$ be a reduced coherent real analytic space and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. We can take the same definition as before; namely, if $\{U_\lambda\}$ is an open covering of X such that on each U_λ we have a resolution of $\mathcal{F}$, we can define for $K \subset U_\lambda$ compact, $p \in \mathbb{N}$ and $\gamma \in \Gamma(K, \mathcal{F})$

$$||\gamma||^p_K = \inf_{\substack{\tilde{\gamma}=(\gamma_1,\cdots,\gamma_q)\\ \beta(\tilde{\gamma})=\gamma}} (||\gamma_1||^p_K + \cdots + ||\gamma_q||^p_K)$$

(5) In the same situation as (4) we can define

$$\mathcal{F}^\infty = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_X$$

where $\mathcal{E}_X$ is the sheaf of germs of C^∞-functions on X *. Since $\mathcal{F}$ is coherent, the stalk $\mathcal{F}_x$ is generated by a finite number of global

* A map $\varphi : X \to \mathbb{R}$ is C^∞ if for any $x \in X$ there exist a neighbourhood W_x of x, an embedding $W_x \to \mathbb{R}^n$ as a locally closed analytic set and $\varphi_{|W_x}$ extends to a smooth function on some neighbourhood of W_x in $\mathbb{R}^n$. If $\mathcal{E}_\varphi$ is the set of such extensions we can define, for a compact set $K \subset W_x$,

$$||\varphi||^p_K = \inf_{\bar{\varphi} \in \mathcal{E}_\varphi} ||\bar{\varphi}||^p_K$$

sections (theorem A). Hence we can construct an open covering $\mathcal{U} = \{U_\lambda\}$ and for each λ we can find $f_1^\lambda, \dots, f_{q(\lambda)}^\lambda$ in $\Gamma(X, \mathcal{F})$ such that they generate $\mathcal{F}_x$ as $\mathcal{O}_{X,x}$-module for each $x \in U_\lambda$. Let γ be an element in $\Gamma(U_\lambda, \mathcal{F}^\infty)$. Then we can write (not in a unique way) $\gamma = \sum_{i=1}^{q(\lambda)} \alpha_i f_i^\lambda$ with $\alpha_i \in C^\infty(U_\lambda)$. In fact this can be done locally by definition of $\mathcal{F}^\infty$ and then can be globalized by using a C^∞ partition of unity. For $K \subset U_\lambda$ and $p \in \mathbb{N}$ we can define

$$||\gamma||_{K,\lambda}^p = \inf_{\alpha_1,\cdots,\alpha_{q(\lambda)}} \left(\sum_{i=1}^{q(\lambda)} ||\alpha_i||_K^p \right)$$

(The inf is taken on all the system of coefficients $\alpha_1, \dots, \alpha_{q(\lambda)}$ describing γ with respect to the chosen generators $f_1, \dots, f_{q(\lambda)}$). This is a local system of seminorms: we shall always use this one to define the weak and the Whitney topology on $\Gamma(U, \mathcal{F}^\infty)$, if we do not specify any more.

Remark 1.9. The morphisms between coherent sheaves of $\mathcal{O}_X$-modules induce continuous maps between the spaces of sections, (see [GR] for the complex case: the same proof works in the real one).

2. A WHITNEY APPROXIMATION THEOREM

This section is devoted to the proof of a Whitney - like approximation theorem for smooth functions defined on a real analytic space X.

If X is coherent we shall get in the next section a similar result for sections of any coherent sheaf of $\mathcal{O}_X$-moduls.

Our proof is similar to the classical one that can be found in [W], [N], [T6], [T8]. Under the hypothesis: X is an analytic submanifold of $\mathbb{R}^n$ and $\mathcal{F}$ a subsheaf of $\mathcal{O}$. Theorem 2.9 is proved in [BKS], where the Whitney topology is called *Very Strong Topology.*

We shall use the following standard notations for $x \in \mathbb{R}^n$ (or $\mathbb{C}^n$), φ a C^∞ function on $\mathbb{R}^n$ (or $\mathbb{C}^n$), $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$ in $\mathbb{N}^n$:

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \qquad \alpha! = \alpha_1! \cdots \alpha_n!$$
$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} \quad (\text{if } \beta_j \le \alpha_j \text{ for } j = 1, \ldots, n)$$
$$|x| = \max_j |x_j| \qquad ||x|| = \Big(\sum_j |x_j|^2\Big)^{\frac{1}{2}}$$
$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \qquad D^\alpha \varphi = \frac{\partial^{\alpha_1+\cdots+\alpha_n}\varphi}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

Let $(X, \mathcal{O}_X)$ be a real analytic space, not necessarily coherent; we suppose that $(X, \mathcal{O}_X)$ is the real part of a reduced complex analytic space $(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

This means that there exists a complex analytic space $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ which is defined over $\mathbb{R}$, and an antiinvolution $\sigma : \tilde{X} \to \tilde{X}$ such that $(X, \mathcal{O}_X)$ is isomorphic to the real analytic space $X' = \{x \in \tilde{X} : \sigma(x) = x\}$ endowed with the structure sheaf $\mathcal{O}_X$ consisting of all σ-invariant germs.

In this situation X has in $\tilde{X}$ an invariant neighbourhood $U = \sigma(U)$ which is a Stein space. So, in the following, we shall assume that $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is a reduced Stein space defined over $\mathbb{R}$. In the case when the real part is coherent we can assume that $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is its complexification (see [T10] and [T11]).

Consider three compact sets

$$H_1 \subset \overset{\circ}{H}_2 \subset H_2 \subset \overset{\circ}{H}_3 \subset H_3 \subset X$$

Definition 2.1. *A complex neighbourhood $\tilde{U}_1$ of H_1 is called a vertical neighbourhood, relatively to H_2, H_3, if for any C^∞ function $\varphi : X \to \mathbb{R}$ such that supp $\varphi \subset H_3$, $\varphi|_{H_2} \equiv 0$, and any $\varepsilon > 0$ and $p \in \mathbb{N}$, there exists an analytic function g on X such that:*

(1) $||g - \varphi||^p_{H_3} < \varepsilon$

(2) g is the restriction of a holomorphic function $G : \tilde{X} \to \mathbb{C}$ such that $|G(z)| < \varepsilon$ for $z \in \tilde{U}_1$

Remark 2.2. In the above situation, if $H_1 \subset \tilde{U}_1' \subset \tilde{U}_1$, $\tilde{U}_1'$ is also a vertical neighbourbood.

Lemma 2.3. *Let* $(\tilde{X}, \mathcal{O}_{\tilde{X}})$, H_i, $i = 1,2,3$. $\tilde{U}_1$ *as before. Let* $\tilde{f} : (\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \to (\tilde{X}, \mathcal{O}_{\tilde{X}})$, *be a complex analytic map. Assume* $\tilde{f}$ *and* $(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ *are defined over* $\mathbb{R}$, $(Y, \mathcal{O}_Y)$ *is the the real part of* $(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ *and consider* $f = \tilde{f}|_Y$. *Assume there is an open subset* $Y' \subset Y$ *such that* f *defines an isomorphism between* Y' *and a closed analytic subset* $X' = f(Y')$ *of an open set* $W \supset H_3$ *of* X. *Define* $H_i' = f^{-1}(H_i \cap X')$ *for* $i = 1,2,3$. *Then* $\tilde{f}^{-1}(\tilde{U}_1)$ *is a vertical neighbourhood of* H_1' *relative to* H_2', H_3'.

Proof. Let $\varphi : Y \to \mathbb{R}$ be a C^∞ function such that supp $\varphi \subset H_3'$, and $\varphi_{|H_2'} \equiv 0$. Clearly the function $\psi = \varphi \circ f^{-1} : X' \to \mathbb{R}$ can be extended to a C^∞ function on X (denoted also by ψ) such that supp $\psi \subset H_3$ and $\psi|_{H_2} \equiv 0$. If $G : X \to \mathbb{R}$ is an analytic approximation of ψ and its holomorphic extension $\tilde{G}$ is small on $\tilde{U}_1$, then $\tilde{G} \circ \tilde{f}$ approximates φ and is "small" on $\tilde{f}^{-1}(\tilde{U}_1)$. ■

Now we define vertical neighbourhoods for $\mathbb{R}^n$, considered as the real part of $\mathbb{C}^n$.

