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Rational models of solvmanifolds with Kählerian structures.

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Abstract

We investigate the existence of symplectic non-Kählerian structures on compact solvmanifolds and prove some results which give strong necessary conditions for the existence of Kählerian structures in terms of rational homotopy theory. Our results explain known examples and generalize the Benson-Gordon theorem [BG2] (our method allows us to *drop the assumption of the complete solvability of* G).

1 Introduction

There has been an interest recently in examples of compact symplectic manifolds with no Kähler structures ([Ab, Gmf, CFG, FLS, McD, LuO] and others). With the exception of [McD, FLS] and the surgery technique of [Gmf], known examples are nilmanifolds coming from the following general theorem proved by Benson and Gordon [BG1] and Hasegawa [Has].

Let M be a compact $K(\Gamma,1)$*−manifold where* Γ *is a discrete, finitely generated, torsion free, nilpotent group. If* M *admits a Kähler structure, then* Γ *is abelian and* M *has the homotopy type of a torus.*

This theorem implies that any *non-toral* symplectic $K(\Gamma,1)-$ manifold with *nilpotent* Γ yields the desired example.

Nevertheless, it would be interesting to look for other aspherical

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manifolds which are symplectic but non-Kählerian. For example, the result of [FLS] suggests looking for such manifolds in the class of *solvmanifolds.* Observe once more that the fundamental group of a solvmanifold *need not be nilpotent,* (moreover, it is known that for a solvmanifold M, the nilpotency of $\pi_1(M)$ implies the diffeomorphicity of M to a nilmanifold [VGS]).

The case of solvmanifolds, however, differs essentially from that of the nilmanifolds for several reasons. In general, the Nomizu theorem concerning the cohomology of a nilmanifold is not available, therefore, the minimal model of a solvmanifold cannot be used directly. Moreover, the known examples of symplectic and Kählerian solvmanifolds show that *all possible homotopy types, Kählerian and non-Kählerian, may occur* (see [FLS, BG2]). The most unexpected example of [BG2] shows that there exists a compact symplectic solvmanifold of the same rational homotopy type as $T^2 \times S^2$, but *non-Kählerian,* because it cannot carry any complex structure [FG].

In [BG2] and [FLS] , the authors investigated the case of a solvmanifold G/Γ of a *completely solvable* Lie group G and established the necessary conditions for the existence of Kählerian structure on such manifolds. The key to the proof of the cited results is the Nomizu-Hattori theorem [H] regarding the cohomology of G/Γ which allowed the authors to establish the *rational model* $(\bar{A}^*(G/\Gamma), d) = (\Lambda L(G)^*, \delta)$. In general, the *Nomizu-Hattori theorem does not hold* (see [R] and [VGS]) and the technique of [BG2] is not available. Moreover, there are many types of solvmanifolds which are not completely solvable [VGS], e.g. solvmanifolds of $(R)-$type, $(E)-$type, mixed types etc.

Nevertheless, the main result of this paper shows that there exists a *graded differential algebra* $(A^*(G/\Gamma), d)$ *which is a rational model of an arbitrary Kählerian solvmanifold and possesses the same algebraic properties as* $(\bar{A}^*(G/\Gamma), d)$. Of course, in the completely solvable case these algebraic properties determine the structure of G (it is semidirect product of abelian and nilpotent parts). We use a different approach based on the Thomas theorem [Th1], [Th2] which describes models of Serre fibrations. The use of this theorems is of independent interest.

To formulate the main result of the paper, we introduce the notion of the *twisted tensor product of graded commutative differential algebras.* Here and in the sequel we use traditional notations of rational homotopy

theory and solvmanifolds. Nevertheless, some of them are explained in the next section.

Let $(\mathcal{R}, \bar{d})$ and $(\mathcal{S}, \delta)$ be graded commutative differential algebras. Introduce the tensor product $\mathcal{R} \otimes \mathcal{S}$ of $\mathcal{R}$ and $\mathcal{S}$ as graded algebras and define the "twisted" derivation d by the formula

$$d(r \otimes 1) = \bar{d}(r) \otimes 1, r \in \mathcal{R}$$

$$d(1 \otimes s) = 1 \otimes \delta(s) + \sum_{i \geq 0} (-1)^{(i+1)deg(s)} \sum_{\xi \geq 1} \phi_i^{\xi}(s) \otimes r_{i+1}^{\xi}, s \in \mathcal{S} \qquad (1)$$

here ϕ_i^{ξ} is a derivation of $(\mathcal{S}, \delta)$ decreasing the degree by i, that is

$$\phi_i^{\xi}(xy) = \phi_i^{\xi}(x)y + (-1)^{i \cdot deg(x)} x \phi_i^{\xi}(y)$$

and $\{r_{i+1}^{\xi}, \xi = 1, 2, \cdots\}$ constitute a basis of $\mathcal{R}^{i+1}$. By definition, the graded differential algebra $(\mathcal{R} \otimes \mathcal{S}, d)$ is called the *twisted tensor product* of $(\mathcal{R}, \bar{d})$ and $(\mathcal{S}, \delta)$. In the sequel we denote the twisted tensor product by

$$(\mathcal{R}, d) \otimes_{\tau} (\mathcal{S}, \delta).$$

Theorem. *Let $M = G/\Gamma$ be a compact solvmanifold carrying a Kählerian structure. There exists a free graded differential algebra $(A^*(E), d)$ satisfying the following properties*

