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On Witt rings of function fields of real analytic surfaces and curves.

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Abstract

Let V be a paracompact connected real analytic manifold of dimension 1 or 2, i.e. a smooth curve or surface. We consider it as a subset of some complex analytic manifold $V_{\mathbb{C}}$ of the same dimension. Moreover by a prime divisor of V we shall mean the irreducible germ along V of a codimension one subvariety of $V_{\mathbb{C}}$ which is an invariant of the complex conjugation. This notion is independent of the choice of the complexification $V_{\mathbb{C}}$. In the one-dimensional case prime divisors are just points, in the two-dimensional - analytic curves or elliptic points (intersections of two conjugated complex analytic curves). Every such divisor induces a discrete valuation on the field $\mathcal{M}$ of meromorphic functions on V - the order of the zero or minus the order of the pole of the function. Therefore it induces the so called residue homomorphisms (first and second) of the Witt group of the field $\mathcal{M}$ to the Witt group of the residue field - the function field of the divisor.

The main goal of this paper is to show that the intersection of kernels of all second residue homomorphisms associated to prime divisors is isomorphic to the Witt group of the Riemannian bundles on V.

As an example of an application of this result we provide the new proof of the Artin-Lang property for one and two dimensional real analytic manifolds (both compact and noncompact), which is neither based on the description of all posible orderings of the field of meromorphic functions nor on the compactification of the variety.

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1 Introduction

Let V be a paracompact connected real analytic manifold of dimension 1 or 2, i.e. a smooth curve or surface. We consider it as a subset of some complex analytic manifold $V_{\mathbb{C}}$ of the same dimension. Moreover by a prime divisor of V we shall mean an irreducible germ of codimension one subvariety of $V_{\mathbb{C}}$ along V which is invariant under the complex conjugation. This notion is independent of the choice of the complexification $V_{\mathbb{C}}$. In one-dimensional case prime divisors are just points, in two-dimensional - analytic curves or elliptic points (intersections of two conjugated complex analytic curves). Every such divisor induces a discrete valuation on the field $\mathcal{M}$ of meromorphic functions on V - the order of the zero or minus the order of the pole of the function. Therefore it induces the so called residue homomorphisms (first and second) of the Witt group of the field $\mathcal{M}$ to the Witt group of the residue field - the function field of the divisor.

The main goal of this paper is to show that the intersection of the kernels of all second residue homomorphisms associated to prime divisors is isomorphic to the Witt group of the Riemannian bundles on V.

As an example of an application of this result we provide a new proof of the Artin-Lang property for one and two dimensional real analytic manifolds.

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2 Notation

2.1 Preliminaries

We consider in this paper the Witt rings over integral domains and over ringed spaces as defined by Knebusch (see [11, 17] or [13]). The Witt ring is a Grothendieck ring of nondegenerated bilinear forms on projective modules or respectively of Riemannian bundles modulo the metabolic (split) ones.

Let $a_1, \dots a_n$ be nonzero elements of a field K. We recall that

$\langle a_1, \dots a_n \rangle$ denotes the class corresponding to the following quadratic form:

$$a_1 T_1^2 + \dots + a_n T_n^2.$$

The tensor products

$$\langle 1, a_1 \rangle \otimes \dots \otimes \langle 1, a_n \rangle$$

are called n-fold Pfister forms and are denoted in the abbreviated form

$$\langle\langle a_1, \dots a_n \rangle\rangle.$$

2.2 Witt rings of vector bundles

Let E be an analytic vector bundle over a connected real analytic manifold V. We say that E is Riemannian if it is provided with an analytic inner product i.e. an analytic section b of the bundle $E^* \otimes E^*$ such that for every point $x \in V$ b_x is a symmetric regular bilinear form. b is called positive definite or respectively negative definite if such is b_x for every point x.

For example if (E, b) is positive definite then $(E, -b)$ is negative definite. We shall abbreviate $(E, -b)$ to $-E$.

We remark that, in contrast to the complex analytic category, every real analytic vector bundle of finite rank is generated by finite number of global sections. Therefore there is an isomorphism between the Witt ring of Riemannian vector bundles on V and the Witt ring of regular bilinear forms on projective modules over the ring of global real analytic functions on V. We shall denote both objects by $W(\mathcal{O}(V))$.