Lemma 2.4. *Let* Ω *be an open set in* $\mathbb{R}^n$. *Let* H_i, $i = 1,2,3$, *be three compact subsets of* Ω *such that* $H_i \subset \overset{\circ}{H}_{i+1}$ *for* $i = 1,2$. *Define* $\delta = d(H_1, \Omega - H_2)$. *Then for any* $\alpha \in (0,1)$ *the set:*

$$\tilde{U}_\alpha = \{z \in \mathbb{C}^n : \textit{ for any } y \in \mathbb{R}^n - H_2,\ |\mathcal{R}(z - y)| > \alpha\delta\}$$

is a vertical neighbourhood of H_1 *relative to* H_2, H_3, *(where* $\mathcal{R}(\ )$ *means the real part of* $(\ )$*).*

Proof. Let $\varphi : \Omega \to \mathbb{R}$ be a C^∞ function such that $\varphi_{|H_2} \equiv 0$ and supp $\varphi \subset H_3$. For any $\lambda \in (0, +\infty)$, we define

$$I_\lambda(\varphi)(x) = c\lambda^{\frac{1}{2}n} \int_{\mathbb{R}^n} \varphi(y) \exp\ \{-\lambda ||x - y||^2\} dy \tag{1}$$

where $c \cdot \int_{\mathbb{R}^n} \exp\ (-||x^2||) dx = 1$, that is $c = \pi^{-\frac{1}{2}n}$.

We have

$$I_\lambda(\varphi)(x) = c\lambda^{\frac{1}{2}n} \int_{\mathbb{R}^n} \varphi(x - y) \exp\{-\lambda ||y||^2\} dy \tag{2}$$

and hence, for any $\alpha \in \mathbb{N}^n$

$$D^\alpha(I_\lambda(\varphi))(x) = c\lambda^{\frac{1}{2}n} \int_{\mathbf{R}^n} (D^\alpha\varphi)(x-y)\exp\{-\lambda||y||^2\}dy =$$

$$= c\lambda^{\frac{1}{2}n} \int_{\mathbf{R}^n} (D^\alpha\varphi)(y)\exp\{-\lambda||x-y||^2 dy\}$$

From (2) and (3) we deduce:

$$D^\alpha(I_\lambda(\varphi))(x) - D^\alpha(\varphi)(x) = \\ c\lambda^{\frac{1}{2}n} \int_{\mathbf{R}^n} ((D^\alpha\varphi)(y) - (D^\alpha\varphi)(x))\exp\{-\lambda||x-y||^2\}dy \tag{4}$$

and

$$|D^\alpha(I_\lambda(\varphi))(x) - D^\alpha(\varphi)(x)| = \\ = \left| c\lambda^{\frac{1}{2}n} \int_{||x-y||<\delta} ((D^\alpha\varphi)(y) - (D^\alpha\varphi)(x)) \exp\{-\lambda||x-y||^2\}dy + \right. \\ \left. + c\lambda^{\frac{1}{2}n} \int_{||x-y||\geq\delta} ((D^\alpha\varphi)(y) - (D^\alpha\varphi)(x))\exp\{-\lambda||x-y||^2\}dy \right| \tag{5}$$

Relation (5) proves that for any $p \in \mathbb{N}$ we have

$$\lim_{\lambda\to\infty} ||I_\lambda(\varphi) - \varphi||^p_{H_3} = 0 \tag{6}$$

In fact for any $\varepsilon > 0$ we may suppose δ small enough to ensure that the first integral in (5) is less than $\frac{\varepsilon}{2}$. (We use the fact that φ has compact support and hence $D^\alpha\varphi$ is uniformely continuous).

Given the positive number δ, we may find $\lambda \in \mathbb{R}$ big enough to ensure that the second integral in (5) has absolute value less than $\frac{\varepsilon}{2}$ (because of the nature of the "bump function" $\exp\{-\lambda\|x-y\|^2\}$).

So $I_\lambda(\varphi)$ approximates φ in the compact-open topology.

Coming back to the definition of $I_\lambda(\varphi)$, we remark that the variable x occours only in $\exp\{-\lambda\|x-y\|^2\}$ which is holomorphic on $\mathbb{C}^n$. Moreover φ has compact support, hence we deduce that the function

$$I_\lambda(\varphi)(z) = c\lambda^{\frac{1}{2}n} \int_{\text{supp } \varphi} \varphi(y)\exp\{-\lambda\|z-y\|^2\}dy \tag{7}$$

is holomorphic for any $z \in \mathbb{C}^n$, and in particular is analytic on Ω.

To complete the proof it is enough to verify the following: if $\varphi_{|H_2} \equiv 0$, then for any $\varepsilon > 0$, $\alpha \in (0,1)$ there exists a λ_0 such that if $\lambda > \lambda_0$, we have:

$$|(I_\lambda(\varphi))(z)| < \varepsilon \tag{8}$$

for any $z \in \tilde{U}_\alpha$

Fix $\alpha \in (0,1)$: there exists $\sigma > 0$ such that for $z \in \tilde{U}_\alpha$, if $d((\mathcal{R}(z), H_1) < \sigma$, then $\varphi(\mathcal{R}(z)) = 0$.

This implies that for $x \in \tilde{U}_\alpha \cap \mathbb{R}^n$, we can evaluate $(I_\lambda(\varphi))(x)$ by the formula

$$(I_\lambda(\varphi))(x) = c\lambda^{\frac{1}{2}n} \int_{\text{supp } \varphi\cap\{\|x-y\|>\sigma\}} \varphi(y)\exp\{-\lambda\|x-y\|^2\}dy \tag{9}$$

and, as remarked before, for any ε, if λ is big enough, then $|(I_\lambda\varphi)(x)| < \varepsilon$.

Finally note that, since in (7) the variable z occours only in an exponential function, only its real part is significant for the norm of $I_\lambda(\varphi)(x)$; we use here the fact that $|e^{a+ib}| = |e^a|$).

The last remark ensures that the inequality $|I_\lambda(\varphi)(z)| < \varepsilon$ holds for any $z \in \tilde{U}_\alpha$ and this completes the proof. ∎

Now we generalize Lemma 2.4 to real analytic subsets of $\mathbb{R}^n$.

Let $X \subset \Omega \subset \mathbb{R}^n$ be a real analytic set in the open set Ω of $\mathbb{R}^n$. We shall suppose X to be the real part of a complex space $\tilde{X} \subset \tilde{\Omega} \subset \mathbb{C}^n$ defined over $\mathbb{R}$.

Under these hypothesis we have:

Lemma 2.5. *Let H_i, $i = 1,2,3$ be three compact subsets of X such that $H_i \subset \overset{\circ}{H}_{i+1}, H_i = \overline{\overset{\circ}{H}_i}$, $\delta = d(H_1, X - H_2)$, $\alpha \in (0,1)$ and*

$$\tilde{U}_\alpha = \{x \in \tilde{X} : \ |\mathcal{R}(x - w)| > \alpha\delta \text{ for any } w \in X - H_2\}$$

Then, for any $\alpha \in (0,1)$, $\tilde{U}_\alpha$ is a vertical neighbourhood of H_1, relatively to H_2, H_3.

Proof. For $i = 1,2,3$, let us define:

$A_i = \{ x \in \mathbb{R}^n - (X - \overset{\circ}{H}_i) :$ there exists $y \in H_i$ such that $d(x,y) < d(y, \partial H_i)\}$,

where d is the usual metric in $\mathbb{R}^n$. It is easy to verify that:

(1) A_i is open and $A_i \cap X = \overset{\circ}{H}_i$.

In fact A_i is union of the balls $B(y, \rho_y)$ with radius $\rho_y = d(y, \partial H_i)$ for $y \in \overset{\circ}{H}_i$. The condition $A_i \cap X = \overset{\circ}{H}_i$ follows from the definition of A_i.

(2) $\overline{A_i}$ is compact and $\overline{A_i} \cap X = H_i$.

This equality is an easy consequence of the hypotesis $H_i = \overline{\overset{\circ}{H}_i}$. The compactness follows from the fact that $\overline{A_i}$ is closed and bounded.

(3) $\overline{A_i} \subset A_{i+1}$, $i = 1,2$.

This inclusion is a consequence of the hypothesis $H_i \subset \overset{\circ}{H}_{i+1}$; it implies that $d(y, \partial H_i) < d(y, \partial H_{i+1})$ if $y \in \overset{\circ}{H}_i$.

Let now $\varphi : X \to \mathbb{R}$ be a C^∞ function such that supp $\varphi \subset H_3$, $\varphi|_{H_2} \equiv 0$ and fix $\alpha \in (0,1)$. We claim that there exists a C^∞ extension $\Phi : \mathbb{R}^n \to \mathbb{R}$ of φ such that supp $\Phi \subset A_3$, and $\Phi|_{A_2} \equiv 0$.

The existence of such a Φ can be proved using a partion of unity or as a particular case of the Whitney extension theorem (see [W]).

Take $\alpha' \in (0,1)$ and let δ' be the distance $d(\partial A_1, A_2)$; from Lemma 2.4 we get that the set:

$$\tilde{A}_1^{\alpha'} = \{x \in \mathbb{C}^n : |\mathcal{R}(x-y) > \alpha'\delta' \text{ for any } y \in \mathbb{R}^n - \overline{A_2}\}$$

is a vertical neighbourhood of $\overline{A_1}$ relatively to $\overline{A_2}, \overline{A_3}$.