(i) $(A^*(E), d)$ is a Lefschetz algebra and a rational moder for E;

(ii) $(A^*(E), d) = (\Lambda X, \bar{d} = 0) \otimes_{\tau} (\Lambda Y, \delta)$, X and Y are finite dimensional vector spaces of elements of degree 1, and $(\Lambda Y, \delta)$ is a minimal differential algebra;

(iii) for the bigrading $\Lambda^{ij} = \Lambda^i X \otimes \Lambda^j Y$ the equalities hold:

$$H^1(A^*) = \Lambda^{1,0}, H^{n-1}(A^*) = \Lambda^{k,l-1}$$

(iv) the Lefschetz element $\omega \in \Lambda^2(X \oplus Y)$ can be chosen in the form

$$\omega = \omega^{2,0} + \omega^{0,2}$$

and $\omega^{2,0}$ and $\omega^{0,2}$ are non-degenerate 2−forms on X^* and Y^*, which are closed and non-exact with respect to d and δ.

(v) dimX and dimY are even and X^* and Y^* are $\omega-$orthogonal.

To illustrate the usefulness of theorem 3 we also show that it generalizes the Benson-Gordon theorem [BG2]. Here and in the sequel $L(G)$ denotes the Lie algebra of the Lie group G.

Corollary. *If G is completely solvable and G/Γ is a solvmanifold that admits a Kahler structure, then*

(i) there is an abelian complement $\mathcal{A}$ in $L(G)$ of the derived algebra $\mathcal{N} = [L(G), L(G)]$;

(ii) $\mathcal{A}$ and $\mathcal{N}$ are even dimensional;

(iii) the center of $L(G)$ intersects $\mathcal{N}$ trivially;

(iv) the Kahlerian form is cohomologous to a left invariant sympletic form $\omega = \omega_0 + \omega_1$, where $\mathcal{N} = \ker(\omega_0)$ and $\mathcal{A} = \ker(\omega_1)$;

(v) both ω_0 and ω_1 are closed but not exact in $L(G)$ and also in $\mathcal{N}$ and $\mathcal{A}$;

(vi) the adjoint action of $\mathcal{A}$ on $\mathcal{N}$ is by infinitesimal symplectomorphisms.

Remark.

(i) Corollary follows rather from *the proof of Theorem 3.*

(ii) The properties (ii)-(v) stated in the Theorem are the algebraic properties established in [BG2] for $(\Lambda L(G)^*, \delta)$ in the completely solvable case.

In the last section of the paper we construct explicitly an example of a sympletic solvmanifold which is *not completely solvable.* We deduce that this solvmanifold is not Kählerian. We use the algebraic properties of the model $(A^*(G/\Gamma), d)$ established in the Theorem for Kählerian solvmanifolds to describe the rational homotopy type of *any* 4-dimensional Kählerian solvmanifold. This description explains examples in [FG] and [BG2] from the rational homotopy point of view.

2 Preliminaries

This work is devoted, in fact, to algebraic properties of free models of Kählerian compact solvmanifolds G/Γ, that is free graded differential algebras $A^*(G/\Gamma)$, quasiisomorphic to the Rham algebra $\Omega_{DR}(G/\Gamma)$. Thus, we are in the *framework of rational homotopy theory*, although our models are not minimal in general. Since there are many books and research articles on rational homotopy theory, we assume the reader to be familiar with it, referring to [Ha], [Le], [Ta] and [DGMS].

In the sequel we consider the category $R-DGA$ of graded differential algebras over the field of the real numbers (although the algebraic results are valid for the arbitrary field of a zero characteristic). By definition, a *model* of a manifold M is a *free graded differential algebra* $A^*(M)$ such that there exists either a homomorphism,

$$\alpha : A^*(M) \to \Omega_{DR}(M)$$

or a homomorphism

$$\beta : \Omega_{DR}(M) \to A^*(M)$$

inducing isomorphism in cohomology. Of course, $A^*(M)$ need not be minimal and therefore is not unique. We use the notion of the *minimal model* in the usual sense. The cohomology functor for $R-DGA$ is denoted by H^*. By definition we call a free finitely-graded differential algebra

$$A = \oplus_{i=0}^{n} A^i$$

oriented if $H^n(A) \neq 0$.

Definition.