The Riemannian bundle (E, b) is called

• hyperbolic if it splits, i.e. there exists an analytic bundle E_1 such that

$$E = E_1 \oplus E_1^*, \quad b((\alpha, \beta), (\alpha_1, \beta_1)) = \beta(\alpha_1) + \beta_1(\alpha),$$

• metabolic if it is stable hyperbolic i.e. there exit such hyperbolic bundles (H_1, h_1) and (H_2, h_2) that

$$E \perp H_1 = H_2, \quad h_2((\alpha, \beta), (\alpha_1, \beta_1)) = b(\alpha, \alpha_1) + h_1(\beta, \beta_1).$$

2.3 Homomorphisms of Witt rings

We recall the basic facts.

Every ring homomorphism

$$i : K \longrightarrow L$$

induces the ring homomorphism of Witt rings

$$i^* : W(K) \longrightarrow W(L),$$

$$i^*(M,b) = (M \otimes_K L, i \circ b), \quad i \circ b((\alpha \otimes a), (\alpha_1 \otimes a_1)) = aa_1 \cdot i(b(\alpha, \alpha_1)).$$

Analogically, if (E, b) is a Riemannian vector bundle, which is generated by global sections, and i is an inclusion of the ring $\mathcal{O}(V)$ into its field of quotients (the field of meromorphic functions) $\mathcal{M}(V)$ then $i^*(E,b) = (\Gamma(E) \otimes_{\mathcal{O}(V)} \mathcal{M}(V), i \circ b)$.

The so called residue homomorphisms are other examples of mappings of Witt rings. Let

$$v : K \longrightarrow \Gamma \cup \{\infty\}$$

be a discrete valuation on a field K. Then

$$R = \{a \in K : v(a) \geq 0\}$$

is a discrete valuation ring with the maximal ideal

$$m = \{a \in K : v(a) > 0\}.$$

Any generator π of the ideal m is called the uniformizer of the valuation. Every element of the field K may be uniquely written as product $\pi^k a$, where $k \in Z$, and $a \in R \backslash m$. The first and second residue homomorphism are defined as follows:

$$\partial^i : W(K) \longrightarrow W(R/m), \qquad i = 1, 2,$$

$$\partial^i \langle \pi^k \cdot a \rangle = \begin{cases} \langle \overline{a} \rangle & \text{if } k + i \text{ is odd} \\ 0 & \text{otherwise} \end{cases},$$

where $\overline{a}$ is an image of a in the residue field R/m.
We remark that the residue homomorphisms commute with the multiplication by elements from $W(R)$ and furthermore their kernels do not depend on the choice of the uniformizer π.

3 Main results

Let V be a paracompact connected real analytic manifold of dimension 1 or 2 and let $\mathcal{P}$ be the set of all its prime divisors

Theorem 1. *The following sequence of Witt groups is exact*

$$0 \to W(\mathcal{O}(V)) \xrightarrow{i^*} W(\mathcal{M}(V)) \xrightarrow{\bigoplus \partial_p} \bigoplus_{p \in \mathcal{P}} W(\mathcal{M}(p))$$

where ∂_p is a second residue homomorphism associated to the prime divisor p and $\mathcal{M}(p)$ is the field of meromorphic functions on p.

We remark that $\mathcal{M}(p)$ is isomorphic to
- the field of real numbers if p is a point,
- the field of convergent Laurent series in one variable over the field of complex numbers if p is an elliptic point (see lemma 4),
- the field of global meromorphic functions on the real line or on the circle if p is an analytic curve.

Let $hdim\ V$ be the homotopical dimension of V i.e. the minimum of the dimensions of CW-complexes which are homotopicaly equivalent to V. If V is one-dimensional then

$$hdim\ V = \begin{cases} 0 & \text{if } V = \mathbb{R}, \\ 1 & \text{if } V = S^1. \end{cases}$$

If V is two-dimensional but noncompact then

$$hdim\ V = \begin{cases} 0 & \text{if } V = \mathbb{R}^2, \\ 1 & \text{otherwise.} \end{cases}$$

If V is two-dimensional and compact then $hdim\ V = 2$.