This implies that for any $\varepsilon > 0$, $p \in \mathbb{N}$ and $\alpha' \in (0,1)$ there exists an analytic approximation $G : \mathbb{R}^n \to \mathbb{R}$ such that:

a) $||G - \Phi||^p_{\bar{A}_3} < \varepsilon$

b) G is the restriction of a holomorphic function $\tilde{G} : \mathbb{C}^n \to \mathbb{C}$ such that $|\tilde{G}(z)| < \varepsilon$ if $z \in \tilde{A}_1^{\alpha'}$.

It is easy to verify that we can choose α' in such a way that $\tilde{U}_\alpha \subset \tilde{A}_1^{\alpha'}$ and hence $\tilde{G}|_{\tilde{X}}$ gives the approximation of φ : so $\tilde{U}_\alpha$ is a vertical neighbourhood for H_1, relative to H_2, H_3. ■

Definition 2.6. *If H_i, $i = 1,2,3$, are compact sets satisfying the conditions of lemma 2.5 the neighbourhoods $\tilde{U}_\alpha$ of H_1 defined above shall be called the canonical vertical neighbourhoods of H_1.*

Vertical neighbourhoods can be defined also for real analytic spaces which are not subsets of some $\mathbb{R}^n$.

Let $(X, \mathcal{O}_X)$ be a real analytic space and assume it is the real part of a Stein space $(\tilde{X}, \mathcal{O}_{\tilde{X}})$.

Let $\{K_n\}_{n\in N}$ be a sequence of compact sets in X with the following properties:

(1) $K_n = \overline{\mathring{K}_n}$ and $\mathring{K}_{n+1} \supset K_n$ for any n.

(2) $\bigcup_n K_n = X$.

For any $p \in \mathbb{N}$ consider the compact sets K_p, K_{p+1}, K_{p+4}; they shall play the same role as H_1, H_2, H_3 before. We wish to prove the following

Lemma 2.7. For any $p \in \mathbb{N}$ there exists a vertical neighbourhood $\tilde{K}_p$ of K_p in $\tilde{X}$, with respect to K_{p+1}, K_{p+4} in such a way that $\tilde{K}_{p+1} \supset \tilde{K}_p$ for any $p \in \mathbb{N}$.

Proof. From the general theory of analytic spaces we can easily deduce that for each $p \in \mathbb{N}$ there exists a holomorphic map $\tilde{f}_p : \tilde{X} \to \mathbb{C}^{n_p}$ such that

(1) $\tilde{f}_p$ is defined over $\mathbb{R}$,

(2) $\tilde{f}_p$ defines an isomorphism between an open neighbourhood $\tilde{U}_p$ of K_{p+4} in $\tilde{X}$ and a complex analytic subset $\tilde{V}_p = \tilde{f}_p(\tilde{U}_p)$ of an open set Ω_p of $\mathbb{C}^{n_p}$,

(3) for any p, $\tilde{U}_{p+1} \supset \tilde{U}_p$,

(4) $d(\tilde{f}_p(\partial K_p), \tilde{f}_p(\partial K_{p+1})) = \delta > 0$.

In fact, from [N] we know that for any compact set $H \subset \tilde{X}$ there exists a holomorphic map $g : \tilde{X} \to \mathbb{C}^n$ for some $n \in \mathbb{N}$, which is an isomorphism onto its image when restricted to a suitable open neighbourhood of H.

If $H \subset X$, it is easy to verify that g can be chosen defined over $\mathbb{R}$, Finally condition (4) is obtained by multiplying $\tilde{f}_p$ by a suitable positive constant.

Define now $\tilde{g}_p = (\tilde{f}_1, \dots, \tilde{f}_q) : \tilde{X} \to \mathbb{C}^{n_1+\cdots+n_q}$. Denote by $\tilde{U}^1_\alpha$ the canonical vertical neighbourhoods in $\tilde{V}_1 = \tilde{f}_1(\tilde{U}_1)$ of $H_1 = \tilde{f}_1(K_1)$ with respect to $H_2 = \tilde{f}_1(K_2)$ and $H_3 = \tilde{f}_1(K_5)$ and define $\tilde{K}^\alpha_1 = \tilde{f}_1^{-1}(\tilde{U}^1_\alpha)$ using Lemma 2.3. it is easy to see that, for any $\alpha \in (0,1)$, $\tilde{K}^\alpha_1$ is a vertical neighbourhood of K_1 in $\tilde{X}$ with respect to K_2, K_5.

Now we define the vertical neighbourhoods of K_2 relative to K_3, K_6.

Consider the map $\tilde{g}_2 = (\tilde{f}_1, \tilde{f}_2) : \tilde{X} \to \mathbb{C}^{n_1+n_2}$; $\tilde{g}_2$ is an isomorphism on a neighbourhood of K_6 in $\tilde{X}$, $\tilde{U}_2 \supset \tilde{U}_1$ and $\tilde{V}_2 = \tilde{g}_2(\tilde{U}_2)$ is a complex analytic subset of an open set Ω_2 of $\mathbb{C}^{n_1+n_2}$. Take the compact sets $H^2_2 = g_2(K_2)$, $H^2_3 = g_2(K_3)$, $H^2_6 = g_2(K_6)$ and let $\tilde{U}^2_\alpha$ be the canonical vertical neighbourhoods of H^2_2 with respect to H^2_3, H^2_6 in $\tilde{V}_2 = \tilde{g}_2(\tilde{U}_2)$.

Define $\tilde{K}^\alpha_2 = \tilde{g}_2^{-1}(\tilde{U}^2_\alpha)$: as we remarked before, for any $\alpha \in (0,1)$, $\tilde{K}^\alpha_2$ is a vertical neighbourhood of K_2, with respect to K_3, K_6, in $\tilde{X}$.

In a similar way we define the $\tilde{K}^\alpha_3$, $\tilde{K}^\alpha_4, \dots$ as vertical neighbourhoods of $K_3, K_4, \dots$. To complete the proof we have to verify that

$$\tilde{K}^\alpha_{p+1} \supset \tilde{K}^\alpha_p. \qquad (*)$$

It is enough to verity (*) for $p = 1$, the general case follows from the same argument. Recall that:

$$\tilde{U}_1^\alpha = \{x \in \tilde{f}_1(\tilde{U}_1) : |\mathcal{R}(x - w)| > \alpha\delta \text{ for any } w \in \tilde{f}_1(\tilde{U}_1 \cap (X - K_2))\}$$

$$\tilde{U}_2^\alpha = \{x \in \tilde{g}_2(\tilde{U}_2) : |\mathcal{R}(x - w)| > \alpha\delta \text{ for any } w \in \tilde{g}_2(\tilde{U}_2 \cap (X - K_3))\}$$

and $\tilde{K}_1^\alpha = \tilde{f}_1^{-1}(\tilde{U}_1^\alpha)$, $\tilde{K}_2^\alpha = \tilde{g}_2^{-1}(\tilde{U}_2^\alpha)$.

We can easily remark the following:

a) if $x, y \in \tilde{X}$ then $d(\tilde{f}_1(x), \tilde{f}_1(y)) \leq d(\tilde{g}_2(x), \tilde{g}_2(y))$ and $|\mathcal{R}(\tilde{f}_1(x) - \tilde{f}_1(y))| \leq |\mathcal{R}(\tilde{g}_2(x) - \tilde{g}_2(y))|$.

b) if $x \in K_2$ then $d(\tilde{g}_2(x), \tilde{g}_2(\partial K_2)) \leq d(\tilde{g}_2(x), \tilde{g}_2(\partial K_3))$.

From the hypothesis $\delta = d(\tilde{f}_1(\partial K_1), \tilde{f}_2(\partial K_2)) = d(\tilde{f}_2(\partial K_2), \tilde{(}\partial K_3))$ and these two remarks, (*) follows. ∎

Note that Lemma 2.7 is trivial for $X = \mathbb{R}^n$: in this case we have the canonical vertical neighbourhoods.

Definition 2.8. *A sequence $\{\tilde{K}_p\}_{p \in \mathrm{N}}$ of neighbourhoods of compact sets as in Lemma 2.7 shall be called a consistent sequence of vertical neighbourhoods.*

Theorem 2.9. *Let $(X, \mathcal{O}_X)$ be the real part of a complex space $(\tilde{X}, \mathcal{O}_{\tilde{X}})$. Denote by $\mathcal{E}_X$ the sheaf of germs of C^∞ functions on X. Then, for any open set $U \subset X$, $\Gamma(U, \mathcal{O}_X)$ is dense in $\Gamma(U, \mathcal{E}_X)$ for the Whitney topology and hence for the weak topology.*

Proof. Consider a sequence $\{K_n\}_{n \in \mathrm{N}}$ of compact sets in X such that, $\overset{\circ}{K}_{n+1} \supset K_n$, $\bigcup_n K_n = X$, $K_n = \overline{\overset{\circ}{K}_n}$.

Let $\{\tilde{K}_n\}_{n \in \mathrm{N}}$ be a consistent sequence of vertical neigbourhoods as defined in Lemma 2.7; we recall that for each p, $\tilde{K}_p$ is a vertical neighbourhood of K_p with respect to K_{p+1}, K_{p+4}.