(i) A finitely-graded free differential algebra $(A,d) = (\oplus_{i=0}^{2n} A^i, d)$ *is called symplectic, if there exists an element* $[\omega] \in H^2$, *such that* $[\omega]^n \neq 0, \omega \in A^2$;

(ii) An algebra (A,d) *satisfying (i) is called Lefschetz, or satisfies the hard Lefschetz condition if all homomorphisms*

$$L_{\omega^r} : H^{n-r} \to H^{n+r}, L_{\omega^r}([h]) = [\omega^r][h], r \geq 1$$

are isomorphisms. The element $\omega \in A^2$ *is called Lefschetz.*

Observe that symplectic algebras satisfying Poincaré duality are models of compact symplectic manifolds and Lefschetz algebras are those of compact Kählerian manifolds. This fact will play the key role in Section 3.

One of the main ingredients of proofs in this article is the *Thomas theorem* concerning the models of rational Serre fibrations [Th 1] and [Th 2], therefore, we reproduce its exact formulation (see [Th 1] for the proof).

Let

$$F \to E \to B \qquad (*)$$

be a Serre fibration with F and B of finite cohomology type. It is a known fact that the fundamental group $\pi_1(B)$ acts naturally on $H^j(F), j \geq 1$, that is, there exists a representation of $\pi_1(B)$ in $Aut(H^*(F))$ (see, e.g.[Se]). We presume that, E, F, B are *path connected.*

Recall that by definition an action of a group G on an abelian group C is called *nilpotent* if the following conditions are satisfied:

(i) G acts on C by automorphism;

(ii) the inductively defined sequence

$$\Gamma_1 C = C_1 \supset \cdots \supset \Gamma_{n+1} C = \{gc - c, g \in G, c \in \Gamma_n C\}$$

vanishes for some $n \geq 1$ ($\{\cdots\}$ denotes the $ZG-$submodule in C generated by $gc - c$).

[TH]-Theorem *(J.-C. Thomas). Let $(*)$ be a Serre fibration satisfying the previous restrictions. Suppose that $\pi_1(B)$ acts nilpotently on $H^j(F), j \geq 1$. Then there exists a model $(A^*(E), d)$ which is a twisted tensor product*

$$(A^*(E), d) = (\mathcal{M}_B, d_B) \otimes_\tau (\mathcal{M}_F, d_F)$$

of the minimal models $(\mathcal{M}_B, d_B)$ and $(\mathcal{M}_F d_F)$ of B and F respectively. The twisting r is of the form (3).

Remark. We prefer to write the twisting r in the form [TeS].

In the sequel we denote the Lie algebra of a Lie group G by $L(G)$. The dual of the vector space X is denoted by X^*.

Following [BG2] we consider as *solvmanifolds* only homogeneous spaces G/Γ, where G is a solvable simply connected Lie group and Γ is a lattice in G, that is a discrete co-compact subgroup (see[Au, M] for general theory of solvmanifolds).

We introduce *nilmanifolds* as homogeneous spaces N/Γ, where N is a simply connected nilpotent Lie group and Γ a lattice in N.

To compare our results with [BG2,FLS], we recall the definition of a *completely solvable* Lie group: a Lie group G is completely solvable, if all endomorphisms $adV : L(G) \to L(G), V \in L(G)$ possess only *real* eigenvalues. Of course, there are many solvable Lie groups which *are not completely solvable* (see e.g. [Au], [VGS]).

To prove the main result of this paper we need the following fact, which is well-known and can be found in the explicit form in [O]:

The minimal model of any compact nilmanifold N/Γ is of the form

$$(\mathcal{M}_{N/\Gamma}, d) \simeq (\Lambda L(N)^*, \delta)$$

where δ is a standard derivation determining the cohomology of $L(N)$:

$$\delta\alpha(X_0, \cdots X_k) = \sum_{i<j} (-1)^{i+j-1} \alpha([X_i, X_j], X_0, \cdots, \widehat{X}_i, \widehat{X}_j, \cdots X_k), \quad (2)$$

$X_0, \cdots X_k \in L(N), \alpha \in \Lambda^k L(N)^*$.

In Section 4 we use the notion of *the higher-order Massey product*, referring to [K] for the definition.

A free graded algebra over a graded vector space X is denoted by ΛX, the degree of an element $x \in X$ is denoted by $deg(x)$.

Analyzing examples in the last section of the paper, we refer to the Kodaira classification of compact complex surfaces [BPV] and [FM]. We use the Kodaira classification table in the form [BPV] (p.188). All definitions necessary to understand this classification are also contained in [BPV]. We denote the geometric genus of a surface by the symbol π_g and the irregularity number by q.

3 Proof of the theorem and corollary

Let $(\Lambda X \otimes \Lambda Y, d)$ satisfy the conditions of the Theorem. In the sequel we assume X and Y to be finite dimensional and fix the bases $x_1, \cdots x_k$, $y_1, \cdots, y_l$ of X and Y respectively. We begin with

Lemma 1. *Let $(\Lambda X \oplus \Lambda Y, d)$ satisfy the condition (ii) of the Theorem. Suppose that $(\Lambda X \oplus \Lambda Y, d)$ is oriented. Then, necessarily, the derivation d is of the form*

$$dy_j = x^{(j)} + y_{s_j t_j} + \sum_{s \neq j} \widetilde{x}_s y_s + \bar{x}_j y_j, j = 1, \cdots l,$$

where

$$\sum_{j=1}^{l} \bar{x}_j = 0, \tag{3}$$

$$x^{(j)} \in \Lambda^2 X, y_{s_j t_j} \in \Lambda^2 Y, deg(\widetilde{x}_s) = 1, deg(y_s) = 1, deg(\bar{x}_j) = 1$$