Theorem 2. *Let $f_1, \ldots, f_k$ be a set of real analytic functions on V.*
- *If $k > hdim V$ and the equivalence class of the Pfister form $\langle\langle f_1, \ldots, f_k \rangle\rangle$ belongs to $i^*(W(\mathcal{O}(V)))$ then either it is hyperbolic or it is equivalent to the form $2^k\langle 1 \rangle$.*
- *If $V = \mathbb{R}$ or $k > dim\ V$ and the equivalence class of the Pfister form $\langle\langle f_1, \ldots, f_k \rangle\rangle$ is a torsion element of $W(\mathcal{M}(V))$ then it is hyperbolic.*

Corollary 1. *Let $f_1, \ldots, f_k$, $k \geq hdim\ V$, be a set of real analytic functions on V such that in every point of V at least one of them is nonpositive, then the Pfister form $\langle\langle 1, f_1, \ldots, f_k\rangle\rangle$ is hyperbolic. Moreover if $V = \mathbb{R}$ or $k > \dim V$ then also $\langle\langle f_1, \ldots, f_k\rangle\rangle$ is hyperbolic.*

We remark, that if a Pfister form $\langle\langle a_1, \ldots, a_k\rangle\rangle$ is hyperbolic then there exist such $T_{j_1,\ldots,j_k}$ that

$$-1 = T_{1,\ldots,0}^2 a_1 + T_{0,1,\ldots,0}^2 a_2 + \ldots + T_{1,\ldots,1}^2 a_1 \ldots a_k$$

i.e. -1 belongs to the preordering generated by a_i's. Therefore it follows immediately from the corollary that if in every point of V at least one of f_i's is nonpositive then -1 belongs to the preordering generated by f_i's hence in every ordering of the field $\mathcal{M}$ at least one f_i is negative. Thus we obtain, as a direct consequence, Artin-Lang property for the ring of global real analytic functions on V. For other proofs of this fact, which are based on quite different ideas, the reader is referred to [9, 15] for V compact and [1, 4] for V noncompact.

Corollary 1 gives also some estimations on the number of squares necessary to represent a positive definite analytic function. For more exact calculations which are also based on the theory of vector bundles the reader is referred to [8].

The above results are proved in sections 7 (theorem 1), 8 (theorem 2) and 9 (corollary 1). The proofs are based on "tools" developed in sections 4,5 and 6.

Next we draw some more consequences of the above theorems and describe the structure of the $W(\mathcal{O}(V))$ in more detail. Let $sgn(E, b)$ be the signature of the bilinear form b_x for any given point $x \in V$.

Corollary 2. *If $hdim\ V = 0$ then the mapping*

$$sgn : W(\mathcal{O}(V)) \longrightarrow \mathbb{Z}$$

is a ring isomorphism.

Let $|det(E, b)|$ be the absolute value of the determinant of the bilinear form $i \diamond b$. We introduce the ring structure on the direct sum of $\mathbb{Z}$ and the factorgroup of the multiplicative group of nonzero functions that are locally squares and the multiplicative group of functions that are squares

$(\mathcal{O}(V)^*_{lsq}/\mathcal{O}(V)^2)$ (the subscript lsq stands for *locally a square*) by the rules

$$(m,a)+(n,b)=(m+n,ab),$$
$$(m,a)\cdot(n,b)=(mn,a^{|n|}b^{|m|}).$$

(Compare the construction of the ring $V(Q)$ and the ring homomorphism r in [5] s.1.)

Corollary 3. *If $hdim\ V=1$ then the mapping*

$$(sgn,|det|): W(\mathcal{O}(V)) \longrightarrow \mathbb{Z}\oplus(\mathcal{O}(V)^*_{lsq}/\mathcal{O}(V)^2)$$

is a ring isomorphism.

Corollary 4. *If $hdim\ V=2$ then the mapping*

$$(sgn,|det|): W(\mathcal{O}(V)) \longrightarrow \mathbb{Z}\oplus(\mathcal{O}(V)^*_{lsq}/\mathcal{O}(V)^2)$$

is a ring epimorphism, which kernel consists of all Riemannian bundles E, such that either the equivalence class of $i^*(E)$ contains a Pfister form $\langle\langle 1,-g\rangle\rangle$, where g is a positive definite analytic function which cannot be represented as a sum of two squares, or it is hyperbolic.

4 Splitting Lemma

We start with the following fact (compare [13] §V.2).