By a partition of unity we may construct a sequence of C^∞ functions $\varphi_p : X \to \mathbb{R}$, $p \geq 0$, such that

a) supp $\varphi_p \subset K_{p+2}$

b) $\varphi_p(x) = 0$ in a neighbourhood of K_{p-1}

c) $\varphi_p(x) = 1$ in a neighbourhood of $L_p = K_{p+1} - \overset{\circ}{K}_p$.

Take a C^∞ function $\gamma : X \to \mathbb{R}$, choose a sequence $\{\varepsilon_p\}$ of positive numbers and a sequence $\{m_p\}$ of natural numbers: we have to find an analytic function $g : X \to \mathbb{R}$ such that for any $p \geq 0$ one has

$$||g - \gamma||_{L_p}^{m_p} < \varepsilon_p \tag{1}$$

We can assume $m_{p+1} \geq m_p$ for any $p \geq 0$. Define, for $p \geq 0$

$$M_{p+1} = 1 + ||\varphi_p||_{K_{p+2}}^{m_p} \tag{2}$$

and choose positive numbers δ_p in such a way that

$$2\delta_{p+1} \leq \delta_p \text{ and } \sum_{q \geq p} \delta_p M_{q+1} \leq \frac{1}{4}\varepsilon_p. \tag{3}$$

Now consider the compact sets

$$H_1 = H_2 = \emptyset,\ H_3 = K_2.$$

By definition of a vertical neighbourhood we can find a holomorphic function $g_0 : \tilde{X} \to \mathbb{C}$, defined over $\mathbb{R}$, such that

$$||g_0 - \varphi_0\gamma||_{K_1}^{m_0} = ||g_0 - \gamma||_{K_1}^{m_0} \leq ||g_0 - \varphi_0\gamma||_{K_2}^{m_0} < \delta_0.$$

Using the fact that $\tilde{K}_p$ is a vertical neighbourhood of K_p with respect to K_{p+1}, K_{p+4}, we may find inductively a sequence of holomorphic functions $\{g_p\}$ on $\tilde{X}$, defined over $\mathbb{R}$, such that:

$$||g_p - \varphi_p(\gamma - g_0 - g_1 - \cdots - g_{p-1})||_{K_{p+1}}^{m_p} < \delta_p \tag{4}$$

$$|g_p(z)| < \delta_p \text{ if } z \in \tilde{K}_{p-2} \tag{5}$$

(Condition (5) is empty for $p < 3$). From the conditions on φ_p we deduce

$$||\gamma - \sum_{i=0}^{p} g_i||_{L_p}^{m_p} < \delta_p \tag{6}$$

$$||g_p||_{K_{p-1}}^{m_p} < \delta_p \tag{7}$$

If in (4) we replace p by $p+1$, we obtain

$$||g_{p+1}||_{L_p}^{m_p} \leq ||\varphi_{p+1}(\gamma - \sum_{i=0}^{p} g_i)||_{L_p}^{m_p} + ||g_{p+1} - \varphi_{p+1}(\gamma - \sum_{i=0}^{p} g_i)||_{L_p}^{m_p}$$

$$\leq ||\varphi_{p+1}||_{K_{p+\delta}}^{m_p} \cdot ||\gamma - \sum_{i=1}^{p} g_i||_{L_p}^{m_p} + \delta_{p+1} \leq M_{p+1}\delta_p + \delta_{p+1} \tag{8}$$

But (7) applied to g_{p+1} gives

$$||g_{p+1}||_{K_p}^{m_p} < \delta_{p+1}, \tag{9}$$

since $m_p \leq m_{p+1}$. Finally, from (8) and (9), we deduce

$$||g_{p+1}||_{K_{p+1}}^{m_p} \leq M_{p+1}\delta_p + 2\delta_{p+1} \leq 2\delta_p M_{p+1}$$

and hence

$$||\sum_{i\geq p} g_i||_{K_{p+1}}^{m_p} \leq ||\sum_{i\geq p} g_i||_{K_{p+1}}^{m_p} \leq \sum_{i\geq p} ||g_i||_{K_{i+1}}^{m_i} \leq 2\sum_{i\geq p} \delta_i M_{i+1} < \frac{1}{2}\varepsilon_p. \tag{10}$$

Relations (6) and (10) prove that the series $\sum g_i$ converges on X to a C^∞ function g which approximates γ as wanted.

Moreover condition (5) proves that the series $\sum g_i$ in fact converges as a series of holomorphic functions on the union $\bigcup_p \tilde{K}_p \subset \tilde{X}$.

Since the space of holomorphic functions on a complex space is complete (see [GR]), the function $g = \sum g_i$ is the restriction of a holomorphic function, and hence it is analytic. The theorem is proved. ■

Remark 2.10. Theorem 2.9 holds when $(X, \mathcal{O}_X)$ is the real part of a complex space. No hypothesis on the coherence or on the dimensions of the Zariski tangent spaces are necessary.

3. APPROXIMATION FOR SECTIONS OF A SHEAF

In the following $(X, \mathcal{O}_X)$ will be a real coherent reduced analytic space; no hypothesis on the dimensions of the Zariski tangent spaces are required.

Theorem 3.1. *Let $(X, \mathcal{O}_X)$ be a coherent real analytic space, consider a coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules and denote by $\mathcal{F}^\infty$ the sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E}_X$. For any open set $U \subset X$, $\Gamma(U, \mathcal{F})$ is dense in $\Gamma(U, \mathcal{F}^\infty)$ for the Whitney topology, hence also for the weak topology.*

Proof. Let $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ be a complexification of $(X, \mathcal{O}_X)$. By a theorem of H. Cartan, there exists a neighbourhood (hence a Stein neighbourhood) of X in $\tilde{X}$ and a coherent sheaf $\tilde{\mathcal{F}}$ on it, such that $\tilde{\mathcal{F}}|_X = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\tilde{X}}$. (see [Ca] for the case $X \subset \mathbb{R}^n$; the proof is the same in the general case). So in the following we shall suppose $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ to be a Stein space and $\tilde{\mathcal{F}}$ defined over $\tilde{X}$.

Take a sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact sets such that $\overline{\mathring{K}_n} = K_n$, $K_n \subset \mathring{K}_{n+1}$, $\bigcup K_n = X$ and let $\{\tilde{K}_n\}$ be a consistent sequence of vertical neighbourhoods in $\tilde{X}$. More precisely we assume that for any p $\tilde{K}_p$ is a vertical neighbourhood of K_p with respect to K_{p+1}, K_{p+4}.

We know from Cartan's theorem A, that for any $p \in \mathbb{N}$ there exists a finite set of global sections of $\tilde{\mathcal{F}}$, say $\gamma_1^p, \ldots, \gamma_{n_p}^p$, such that they generate the stalk $\tilde{\mathcal{F}}_x$, for any x in an open neighbourhood $\tilde{D}_p$ of K_p in $\tilde{X}$.

We can assume $\gamma_i^p|_X \in \Gamma(X, \mathcal{F})$ for $1 = 1, \ldots, n_p$ and for any p.

Take now a C^∞ global section $\sigma \in \Gamma(X, \mathcal{F}^\infty)$. Using a suitable partition of unity, we can find C^∞ functions $\{\alpha_i^j\}_{i,j \in \mathbb{N}}$ on X such that

(1) for any j supp $\alpha_i^j \subset K_{j+2} - K_{j-1}$ for $i = 1, \ldots, n_j$,

(2) $\sigma(x) = \sum\limits_{j=1}^{\infty} \sum\limits_{i=1}^{n_j} \alpha_i^j(x) \gamma_i^j(x)$

(The family { supp α_i^j} is locally finite, so (2) makes sense). If we denote $\psi_j = \sum_{i=1}^{n_j} \alpha_i^j \gamma_i^j$ we can write $\sigma = \sum_{j=1}^{\infty} \psi_j$.

We have the following remarks.

(1) Take $\gamma_1, \dots, \gamma_q \in \Gamma(X, \mathcal{F})$ and a sequence of sections of $\mathcal{F}^\infty$ $\psi_n = \sum_{i=1}^{q} \alpha_i^n \gamma_i$. Then, by using norms on $\Gamma(X, \mathcal{F})$ as in exemple (4) of section 1, it is not difficult to prove that $\lim_{n\to\infty} ||\alpha_i^n - a_i||_K^m = 0$ implies $\lim_{n\to\infty} ||\psi_n - \sum_{i=1}^{q} a_i\gamma_i||_K^m = 0.$

(2) Remark (1) implies that it is possible to approximate on compact sets sections of $\mathcal{F}^\infty$ by sections of $\mathcal{F}$. Moreover if vertical neighbourhoods are defined and the original section has the required properties, the section of $\mathcal{F}$ extends to a section of $\tilde{\mathcal{F}}$ which has small norm on the corresponding vertical neighbourhoods.

(3) The space $\Gamma(\tilde{U}, \tilde{\mathcal{F}})$ is a complete space (Example (3) of Section 1).