Proof. Since $H^n(\Lambda X \otimes \Lambda Y, d) \neq 0$ and since $(\Lambda X \otimes \Lambda Y)$ is the exterior algebra,

$$dim H^n(\Lambda X \otimes \Lambda Y) = dim Z^n(\Lambda X \otimes \Lambda Y) = dim \Lambda^n(X \oplus Y) = 1.$$

Suppose that $\Lambda^{n-1}(X \oplus Y) \neq Z^{n-1}(\Lambda(X \oplus Y))$, then there exists $v \in \Lambda^{n-1}(X \oplus Y), dv \neq 0$, $dv \in \Lambda^n(X \oplus Y)$. By the previous remark $dv = \alpha u$, where u is a generator in $\Lambda^n(X \oplus Y)$, corresponding to the non-zero cohomology class, which is a contradiction.

Thus

$$\Lambda^{n-1}(X \oplus Y) = \Lambda^{k-1,l} + \Lambda^{k,l-1} = Z^{n-1}(X \oplus Y) \tag{4}$$

Therefore

$$u_i = x_1 \cdots \widehat{x}_i \cdots x_k y_1 \cdots y_l \in \Lambda^{k-1,l}, i = 1, \cdots, k$$

are cocycles (here and in the sequel $\widehat{x}_i$ denotes the absence of x_i). Calculating du_i explicitly one obtains

$$du_i = x_1 \cdots \widehat{x}_i \cdots x_k \left((-1)^{j-1} \sum_{j=1}^{l} y_1 \cdots y_{j-1} dy_j y_{j+1} \cdots y_l \right) =$$

$$= x_1 \cdots \widehat{x}_i \cdots x_k \left\{ (-1)^{j-1} \sum_{j=1}^{l} y_1 \cdots y_{j-1} \left(x^{(j)} + y_{s_j t_j} + \sum_{s \neq j} \widetilde{x}_s y_s + \bar{x}_s y_s + \bar{x}_j y_j \right) \right.$$

$$\left. y_{j+1} \cdots y_l \right\} =$$

$$= \sum_{j=1}^{l} (-1)^{j-1} \Big\{ x_1 \cdots \widehat{x}_i \cdots x_k y_1 \cdots y_{j-1} x^{(j)} y_{j+1} \cdots y_l + x_1 \cdots$$

$$\widehat{x}_i \cdots x_k y_1 \cdots y_{j-1} y_{s_j t_j} y_{j+1} \cdots y_l + x_1 \cdots \widehat{x}_i \cdots x_k y_1 \cdots y_{j-1} \left(\sum_{s \neq j} \widetilde{x}_s y_s \right) y_{j+1} \cdots$$

$$y_l + x_1 \cdots \widehat{x}_i \cdots x_k y_1 \cdots y_{j-1} \bar{x}_j y_j y_{j-1} \cdots y_l \Big\} = 0$$

Observe that

$$x_1 \cdots \widehat{x}_i \cdots x_k y_1 \cdots y_{j-1} x^{(j)} y_{j+1} \cdots y_l \in \Lambda^{k+1,l} = \{0\}$$

$$x_1 \cdots x_i \cdots x_k y_1 \cdots y_{j-1} y_{s_j t_j} y_{j+1} \cdots y_l \in \Lambda^{k-1,l+1} = \{0\}$$

$$x_1 \cdots \widehat{x}_i \cdots x_k y_1 \cdots y_{j-1} \left(\sum_{s \neq j} \widetilde{x}_s y_s \right) y_{j+1} \cdots y_l = 0.$$

The first two equalities are obvious, the third follows, since $s \in \{1, \cdots, j-1, j+1, \cdots, l\}$. Finally

$$x_1 \cdots \widehat{x}_i \cdots x_k \left(\sum_{j=1}^{l} \bar{x}_j \right) y_1 \cdots y_{j-1} y_j y_{j+1} \cdots y_l = 0$$

which means that

$$\sum_{j=1}^{l} \bar{x}_j \in Ann_{\Lambda^{1,0}}(x_1 \cdots \widehat{x}_i \cdots x_k)$$

Since the above argument is valid for all i,

$$\sum_{j=1}^{l} \bar{x}_j \in \cap_{i=1}^{k} Ann_{\Lambda^{1,0}}(x_1 \cdots \widehat{x}_i \cdots x_k)$$

Since the sum of $\bar{x}_j$ is of degree 1, (3) follows. Lemma 1 is proved.