Lemma 1. *Every analytic vector bundle E over the paracompact connected real analytic manifold M with a indefinite Riemannian product is an orthogonal direct sum of two vector bundles E^+ and E^- with definite Riemannian products.*

Proof. Let $\xi_1,\dots,\xi_m$ be a set of global sections of E which generate each stalk.

We consider the corresponding Gram matrix:

$$A=((\xi_i,\xi_j))_{i,j=1,\dots,m},$$

where $(.,.)$ is the Riemannian product i.e. the bilinear form on sections.

The coefficients of the matrix A are analytic functions on the manifold M. We denote by $A(x)$ the matrix which coefficients are their values at the point $x \in M$. M is connected hence the rank and the signature of $A(x)$ do not depend on the point x. Indeed, they are equal respectively to the rank of the bundle E and to the signature of the Riemannian product. Furthermore, since $A(x)$ is symmetric hence its eigenvectors span the whole space $\mathbb{R}^m$ and the above is valid also for the number of positive and negative eigenvalues (counted with multiplicities). Thus the characteristic polynomial $P(\lambda) = det(A - \lambda I)$ is a product of three polynomials with continuous coefficients

$$P(\lambda) = P^+(\lambda)P^-(\lambda)\lambda^k, \quad k = m - rank\ E,$$

where $P^+(\lambda)$ (resp. $P^-(\lambda)$) has only positive (resp. negative) roots. Furthermore, using Weierstrass preparation theorem -[7] s.4.1 (or just Hensel lemma -[7] s.5.6), one can show that the coefficients of both $P^+(\lambda)$ and $P^-(\lambda)$ are analytic at every point of V.

Let $W^0(x)$, $W^+(x)$ and $W^-(x)$ be the linear subspaces of $\mathbb{R}^m$ spanned by the eigenvectors corresponding respectively to zero, positive and negative eigenvalues. They give rise to the decomposition of the trivial bundle over M:

$$M \times \mathbb{R}^m = W^0 \oplus W^+ \oplus W^-.$$

Indeed, W^0, W^+ and W^- are kernels of endomorphisms of the trivial bundle $M \times \mathbb{R}^m$ induced respectively by A, $P^+(A)$ and $P^-(A)$ (since for every x $A(x)$ has a diagonal Jordan form hence $ker\ A^k = ker\ A$).

Furthermore our Riemannian bundle E is isomorphic to bundle $W^+ \oplus W^-$ with the product given by the matrix A. We remark that W^+ is orthogonal to W^- because the eigenvectors corresponding to the distinct eigenvalues are orthogonal.

Lemma 2. *If the manifold M has a homotopical type of a point then every Riemannian bundle E over M is either hyperbolic or an orthogonal sum of a hyperbolic bundle H and a trivial bundle E_1 with the definite Riemannian product given either by the identity or by minus identity matrix,*

$$E = H \perp E_1.$$

Proof. If the analytic manifold is contractible to a point then every analytic vector bundle is trivial i.e. it is generated by nowhere vanishing global analytic sections. Furthermore if the vector bundle has a definite Riemannian product then a square of a nowhere vanishing global analytic section is a nowhere vanishing global analytic function. Therefore we can choose an orthonormal set of global sections for both components E^+ and E^-, $\xi_1^+, \dots, \xi_k^+$ and $\xi_1^-, \dots, \xi_l^-$. The sections ξ_i^+ and ξ_i^-, $i = 1, \dots, min(k, l)$, generate the hyperbolic bundle while the rest of them generate the trivial bundle with a definite product.

Lemma 3. *If the manifold M has a homotopical type of a 1-dimensional CW-complex then for every Riemannian bundle E over M with nonnegative signature there exist hyperbolic bundles H_1 and H_2 such that*

$$E \perp H_1 = H_2 \perp T \perp L,$$

where T is a trivial bundle with the Riemannian product given by the identity matrix, and L is a negatively defined one-dimensional Riemannian bundle.

Proof. If the analytic manifold is contractible to a one dimensional CW-complex then every analytic vector bundle of dimension greater than 1 has a nowhere vanishing global analytic section. Thus, using the same arguments as above, we obtain that the component E^+ has a decomposition into orthogonal sum of two positive definite Riemannian bundles $E^+ = E_1^+ \perp E_2^+$, where E_1^+ is trivial and E_2^+ one-dimensional.