Now we can repeat the proof of Theorem 2.9 almost word by word. It is enough to replace $\varphi_p(\gamma - g_0 - \cdots - g_{p-1})$ by the section ψ_p, and to use seminorms for sections instead of seminorms for functions. This proves the theorem. ■

Consider now a closed coherent subspace $Y \subset X$. The structural sheaf $\mathcal{O}_Y$ is defined by the exact sequence of coherent sheaves:

$$0 \to \mathcal{J}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

If $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules we can define the restriction of $\mathcal{F}$ to Y in the following way:

Definition 3.2. $\mathcal{F}|_Y = \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F}$

From the above exact sequence we deduce the exacteness of the sequence

$$\mathcal{J}_Y \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{F} \xrightarrow{r} \mathcal{F}|_Y \to 0$$

where r is the restriction map; its kernel is the image in $\mathcal{F}$ of the sheaf $\mathcal{J}_Y \otimes_{\mathcal{O}_X} \mathcal{F}$ (the sheaf of germs of sections vanishing at Y). In this situation we can give a sort of relative version of Theorem 3.1.

Let g be an element in $\Gamma(Y, \mathcal{F}|_Y)$. Denote by $\Gamma(X, \mathcal{F})_g$ the set of sections which extend g to X. This set is not empty because of Cartan's Theorem B. Denote in the same way by $\Gamma(X, \mathcal{F}^\infty)_g$ the extensions of g to X in $\mathcal{F}^\infty$. Then we have:

Theorem 3.3. *Let Y be a closed coherent analtic subset of the coherent analytic space X and $g \in \Gamma(Y, \mathcal{F}|_Y)$; then $\Gamma(X, \mathcal{F})_g$ is dense in $\Gamma(X, \mathcal{F}^\infty)_g$ for the Whitney topology.*

Proof. Let φ be an element in $\Gamma(X, \mathcal{F}^\infty)_g$. Given a neighbourhood B_φ in the Whitney topology we have to find $h \in B_\varphi \cap \Gamma(X, \mathcal{F})_g$.

Let G be an element in $\Gamma(X, \mathcal{F})_g$. Then $(\varphi - G)|_Y = 0$ and so replacing φ by $\varphi - G$ we can suppose g to be the zero-section of $\Gamma(Y, \mathcal{F}|_Y)$.

By considering the exact sequence:

$$\mathcal{J}_Y \otimes_{\mathcal{O}_X} \mathcal{F}^\infty \xrightarrow{\beta} \mathcal{F}^\infty \xrightarrow{\alpha} \mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{F}^\infty \to 0$$

we see that φ is in the image of $\mathcal{J}_Y \otimes_{\mathcal{O}_X} \mathcal{F}^\infty = (\mathcal{J}_Y \otimes_{\mathcal{O}_X} \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{E}_X$. Let $\psi \in \Gamma(X, \mathcal{J}_Y \otimes \mathcal{F}^\infty)$ be a preimage of φ. By Theorem 3.1 ψ can be approximated in the Whitney topology by a section f of $\Gamma(X, \mathcal{J}_Y \otimes_{\mathcal{O}_X} \mathcal{F})$. Then $\beta(f)$ approximates φ because β is continuous and $\beta(f)|_Y = 0$.

Suppose now to have a sheaf homomorphism between two coherent analytic sheaves of $\mathcal{O}_X$-modules $\alpha : \mathcal{F} \to \mathcal{G}$.

Then one has an exact sequence

$$0 \to \ker \alpha \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G}$$

Since $\mathcal{E}_X$ is flat over $\mathcal{O}_X$ we get an exact sequence

$$0 \to (\ker \alpha)^\infty \to \mathcal{F}^\infty \xrightarrow{\alpha^\infty} \mathcal{G}^\infty$$

Remark 3.4. Any exact sequence of $\mathcal{O}_X$-modules is also locally an exact sequence of $\mathcal{O}_n$-modules, because X is locally isomorphic to an analytic subset of some $\mathbb{R}^n$. For $X \subset \mathbb{R}^n$, $\mathcal{E}_X$ is a faithfully flat $\mathcal{O}_n$-module (see [M] Cor.1.12 pg 88); so, by definition of $\mathcal{E}_X$, α is surjective if and only if α^∞ is surjective and $\alpha(\mathcal{F}) = \alpha^\infty(\mathcal{F}) \cap \mathcal{G}$.

Remark 3.5. If the sheaf $\mathcal{F}$ is $\mathcal{O}_X$, then the sheaf $\mathcal{J}_Y^\infty = \mathcal{J}_Y \otimes_{\mathcal{O}_X} \mathcal{E}_X \simeq \mathcal{J}_Y \cdot \mathcal{E}_X$ is the sheaf of germs of C^∞ functions vanishing at Y, because Y is coherent. ([M] pg.95)

Theorem 3.6. *Let X be a coherent analytic space, let $\mathcal{F}, \mathcal{G}$ be coherent sheaves of $\mathcal{O}_X$-modules and $\alpha : \mathcal{F} \to \mathcal{G}$ be a sheaf homomorphism; suppose $g \in \Gamma(X, \mathcal{G})$ be such that $g = \alpha^\infty(\eta)$ with $\eta \in \Gamma(X, \mathcal{F}^\infty)$. Then in each neighbourhood of η for the Whitney topology there exists $f \in \Gamma(X, \mathcal{F})$ such that $\alpha(f) = g$*

Moreover if $Y \subset X$ is a closed coherent subspace and $\eta|_Y$ is in $\Gamma(Y, \mathcal{F}|_Y)$, we can find f such that $f|_Y = \eta|_Y$

Proof. By Remark 3.4 $g = a^\infty(\eta)$ is the image of some element $h \in \Gamma(X, \mathcal{F})$. Then $\eta - h \in \ker \alpha^\infty$ and, if $\eta|_Y$ is analytic, the same is true for $\eta - h|_Y$.

So we can apply Theorem 3.3 to the sheaf ker α and find $h_1 \in \Gamma(X, \ker \alpha)$ very close to $\eta - h$ and such that, $h_1|_Y = \eta - h|_Y$. Hence $f = h + h_1$ is very close to η, $\alpha(f) = g$ and, if $\eta|_Y$ is analytic, $f|_Y = \eta|_Y$
■

As an application of these results we obtain some approximation theorems for solutions of analytic linear systems.

Let U be an open set in $\mathbb{R}^n$ and $X \subset U$ be a coherent analytic set. Consider an analytic linear system on U:

$$\sum_{k=1}^{q} a_{hk}(x) y_k = g_h \quad h = 1, \dots, p \qquad (*)$$

where $a_{hk}, g_h \in C^\omega(U) = F(U, \mathcal{O})$. Then we have the following:

Theorem 3.7. *If (*) has a C^∞ solution, i.e. there exists $\varphi = (\varphi_1, \dots, \varphi_q) \in C^\infty(U)^q$ such that*

$$\sum_{k=1}^{q} a_{hk}(x) \varphi_k = g_h(x) \quad h = 1, \dots, p$$

then for each neighbourhood B_φ of φ in the product Whitney topology of $C^\infty(U)^q$, there exist a solution $f = (f_1, \dots, f_q) \in C^\omega(U)^q$ of (*) that belongs to B_φ.

Moreover we have:

(1) if $X \subset U$ is a coherent analytic set and $\varphi_{k|X} \in \Gamma(X, \mathcal{O}_X)$ for any k, then we can take

$$f_{k|X} = \varphi_{k|X} \quad \text{for } k = 1, \dots, q$$

(2) if for some $l \leq q$, $\varphi_1, \dots, \varphi_l \in C^\omega(U)$ then we can take $f_1 = \varphi_1, \dots, f_l = \varphi_l$}

Proof. Consider the sheaf homomorphism $\alpha : \mathcal{O}^q \to \mathcal{O}^p$ defined by the matrix (a_{hk}).

If each g_h is the zero function the first statement is Theorem 3.1 applied to ker α.

We have $(g_1, \dots, g_p) = \alpha^\infty(\varphi_1, \dots, \varphi_q)$; by Remark 3.4, then $(g_1, \dots, g_p) \in \Gamma(U, Im\ \alpha)$. So the first statement and (1) are consequence of Theorem 3.6.

To prove (2) consider

$$g'_h = \sum_{k=1}^{l} a_{hk}\varphi_k.$$

This is an element in $\Gamma(U, \mathcal{O})$, so we have a system of the same type as (*) given by

$$\sum_{k=l+1}^{q} a_{hk}y_k = g_h g'_h \quad h = 1, \dots, p \tag{**}$$

and for any solution $(\psi_{l+1}, \dots, \psi_q)$ of (**), $(\varphi_1, \dots, \varphi_l, \psi_{l+1}, \dots, \psi_q)$ is a solution of (*). So we reduce to the first statement. ■

Remark 3.8. In Theorem 3.7 we can suppose (same proof) that U is any open set of a coherent real analytic space.