Lemma 2. *Under the conditions of Lemma 1 the inclusion holds:*

$$d(\Lambda^{k-2,l}) \subset \Lambda^{k,l-1}. \tag{5}$$

Proof. Take the element $y_1 \cdots y_l$. Then

$$d(y_1 \cdots y_l) = \sum_{i=1}^{l} (-1)^{i-1} y_1 \cdots dy_i \cdots y_l =$$

$$= \sum_{i=1}^{l} (-1)^{i-1} y_1 \cdots y_{i-1} \left(x^{(i)} + y_{s_i t_i} + \sum_{s \neq i} \widetilde{x}_s y_s + \bar{x}_i y_i \right) y_{i+1} \cdots y_l =$$

$$= \sum_{i=1}^{l} (-1)^{i-1} y_1 \cdots y_{i-1} \bar{x}^{(i)} y_{i+1} \cdots y_l + \sum_{i=1}^{l} (-1)^{i-1} y_1 \cdots y_{i-1} \bar{x}_i y_i y_{i+1} \cdots y_l =$$

$$= \sum_{i=1}^{l} (-1)^{i-1} y_1 \cdots y_{i-1} x^{(i)} y_{i+1} \cdots y_l \in \Lambda^{2,l},$$

because

$$y_1 \cdots y_{i+1} y_{s_i t_i} y_{i+1} \cdots y_l \in \Lambda^{0,l+1} = \{0\},$$

$$y_1 \cdots y_{i-1} \left(\sum_{s \neq i} \widetilde{x}_s y_s \right) y_{i+1} \cdots y_l = 0$$

and

$$\sum_{i=1}^{l} (-1)^{2(i-1)} \bar{x}_i y_1 \cdots y_{i-1} y_i y_{i+1} \cdots y_l = \left(\sum_{i=1}^{l} \bar{x}_i \right) y_1 \cdots y_l = 0$$

from Lemma 1. Lemma 2 is proved.

Lemma 3. *Let $(\Lambda X \otimes \Lambda Y, d)$ satisfy the conditions of Lemma 7. Assume in addition that the Poincaré duality holds. If*

$$H^1(\Lambda X \otimes \Lambda Y, d) = \Lambda^{1,0}, \tag{6}$$

then

$$H^{n-1}(\Lambda X \otimes \Lambda Y, d) = \Lambda^{k-1,l}. \tag{7}$$

Proof. From the Poincaré duality

$$dim H^{n-1}(\Lambda X \times \Lambda Y) = k.$$

From (4)

$$Z^{n-1}(\Lambda X \otimes \Lambda Y) = \Lambda^{k-1,l} + \Lambda^{k,l-1}$$

and by comparing the degrees (the calculation is straightforward, although it is necessary to use the *minimality* of $(\Lambda Y, \delta)$ which guarrantees that $y_{t_j s_j}$ do not contain expressions with y_j):

$$d(\Lambda^{k-1,l-1} \subset \Lambda^{k,l-1}, d(\Lambda^{k,l-2}) \subset \Lambda^{k,l-1}$$

From Lemma 2 (Formula (5))

$$d(\Lambda^{k-2,l}) \subset \Lambda^{k,l-1}$$

and therefore, the subspace of coboundaries $B^{n-1} \subset \Lambda^{k,l-1}$. Thus,

$$dim H^{n-1}(\Lambda X \otimes \Lambda Y) = dim(\Lambda^{k-1,l} + \Lambda^{k,l-1}) - dim B^{n-1} = k$$

Therefore, since $dim\,\Lambda^{k,l-1} = l$, $dim\,\Lambda^{k-1,l} = k$, the assumption $B^{n-1} \neq \Lambda^{k,l-1}$ would imply

$$dim H^{n-1} = dim Z^{n-1} - dim B^{n-1} = k + l - dim B^{n-1} > k.$$

Thus

$$B^{n-1} = \Lambda^{k,l-1} \tag{8}$$

and Lemma 3 is proved.

Proof of the Theorem

We begin the proof showing that there exists a graded differential algebra which is a *model for* G/Γ *and which satisfies the condition (i) of the Theorem.* Then we prove that this algebra possesses algebraic properties (ii)-(v) stated in the Theorem.

Let N be the nilradical of G. From the Mostow theorem [M], $N \subset \Gamma$ is a lattice in N. The latter fact is equivalent to the closeness of $N\Gamma$ in G. Therefore, one can construct the "double fibration"

$$\Gamma N = N/\cap\Gamma \to G/\Gamma \to G/\Gamma N.$$

Since $G/\Gamma = G \times_N (N/N \cap \Gamma)$, the fiber of the locally trivial bundle above possesses the *path-connected* structural group N. From the Serre theorem [Se], $H^*(N/N \cap \Gamma)$ is a trivial $\pi_1(G/N \cap \Gamma)$−module. Since all cohomology algebras here are of a finite type, the Thomas theorem (Section 2) is applicable and the *rational model* $A^*(G/\Gamma)$ *of* G/Γ *is a twisted tensor product*

$$\mathcal{M}_{G/\Gamma N} \otimes_\tau \mathcal{M}_{N/N\cap\Gamma}.$$

It is well known that $G/\Gamma N$ is a torus and since $\Gamma \cap N$ is a lattice, the fiber of the considered fibration is a *compact nilmanifold*. Finally,

$$\mathcal{M}_{G/\Gamma N} = (\Lambda X, d = 0), \mathcal{M}_{N/N\cap\Gamma} = (\Lambda L(N)^*, \delta)$$

(see Section 2) and

$$(A^*(G/\Gamma), d) = (\Lambda X, 0) \otimes_\tau (\Lambda Y, \delta),$$

where (i) and (ii) are satisfied, $Y = L(N)^*$. Suppose that $(\Lambda X \otimes \Lambda Y, d)$ satisfies the conditions of Lemma 1 and dy_j are linearly independent. Then (6) holds, since $d\mid_X = 0$ and $d\mid_Y$ is one-to-one. If some dy_j are linearly dependent, from the obvious base change one can assume that for the new basic vectors $\bar{y}_j$, $d\bar{y}_j = 0$ and thus $\bar{y}_j \in X$. Therefore one can assume the linear independence of dy_j without loss of generality and (7) follows. Now, the condition (iii) follows from Lemma 3.

To prove (iv)-(v), take $y^* \in Y^*$ and observe that

$$(i(y^*)\omega)\omega^{m-1} = \frac{1}{m} i(y^*)\omega^m \in \Lambda^{k,l-1},$$

since $\omega^m \in \Lambda^n = \Lambda^{k+l}$. From (8) $i(y^*)\omega^m$ is a coboundary. Therefore,

$$i(y^*)\omega \notin \Lambda^{1,0}$$

because otherwise the hard Lefschetz condition and Lemma 3 would give a contradiction. Thus there exists $\bar{y}^* \in Y^*$ such that $\omega(y^*, \bar{y}^*) \neq 0$ and therefore $\omega\mid_{Y^*\times Y^*}$ is non-degenerate. Observe that using non-degeneracy, one can choose X^* to be $\omega-$orthogonal to Y^*. Using the appropriate identifications of exterior algebra elements and alternating forms, one obtains the form of the Lefschetz element, that is (iv) and, as a consequence, (v). The Theorem is proved.

Proof of the Corollary

Analyzing the proof of the Theorem one can notice that conditions (iii)-(v) *are valid for any twisted tensor product satisfying the Lefschetz conditions and* (ii). Following [BG2] we assign to G/Γ a model $\bar{A}^*(G/\Gamma)$ which possesses properties (ii)-(v) of $A^*(G/\Gamma)$, although this new model in general differs from the latter. We can observe that for completely solvable Lie groups

$$H^*(G/\Gamma) \simeq H^*(L(G)) = H^*(\Lambda L(G)^*, \delta)$$

from Hattori's theorem [H]. Since

$$(\Lambda L(G)^*, \delta) \simeq (\Omega^{inv}_{DR}(G), d),$$

the cohomology class of a Kahlerian form has a representative $\omega \in (\Lambda^2 L(G)^*, \delta)$. Since $\omega^m \neq 0$ and ω is an alternarive 2-form on $L(G)$, it is non degenerate. Therefore, $(\Lambda L(G)^*, \delta)$ *can be taken as a model of* G/Γ.

Take $\mathcal{N} = [L(G), L(G)]$ and decompose $L(G)^*$ as a vector space sum

$$L(G)^* = \mathcal{A}^* \oplus \mathcal{N}^*,$$

where

$$\mathcal{A}^* = \{\theta \in L(G)^* \mid \theta(\mathcal{N}) = 0\}, \mathcal{N}^* = \{\alpha \in L(G)^* \mid \alpha(\mathcal{A}) = 0\}$$

(here $\mathcal{A}$ is an arbitrary complement to $\mathcal{N}$). Now, taking into account the equalities (the first follows from (2))

$$\delta\beta(U, V) = -\beta([X, Y]), \mathcal{N} = [L(G), L(G)]$$

we have

$$\delta(\mathcal{A}) = 0.$$

Since $\mathcal{N} \subset$ in the nilradical of $L(G)$, the ideal $\mathcal{N}$ is nilpotent and therefore

$$(\Lambda\mathcal{N}^*, \delta \mid_{\mathcal{N}^*})$$

is a *minimal graded differential algebra.* Therefore,

$$(\bar{A}^*(G/\Gamma) = (\Lambda L(G)^*, \delta) = (\Lambda\mathcal{A}^*, \delta = 0) \otimes_\tau (\Lambda\mathcal{N}^*, \delta \mid_{\mathcal{N}^*})$$

(that is, the model for G/Γ can be represented as a twisted tensor product satisfying the condition (ii) of the Theorem, the Lefschetz condition and the conditions of Lemmata 1-3 (the latter is guaranteed by the minimality of the second term in the twisted tensor product). Finally, $(\bar{A}^*(G/\Gamma), d)$ satisfies the properties (i)-(v) of the Theorem and as a consequence, the properties (ii), (iv) and (v) of the Corollary.

Remark. These properties were proved in [BG2] directly for the particular case of $(\Lambda L(G)^*, \delta)$ with completely solvable G.