4. APPROXIMATION FOR SHEAF HOMOMORPHISMS

Let X be a coherent analytic space. If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves of $\mathcal{O}_X$-modules, we know that the sheaf $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is also a coherent sheaf (see [S]). Next proposition gives the relations between $\mathcal{H}om\ (\mathcal{F},\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{E}_X = \mathcal{H}om\ (\mathcal{F},\mathcal{G})^\infty$ and $\mathcal{H}om\ (\mathcal{F}^\infty,\mathcal{G}^\infty)$

Proposition 4.1. $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{F}^\infty,\mathcal{G}^\infty) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})^\infty$

Proof. It is enough to prove the statement locally, since we have a natural map:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})^\infty \to \mathcal{H}om_{\mathcal{E}_X}(\mathcal{F}^\infty,\mathcal{G}^\infty)$$

Step 1. The thesis is true for $\mathcal{F} = \mathcal{O}_X^p$ and $\mathcal{G} = \mathcal{O}_X^q$.

In fact a sheaf homomorphism between $\mathcal{O}_X^p$ and $\mathcal{O}_X^q$ is given by a $p \times q$ matrix whose entries are analytic functions on X.

A similar result is true for $\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X^p,\mathcal{E}_X^q)$. In other words we have:

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^p,\mathcal{O}_X^q) \simeq \mathcal{O}_X^{p\times q}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X^p,\mathcal{E}_X^q) \simeq \mathcal{E}_X^{p\times q}$$
$$\mathcal{O}_X^{p\times q} \otimes_{\mathcal{O}_X} \mathcal{E}_X = \mathcal{E}_X^{p\times q}$$

Step 2. The thesis is true for $\mathcal{F} = \mathcal{O}_X^p$ and general $\mathcal{G}$.

Take a local resolution of $\mathcal{G}$:

$$\mathcal{O}_X^n \to \mathcal{O}_X^q \to \mathcal{G} \to 0$$

and apply the functor $\mathcal{H}om(\mathcal{O}_X^p,-)$. Each homomorphism $\mathcal{O}_X^p \to \mathcal{G}$ can be lifted (see Proposition 6 of Chap. VIII in [G.R.]) to a morphism $\mathcal{O}_X^p \to \mathcal{O}_X^q$, hence we get an exact sequence

$$\mathcal{H}om(\mathcal{O}_X^p,\mathcal{O}_X^n) \to \mathcal{H}om(\mathcal{O}_X^p,\mathcal{O}_X^q) \to \mathcal{H}om(\mathcal{O}_X^p,\mathcal{G}) \to 0$$

Tensoring with $\mathcal{E}_X$; we get:

$$(\mathcal{H}om(\mathcal{O}_X^p,\mathcal{O}_X^n))^\infty \to (\mathcal{H}om(\mathcal{O}_X^p,\mathcal{O}_X^q))^\infty \to (\mathcal{H}om(\mathcal{O}_X^p,\mathcal{G}))^\infty \to 0$$

Doing the same but in opposite order (using again that homomorphisms can be lifted), from

$$\mathcal{E}_X^n \to \mathcal{E}_X^q \to \mathcal{G}^\infty \to 0$$

we get

$$\mathcal{H}om(\mathcal{E}_X^p, \mathcal{E}_X^n) \to \mathcal{H}om(\mathcal{E}_X^p, \mathcal{E}_X^q) \to \mathcal{H}om(\mathcal{E}_X^p, \mathcal{G}^\infty) \to 0$$

The natural map $\mathcal{H}om(\mathcal{O}_X^p, \mathcal{O}_X^n) \otimes \mathcal{E}_X \to \mathcal{H}om(\mathcal{E}_X^p, \mathcal{E}_X^n)$ gives the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{H}om(\mathcal{O}_X^p, \mathcal{O}_X^n) \otimes \mathcal{E}_X & \to & \mathcal{H}om(\mathcal{O}_X^p, \mathcal{O}_X^q) \otimes \mathcal{E}_X & \to & \mathcal{H}om(\mathcal{O}_X^p, \mathcal{G}) \otimes \mathcal{E}_X & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{H}om(\mathcal{E}_X^p, \mathcal{E}_X^n) & \to & \mathcal{H}om(\mathcal{E}_X^p, \mathcal{E}_X^q) & \to & \mathcal{H}om(\mathcal{E}_X^p, \mathcal{G}^\infty) & \to & 0
\end{array}$$

The first two vertical rows are isomorphisms by step 1, so the third one is an isomorphism too.

Step 3. The general case.

Take a local resolution for $\mathcal{F}$

$$\mathcal{O}_X^m \to \mathcal{O}_X^s \to \mathcal{F} \to 0$$

Apply $\mathcal{H}om(-, \mathcal{G})$:

$$\mathcal{H}om(\mathcal{O}_X^m, \mathcal{G}) \leftarrow \mathcal{H}om(\mathcal{O}_X^s, \mathcal{G}) \leftarrow \mathcal{H}om(\mathcal{F}, \mathcal{G}) \leftarrow 0$$

Tensoring with $\mathcal{E}_X$ yields

$$\mathcal{H}om(\mathcal{O}_X^m, \mathcal{G})^\infty \leftarrow \mathcal{H}om(\mathcal{O}_X^s, \mathcal{G})^\infty \leftarrow \mathcal{H}om(\mathcal{F}, \mathcal{G})^\infty \leftarrow 0$$

Doing the same in opposite order yields

$$\mathcal{H}om(\mathcal{E}_X^m, \mathcal{F}^\infty) \leftarrow \mathcal{H}om(\mathcal{E}_X^s, \mathcal{F}^\infty) \leftarrow \mathcal{H}om(\mathcal{F}^\infty, \mathcal{G}^\infty) \leftarrow 0$$

But the natural maps:

$$\mathcal{H}om(\mathcal{O}_X^m, \mathcal{F})^\infty \to \mathcal{H}om(\mathcal{E}_X^m, \mathcal{F}^\infty) \text{ and}$$

$$\mathcal{H}om(\mathcal{O}_X^s, \mathcal{F})^\infty \to \mathcal{H}om(\mathcal{E}_X^s, \mathcal{F}^\infty)$$

are isomorphisms, hence $\mathcal{H}om(\mathcal{F}, \mathcal{G})^\infty \to \mathcal{H}om(\mathcal{F}^\infty, \mathcal{G}^\infty)$ is an isomorphism too. ■

Now we want to study the set of isomorphisms in $\mathcal{H}om(\mathcal{F}, \mathcal{G})$. If α is such an isomorphism, then we have an isomorphism $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{F}, \mathcal{F})$, obtained by composition with α^{-1}, which is also an isomorphism. So we can assume $\mathcal{F} = \mathcal{G}$.

For each $x \in X$ we can consider a minimal resolution of $\mathcal{F}$ in a small neighbourhood U of x

$$\mathcal{O}_U^n \xrightarrow{\tau} \mathcal{O}_U^p \xrightarrow{\sigma} \mathcal{F}|_U \to 0$$

This means:

(1) $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module generated by p sections $(a_1, \ldots, a_p)$ and it cannot be generated by $p-1$ sections

(2) the kernel of $\sigma : \mathcal{O}^p \to \mathcal{F}$ is a subsheaf of $\mathcal{O}^p$ generated by n p-tuples of analytic functions.

If $\alpha : \mathcal{F} \to \mathcal{F}$ is a given isomorphism, then $(\alpha(a_1) = b_1, \ldots, \alpha(a_p) = b_p)$ is another system of generators of $\mathcal{F}_x$. Then we have:

Lemma 4.2. *α can be lifted to a morphism $\tilde{\alpha} : \mathcal{O}^p \to \mathcal{O}^p$ (not in a unique way) and any lifting is an isomorphism on a neighbourhood of x.*

The proof is a consequence of the following more general fact.

Lemma 4.3. *Let (A, m) be a local ring and let M be an A-module of finite type. Let p be the minimal number of elements in M generating M over A. Then any A-homomorphism $f : M \to M$ can be lifted to a homomorphism $\hat{f} : A^p \to A^p$. Moreover if f is an isomorphism any lifting $\hat{f}$ is an isomorphism.*

Proof. M is isomorphic to A^p/N, where N is a submodule and $N \subset mA^p$, since p is minimal. If $(v_1, \ldots, v_p)$ is a system of generators for M over A, f can be expressed (not in a unique way) as a matrix F with entries in A; $F: A^p \to A^p$ lifts f. If both F and G lift f then the columns of the difference $F - G$ are elements of N. In particular if $F = (f_{ij})$ and $G = (g_{ij})$, then $f_{ij} - g_{ij} \in m$ for any i, j. So, if f is invertible and $h = f^{-1}$, F lifts f and H lifts h, we have $f \circ h = id$, hence $F \cdot H = (\delta_{ij} + m_{ij})$ with $m_{ij} \in m$. This implies that both det F and det H are units in A: in particular F is an isomorphism. ■

Lemma 4.4. *Suppose $\alpha: \mathcal{F} \to \mathcal{F}$ is a sheaf isomorphism. For each point x in X there exist a compact neighbourhood H_x of x and a positive constant $\mathcal{E}_x$ such that, if $\beta: \mathcal{F} \to \mathcal{F}$ verifies $||\beta - \alpha||^0_{H_x}$, then β is an isomorphism on a neighbourhood of H_x.*

Proof. Let $\tilde{\alpha}$ be a lifting of a on a neighbourhood U_x of x. By Lemma 4.2 we can suppose $\tilde{\alpha}$ to be an isomorphism on U_x. Define

$$d_0 = |\det(a_{ij}(x))|$$

We can find a compact neighbourhood H_x of x such that for each $y \in H_x$ one has

$$|\det(a_{ij}(y))| \geq \frac{d_0}{2}$$

Then for each matrix of analytic functions $(b_{ij}(y))$ sufficiently near to $(a_{ij}(y))$ we have

$$|\det(b_{ij}(y)) - \det(a_{ij}(y))| < \frac{d_0}{4}$$

for each $y \in H_x$. This means that there exists ε_x such that, if $||\beta - \alpha||^0_{H_x} < \varepsilon_x$, then β has a lifting $(b_{ij}(y))$ such that $|\det(b_{ij}(y))| > \frac{d_0}{4}$ for each $y \in H_x$; so $(b_{ij}(y))$, and hence β, is an isomorphism over a neighbourhood of H_x. ■

Corollary 4.5. For any compact $K \subset X$ there exists a neighbourhood

$$V(K, \varepsilon) = \{\beta \in \Gamma(X, \mathcal{H}om(\mathcal{F}, \mathcal{F}))| \; ||\beta - \alpha||^0_K < \varepsilon\}$$

such that any $\beta \in V(K,\varepsilon)$ is an isomorphism on a neighbourhood of K.

Proof. Cover K by a finite number of H_x and take $\varepsilon = \min\{\varepsilon_x\}$ ■

Finally we have:

Theorem 4.6. *Let X be a coherent real analytic space and $\mathcal{F},\mathcal{G}$ be two coherent sheaves of $\mathcal{O}_X$-modules. Then the set*

$$\mathcal{I}so(\mathcal{F},\mathcal{G}) = \{\beta \in \Gamma(X,\mathcal{H}om(\mathcal{F},\mathcal{G})) | \beta \text{ is an isomorphism}\}$$

is an open set for the Whitney topology.

Proof. As before we can suppose $\mathcal{F} = \mathcal{G}$. Let α be an isomorphism. We have to show that $\mathcal{I}so(\mathcal{F})$ contains a neighbourhood of α.

Let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustive sequence of compact sets. Define ε_i as follows:

- ε_0 is such that if $||\beta-\alpha||^0_{K_0} < \varepsilon_0$ then β is an isomorphism on a neighbourhood of K_0.
- ε_1 is such that if $||\beta-\alpha||^0_{K_1-\overset{\circ}{K}_0} < \varepsilon_1$ then β is an isomorphism on a neighbourhood of $K_1 - \overset{\circ}{K}_0$.
- ...

and so on.

Then $\{K_i,\varepsilon_i\}_{i\in\mathbb{N}}$ defines a neighbourhood of α in the Whitney topology, namely the set of β's such that for any i

$$||\beta-\alpha||^0_{K_i-\overset{\circ}{K}_{i-1}} < \varepsilon_i$$

For any β in such a neighbourhood, β is an isomorphism on a neighbourhood of K_0 and on a neighbourhood of $K_i - \overset{\circ}{K}_{i-1}$ for each i. If all the ε_i are small enough β is injective, hence $\beta \in \mathcal{I}so(\mathcal{F})$.

Theorem 4.7. *The set of isomorphisms in $\Gamma(X,\mathcal{H}om(\mathcal{F}^\infty,\mathcal{G}^\infty))$ is an open set for the Whitney topology:*

Proof. The proofs of Lemma's 4.2 and 4.4, Corollary 4.5 and Theorem 4.6 can be repeated, almost without changes, with $\mathcal{F}^\infty$ instead of $\mathcal{F}$. ■

Corollary 4.8. Let $\varphi: \mathcal{F}^\infty \to \mathcal{G}^\infty$ be an isomorphim, then there exists an isomorphism $f: \mathcal{F} \to \mathcal{G}$ such that f^∞ is an isomorphism and arbitrarily close to φ

Proof. It is an application of Theorem 3.1 together with Proposition 4.1 and Theorems 4.6 and 4.7 below. ■

5. THE ALGEBRAIC CASE

Let $X \subset \mathbb{R}^n$ be a real algebraic set. Denote by $\mathcal{R}_X$ the sheaf of germs of regular functions on X and by $\mathcal{O}_X$ the structural sheaf of X as real analytic set.

Definition 5.1. *Let $\tilde{X}$ be a complexification of X as an affine variety: we say that X is almost regular if for each $x \in X$ the germ $\tilde{X}_x$ coincides with the analytic complexification of the germ X_x.*

Definition 5.2. *A sheaf $\mathcal{F}$ of $\mathcal{R}_X$-modules is called A-coherent if it admits a resolution:*

$$\mathcal{R}_X^p \to \mathcal{R}_X^q \to \mathcal{F} \to 0$$

Consider the natural injection $X \to$ Spec $\Gamma(X, \mathcal{R}_X)$: one can show ([T9]) that a sheaf $\mathcal{F}$ is A-coherent if and only if it extends, of course in a unique way, to a coherent sheaf $\hat{\mathcal{F}}$ over Spec $\Gamma(X, \mathcal{R}_X)$; this extension $\hat{\mathcal{F}}$ shall have an important role in what follows.

Definition 5.3. *An A-coherent sheaf $\mathcal{F}$ is called B-coherent if any $\gamma \in \Gamma(X, \mathcal{F})$ extends to $\Gamma($ Spec $\Gamma(X, \mathcal{R}_X), \hat{F})$*

For an A-coherent sheaf of $\mathcal{R}_X$-modules $\mathcal{F}$ denote by $\mathcal{F}^o$ the sheaf $\mathcal{F} \otimes_{\mathcal{R}_X} \mathcal{O}_X$.

Remark 5.4. If X is almost regular in particular it is coherent as analytic space ([T7]); in this case $\mathcal{O}_X$ is a faithfully flat $\mathcal{R}_X$-module (see [S] Cor.1 pg 11).

If $U \subset X$ is an open set, we can endow $\Gamma(U, \mathcal{F})$ with a local system of seminorms by considering the usual weak topology on $\mathcal{R}(U)$ as in Examples 2) and 3) in §1. Then we have:

Theorem 5.5. *Let X be a real affine algebraic set and $Y \subset X$ be an almost regular algebraic subset. Let $\mathcal{F}$ be a A-coherent sheaf, then $\Gamma(U, \mathcal{F})$ is dense in $\Gamma(U, \mathcal{F})$ is dense in $\Gamma(U, \mathcal{F}^o)$ in the weak topology.*

Moreover if $f \in \Gamma(X, \mathcal{F}^o)$ and $f|_Y = g \in \Gamma(Y, \mathcal{F}|_Y)$, then the set $\Gamma(X, \mathcal{F})_g$ of regular sections extending g is dense in the corresponding set $\Gamma(X, \mathcal{F}^o)_g$.

Proof. It is the same as the proofs of Theorems 3.1 and 3.3 using Stone-Weierstrass instead of Whitney approximation theorem. Remark 5.4 and the fact that $\mathcal{F}$ is A-coherent give us the necessary ingredients to repeat the proofs. ■

Now consider a sheaf homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ between two A-coherent sheaves of $\mathcal{R}_X$-modules. We can define $\alpha^o : \mathcal{F}^o \to \mathcal{G}^o$ by tensorizing with the sheaf $\mathcal{O}_X$. By Remark 5.4 we have

$$\alpha^o(\mathcal{F}^o) \cap \mathcal{G} = \alpha(\mathcal{F})$$

and hence the following result:

Theorem 5.6. *Let U be a Zariski open set in $\mathbb{R}^n$. Consider the linear system*

$$\sum_{k=1}^{q} a_{hk}(x) y_k = g_h \quad h = 1, \dots, p \tag{$*$}$$

where $a_{hk}(x)$ and $g_h(x)$ are regular functions on U for $h = 1, \dots, p$ and $k = 1, \dots, q$. Then any differentiable solution $(f_1, \dots, f_q)$ of $(*)$ can be approximated in the weak topology by a regular solution $(g_1, \dots, g_q)$.

Moreover:

(1) If for an almost regular algebraic set $X \subset U$ we have $f_k|_X \in \Gamma(X, \mathcal{R}_X)$ for $k = 1, \dots, q$, we can take $g_1, \dots, g_q$ in such a way that $g_{k|X} = f_{k|X}$ for $k = 1, \dots, q$

(2) If the first $l < q$ components $f_1, \dots, f_l$ are regular, then we can take $g_1, \dots, g_l = f_l$.