To finish the proof it is enough to repeat the argument in [BG2] (parts (i), (iii) and (vi). Here, of course, the proof is based on the particular choice of $(A^*(G/\Gamma), d)$. For example, the commutativity of $\mathcal{A}$ follows from (v) of the Theorem and the formula

$$0 = \delta\omega(A, B, V) = -\omega([A, B], V) + \omega([A, V], B) - \omega([B, V], A), A, B \in \mathcal{A}, V \in \mathcal{N}$$

(see the proof in [BG2] for details). The Corollary is proved.

4 Examples

It is rather easy to produce examples of solvmanifolds which are not completely solvable. For instance, any 4-dimensional real solvable Lie group possessing lattices is either nilpotent, or is of one of two forms below:

a) $G = R \times_\varphi R^3$, where $\varphi : R \to GL_3(R)$ is a homomorphism satisfying the condition $\varphi(1) \in GL_3(Z)$, or

b) $G = R \times_\varphi N_3$, where N_3 is a maximal unipotent subgroup in $SL_3(R)$ and $\varphi : R \to Aut(N_3)$ is a homomorphism such that $\varphi(1)$ preserves the subgroup $N_3(Q)$ in $SL_3(Q)$ of matrices with rational entries in N_3 (see [VGS] and [Au]).

Following this general description, we produce the following example.

Example 1. *Let G and φ be as in a) above and define ϕ by the formula*

$$\varphi(t) = \begin{pmatrix} \cos t5\pi & \sin t5\pi & 0 \\ -\sin t5\pi & \cos t5\pi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then there exists a lattice Γ in G such that G/Γ is a compact symplectic solvmanifold which is not diffeomorphic to a manifold. This solvmanifold is not completely solvable and has no Kählerian structure.

To prove this assertion, observe that the Lie algebra of G is of the form

$$L(G) = Span(A, X, Y, Z), [A, X] = -Y, [A, Y] = X, [A, Z] = 0$$

and all other brackets are zero. Since

$$L(G) = \mathcal{A} \times_{\phi_*} \mathcal{B}, \mathcal{A} = Span(A), \mathcal{B} = Span(X, Y, Z),$$

where $\mathcal{A}$ and $\mathcal{B}$ are abelian, $L(G)$ is solvable. It is non-nilpotent, since $ad(A)^n(X) \neq 0$ for any n. Obviously $ad(A)$ has imaginary eigenvalues and G is not completely solvable. Since $\phi(1)$ has integer entries, G possesses a lattice. If we generate the lattice in R, say, by $\frac{1}{5}$, we obtain a lattice Γ which is a non-nilpotent group extending Z^4 by a group $Z/5Z$. And therefore G/Γ is not diffeomorphic to a nilmanifold.

Observe that G/Γ is *symplectic*. To show this, consider the complex $(\Lambda L(G)^*, \delta)$. From Section 2 one can write

$$(\Lambda L(G)^*, \delta) = (\Lambda(a, x, y, z), \delta),$$

$$\delta(a) = 0, \delta(x) = -ay, \delta(y) = ax, \delta(z) = 0.$$

One can check that

$$\omega = az + xy$$

is a symplectic form on G/Γ (one identifies $\Lambda L(G)^*$ with left-invariant forms on G and projects this forms on G/Γ).

We cannot use the Hattori theorem to prove the non-existence of a Kählerian structure, since G/Γ is not completely solvable. To prove the latter assertion we appeal to the Kodaira classification of complex surfaces. Assuming G/Γ to be Kählerian and hence complex one performs the following argument. Since G/Γ is parallelizable, its Euler characteristic and the first Pointriagin number vanish, which implies the vanishing of the Chern classes c_1 and c_2. Therefore, the geometric genus $p_g(G/\Gamma) = 0$. Then the Kodaira classification of complex surfaces implies G/Γ to be algebraic [BPV]. One can notice that if $\Gamma = A \times_\varphi B$ is a semidirect product of two abelian groups determined by the homomorphism $\varphi : A \to Aut(B)$, then $\Gamma/[\Gamma, \Gamma]$ consists of elements of the form

$$(e, b \cdot \varphi(a)(b_1) \cdot \varphi(a_1)(b^{-1}) \cdot b_1^{-1})$$

Calculating this expression for Γ and φ determining our particular solvmanifold, one can check directly that

$$rank(\Gamma/[\Gamma, \Gamma]) = 2,$$

which implies that $b_1(G/\Gamma) = 2$. From the Kodaira classification table [BPV] (p. 188) G/Γ cannot be a surface of general type. Then, since the geometric genus vanishes and the second Betti number is 2, one obtains

$q(G/\Gamma) = h^1(\mathcal{O}_{G/\Gamma}) = 1$. Using the same argument as in [FM] (p. 30), one eliminates ruled surfaces. The only possibility remains to consider is that of hyperelliptic surfaces. In the latter case,

$$G/\Gamma = (B \times F)/H,$$

where B and F are elliptic curves and H is a finite group acting freely on $B \times F$ and the genera $g(B), g(F) \geq 1$. Since H is always of the form $H = Z/mZ, m = 2,3,4,6$ [FM], the possible fundamental group of $(B \times F)/H$ is an extension of Z^4 by Z/mZ with $m = 2,3,4,6$ and from our construction

$$\pi_1(G/\Gamma) \neq \pi_1((B \times F)/H)$$

which implies the proof.