Proof. By Theorem 3.7 we may suppose $(f_1,\dots,f_q)$ is an analytic solution of $(*)$. As we remarked before, the matrix (a_{hk}) defines an exact sequence of coherent sheaves

$$\begin{array}{ccccccc} 0 & \to & \ker\alpha & \to & \mathcal{R}_U^q & \overset{\alpha}{\to} & \mathcal{R}_U^p \\ & & & & \downarrow & & \\ & & & & \operatorname{Im}\alpha & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

where ker α and Im α are A-coherent because they admit a complexification. Consider the corresponding exact sequence obtained by applying $\otimes_{\mathcal{R}_U}\mathcal{O}$. We can apply Theorem 5.5 to ker α and ker α^o. So any analytic solution of the homogeneous system has a regular approximation.

For the general case we can use the fact that $\alpha^o(\mathcal{F}^o)\cap\mathcal{G}=\alpha(\mathcal{F})$, and we have surjectivity for sections because $\mathcal{R}_U^p$ is B-coherent and so also Im α is B-coherent, being a subsheaf of $\mathcal{R}_U^p$.

So if $(f_1,\dots,f_q)$ is an analytic solution of $(*)$ we have $(g_1,\dots,g_p)=\alpha^o(f_1,\dots,f_q)$ and hence $(g_1,\dots,g_p)=\alpha(h_1,\dots,h_q)$ with $h_1,\dots,h_q\in\Gamma(U,\mathcal{R}_U^q)$, because $\mathcal{O}_X$ is faithfully flat on $\mathcal{R}_X$, (see Remark 5.4).

So $(f_1-h_1,\dots,f_q-h_1)$ $\in$ker α^o and we conclude as before.

If $f_1,\dots,f_l$ are regular, then $(f_{l+1},\dots,f_q)$ is a solution of the system

$$\sum_{k=l+1}^{q} a_{hk}y_k = g_h-\sum_{k=1}^{l}a_{hk}f_k \quad h=1,\dots,p \tag{$**$}$$

and then has a regular approximation: this proves that (2) may be satisfied.

For the assertion (1) we can use a more direct argument instead of repeating the proof of Theorem 3.7.

Suppose $f_{k|X}$ to be regular for each k and let F_k be a regular function on U which extends $f_{k|X}$. Then f_k-F_k is an analytic function vanishing on X. Since X is almost regular, if $p_1,\dots,p_\nu$ are generators for the ideal $I(X)\subset\mathcal{R}_U$, we can write:

$$f_k - F_k = \sum_{j=1}^{\nu} \beta_{jk} p_j \quad k = 1, \ldots, q$$

Then by applying α^o to the vector $(f_1 - F_1, \ldots, f_q - F_q))$ we find that the set $\{\beta_{jk}\}_{\substack{j=1,\ldots,\nu \\ k=1,\ldots,q}}$ is an analytic solution of the system:

$$\sum_{k=1}^{q} \sum_{j=1}^{\nu} a_{hk} p_j \beta_{jk} = g_h - \sum_{k=1}^{q} a_{hk} F_k \quad h = 1, \ldots, p \qquad (***)$$

Hence the β_{jk} can be approximated in the weak topology by regular functions b_{jk} such that the set $\{b_{jk}\}$ is a regular solution of $(***)$. Consider for $k = 1, \ldots, q$ the regular function $G_k = \sum_{j=1}^{\nu} b_{jk} p_j$; it vanishes on X , approximates $f_k - F_k$ and by construction

$$\sum_{k=1}^{q} a_{hk} G_k = g_h - \sum_{k=1}^{q} a_{hk} F_k \quad h = 1, \ldots, p$$

so $(G_1 + F_1, \ldots, G_q + F_q)$ is the required approximation of $(f_1, \ldots, f_q)$ ■

6. ALGEBRAIC AND ANALYTIC VECTOR BUNDLES

Before giving some consequences of the theorems in §3 and §4 let us recall shortly some definitions and results about generalized vector bundles and about duality between them and the coherent sheaves. We refer to [F1], [F2] and [P] for the complex case and to [Ct] and [T9] for the real analytic and algebraic case respectively. A complete survey on this subject shall appear in [T12].

Let $\mathbb{K}$ be the field $\mathbb{C}$ or $\mathbb{R}$, and $(X, \mathcal{O}_X)$ be an analytic set in $\mathbb{K}^m$. If $\mathbb{K} = \mathbb{R}$ assume X to be coherent.

Given a matrix $\alpha(x) = (a_{ij}(x))_{\substack{i=1,\ldots,q \\ j=1,\ldots,p}}$ with entries in $\Gamma(X, \mathcal{O}_X)$, we can think of it as a map:

$$X \times \mathbb{K}^p \to X \times \mathbb{K}^q$$

$$(x,t) \quad \to (x, \alpha(x)t)$$

Definition 6.1. *A linear analytic bundle is the set*

$$F = \ker\, \alpha = \{(x,t) \in X \times \mathbb{K}^p | t \in \ker\, \alpha(x)\}.$$

Let π_F be the projection $F \to X$. A *morphism* of linear analytic bundles $F \subset X \times \mathbb{K}^p$, $G \subset X \times \mathbb{K}^r$ is an analytic map $\varphi : F \to G$ such that:

(1) The diagram

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ \pi_F \downarrow & & \downarrow \pi_G \\ X & \xrightarrow{\text{id}} & X \end{array}$$

commutes

(2) For any $x \in X$, $\varphi_{|\pi_F^{-1}(x)} : \pi_F^{-1}(x) \to \pi_G^{-1}(x)$ is linear.

Denote by $\mathcal{L}(X)$ the category of generalized analytic bundles over X. More generally we could define an abstract notion of analytic linear bundle as a triple (F, π, X) locally isomorphic to a linear bundle as in Definition 6.1.

Remark 6.2. If $\alpha(x)$ has constant rank we have the usual notion of locally trivial vector bundle.

A matrix $(b_{ij}(x))$ with entries in $\Gamma(X, \mathcal{O}_X)$ defines also a morphism $\beta : \mathcal{O}_X^p \to \mathcal{O}_X^q$. Its cokernel $\mathcal{F}$ is a coherent sheaf over X.

Proposition 6.3. *There is a "duality" associating to each linear analytic bundle $F = \ker\, \alpha$ the coherent sheaf $\mathcal{F} = \operatorname{coker}{}^t\alpha$. If $\mathcal{D}(F)$ is the sheaf associated to the presheaf*

$$X \supset U \to \mathcal{H}om(F|_U, U \times \mathbb{K})$$

then $\mathcal{D}(F) = \operatorname{coker}{}^t\alpha$

Similar results, with a few changes, are true in the real algebraic case: in this case one can show that the duality is well defined between the category of B-coherent sheaves and a subcategory of $\mathcal{L}(X)$. Details can be found in [Ct] and [T9].

Now we can apply the results of §3 and §4.

In the following by analytic (C^∞) sections of $\mathcal{F}$ we shall mean sections of $\mathcal{D}(F)$ ($\mathcal{D}(F)^\infty$).

Let Y be a coherent analytic subset of X: we can consider smooth sections of $\mathcal{D}(F)$ wanishing on Y; again by Malgrange theorem these are precisely the sections in the image of $\mathcal{J}^\infty \otimes \mathcal{D}(F)$. Then we have:

Theorem 6.4. *Let X be a coherent real analytic space, Y be a coherent subspace and F be a linear analytic bundle; let σ be a smooth section of F which is analytic on Y.*

Then in each neighbourhood of σ for the Whitney topology, one can find an analytic section s such that $s_{|Y} = \sigma_{|Y}$

Theorem 6.5. *Let X be a compact real affine variety, Y be an almost regular subvariety and F be a linear algebraic bundle; let σ be a smooth section of F which is regular on Y.*

Then in each neighbourhood of σ in the weak topology, one can find a regular section s such that $s_{|Y} = \sigma_{|Y}$.

Remark 6.6. Theorems 6.4 and 6.5 are generalizations of the results about approximation of sections in a locally trivial analytic or algebraic vector bundle. (see [BTI], [BT2], [BT3])

Finally, by duality, Corollary 4.8 gives a classification theorem for linear analytic bundles, which extends the classical results for the complex case (see [G1], [G2], [G3]) and for the real case (see [T1], [T2], [T3]).

Theorem 6.7. *Let F, G be linear analytic bundles over the coherent set $X \subset \mathbb{R}^n$. Let $\varphi: F \to G$ be a smooth isomorphism, i.e. there is a commutative diagram*

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ \pi_F \downarrow & & \downarrow \pi_G \\ X & \xrightarrow{\text{id}} & X \end{array}$$

where φ is the restriction of a C^∞ map from $\mathbb{R}^n \times \mathbb{R}^q$ to $\mathbb{R}^n \times \mathbb{R}^p$ which is invertible and linear on the fibres. Then there exists $f : F \to G$ which is an analytic bundle isomorphism and is arbitrarily close to φ.

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