Example 2.

Proposition 1. *Let $M = G/\Gamma$ be a compact 4-dimensional solvmanifold. If M admits a Kähler structure, then M has the rational homotopy type either of a torus T^4, or of $S^2 \times T^2$.*

Proof. Assuming M to be Kähler and hence complex, we appeal to the Kodaira classification of complex surfaces. From the table in [BPV], either $b_1(M) = 4$, or $b_1(M) = 2$, since $b_1(M) = 1$ and $b_1(M) = 3$ are eliminated by the Kähler condition, and $b_1(M) = 0$ cannot be realized by a compact solvmanifold of the type a) or b) above.

Now, apply the Theorem proved in this paper. Applying the *proof* of it to the Lie groups G of the form a) b), one obtains

$$(A^*(G/\Gamma), d) = (\Lambda(a,b,x,y), d),$$

and from parts (iii) and (v) it follows that either

$$d(a) = d(b) = d(x) = d(y) = 0,$$

or

$$d(a) = d(b) = 0, d(x) \neq 0, d(y) \neq 0.$$

We claim now, that part (iv) shows that the differential d in $(A^*(G/\Gamma), d)$ *up to isomorphism* is either of the form

$$d(a) = d(b) = 0, d(x) = -ay, d(y) = ax,$$

or

$$d(a) = d(b) = 0, d(x) = ax, d(y) = -ay.$$

To prove the latter assertion, it is enough to make on obvious variable change, taking into consideration the identities

$$d^2 = 0, \text{ and } d(xy) = 0 \tag{9}$$

(the latter follows from (iv), since the Lefschetz element is necessarily of the form $ab + xy$. Calculations are straightforward, therefore we won't reproduce them. Note only, that it is convenient to rewrite d, say, in the form

$$d(x) = a_{11}x + a_{12}y, d(y) = a_{21}x + a_{22}y, a_{ij} \in Span(a, b)$$

and write explicitly the conditions on a_{ij} following from (9). Then the appropriate change of variables can be seen easily.

Now, it is enough to notice that the cohomology algebra $H^*(\Lambda(a, b, x, y), d)$ in both cases is generated by elements $a, b, ab, xy, axy, bxy, abxy$. Then some straghtforward computation shows that $H^*(\Lambda(a, b, x, y), d)$ has the minimal model given by the formula

$$(\mathcal{M}, d_1) = (\Lambda(a, b,) \otimes R[X_1] \otimes \Lambda(y_1), d_1),$$

$$d(a) = d(b) = d(X_1) = 0, d(y_1) = X_1^2, deg(X_1) = 2, deg(y_1) = 3.$$

Since we assumed G/Γ to be Kählerian, its minimal model is a *formal consequence* of the cohomology algebra $H^*(G/\Gamma)$ [DGMS]. Then, the uniqueness theorem for the minimal model (which *does not depend* on the nilpotency condition) implies.

$$\mathcal{M} \simeq \mathcal{M}_{G/\Gamma} \simeq \mathcal{M}_{T^2 \times S^2}.$$

The proposition is proved.

Example 3.

Many more examples can be constructed. We restrict ourselves to indicating one more example in dimension 8 which is a modification of an example in [FLS], but which also cannot be handled by the techniques of [FLS].

Proposition 2. *Let $L(G)$ ba a Lie algebra defined as follows*

$$L(G) = Span(A, B, X_1, X_2, X_3, X_4, X_5, X_6),$$

$$[A, X_1] = X_1, [A, X_2] = -X_2, [A, X_3] = X_3, [A, X_4] = -X_4$$

$$[A, X_5] = \alpha X_6, [A, X_6] = -\alpha X_5, [B, X_3] = X_1, [B, X_4] = X_2, \alpha \in R$$

(the other brackets are assumed to be zero).

Let G be a simply-connected Lie group corresponding to $L(G)$. Then:

(i) G is a solvable non-nilpotent and not completely solvable Lie group;

(ii) G contains a lattice Γ for some particular α;

(iii) the homogeneous space G/Γ is a compact symplectic solvmanifold with no Kähler structure

The proof of this proposition is given in [Tr] and is based on the *quite different argument* combined with a long calculation using *quadruple Massey products* [K]. We omit it here.

Remark. Our examples are analogous to Example 1 in [BG2] and Theorems 1-3 in [FG] when G/Γ is not completely solvable.

Examples presented here give some evidence for the following

Conjecture. *A compact solvmanifold G/Γ of $(R)-$type is Kählerian if and only if it is a homotopy torus.*

By definition [VGS], a solvmanifold G/Γ is of $(R)-$type, if for any $V \in L(G)$ all linear operators $adV : L(G) \to L(G)$ have only imaginary eigenvalues.

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