We put $H_1 = E_2^+ \perp (-E_2^+)$ which obviously is hyperbolic. We note that the orthogonal sum $E_2^+ \perp E_2^+$ is a trivial bundle. Indeed the rank of the bundle is 2 hence it has a nowhere vanishing global analytic section $\xi = [\xi_1, \xi_2]$. The section $\xi^* = [\xi_2, -\xi_1]$ is also nowhere vanishing. Moreover it is orthogonal to ξ:

$$(\xi, \xi^*) = (\xi_1, \xi_2) + (\xi_2, -\xi_1) = 0.$$

Thus ξ and ξ^* are linearly independent in every point and give the trivialization of $E_2^+ \perp E_2^+$.

Therefore $E^+ \perp E_2^+$ is a trivial bundle. On the other hand $E^- \perp (-E_2^+)$ is a negative definite Riemannian bundle hence it splits

$$E^- \perp (-E_2^+) = E_1^- \perp E_2^-,$$

where E_1^- is trivial and E_2^- one-dimensional. We put $L = E_2^-$. The rank of $E^+ \perp E_2^+$ is greater than the rank of E_1^- hence we may put

$$H_2 = E_1^- \perp (-E_1^-), \quad E^+ \perp E_2^+ = (-E_1^-) \perp T.$$

It is easy to check that

$$E \perp H_1 = (E^+ \perp E_2^+) \perp (E^- \perp (-E_2^+)) = H_2 \perp T \perp L.$$

5 Injectivity of i^*

If V is compact then the ring of global analytic functions is a regular domain and the injectivity of i^* follows from the results of Ojanguren (see [14] th.17).

If V is noncompact then due to lemma 3 it is enough to consider a bundle $E = E^+ \perp E^-$ where both $E^\pm$ are one-dimensional and the positive definite component is trivial. Assume that $\Gamma(E) \otimes \mathcal{M}$ is hyperbolic then the bundle E has a global nonzero analytic section η of "length" zero. Let η^+ and η^- be the components of η in E^+ and E^-. E^+ is trivial hence $\eta^+ = f \cdot \xi$ where f is a nonzero analytic function and ξ is a nonvanishing global section of length 1. Thus

$$0 = (\eta, \eta) = (\eta^+, \eta^+) + (\eta^-, \eta^-) = f^2 + (\eta^-, \eta^-),$$

and $\frac{1}{f}\eta^-$ is a meromorphic section of square -1. Since E^- is one-dimensional Riemannian bundle hence $\frac{1}{f}\eta^-$ is analytic and nowhere-vanishing. Therefore E^- is trivial and E is hyperbolic.

6 Prime divisors

We recall that we consider the real analytic manifold V as a subset of some complex analytic manifold $V_{\mathbb{C}}$ of the same dimension, and that a prime divisor is a germ along V of a codimension one subvariety of $V_{\mathbb{C}}$ which is invariant under the complex conjugation and irreducible over $\mathbb{R}$. If V is one-dimensional then every codimension 1 irreducible subvariety is just a point hence all prime divisors are (real) points of V. If V is two-dimensional then every codimension 1 subvariety is a curve. It might be either a real analytic curve or two complex conjugated curves. In the second case the subvariety has only one real point - the intersection point of the branches. Therefore we call divisors of such type elliptic points.

Lemma 4. *The function field of an elliptic point p is isomorphic to the field of convergent Laurent series in one variable over the field of complex numbers.*

Proof. Let x be the only real point of the divisor p. Thus p is a germ of two conjugated curves at x. Since V possesses the fundamental system of Stein neighbourhoods in $V_{\mathbb{C}}$ (compare [3, 6]) hence every germ of an analytic function at x can be extended modulo the ideal $I_x(p)$ consisting of all germs vanishing at the prime divisor p to a global analytic function. Therefore

$$\mathcal{O}(V)/I(p) = \mathcal{O}_x/I_x(p).$$

Next we apply the theory of Puiseux expansions and obtain that the field of quotients of $\mathcal{O}_x/I_x(p)$ is isomorphic to the field of convergent Laurent series in one variable over the field of complex numbers.
Namely, let $f \in \mathcal{O}_x$ be a generator of the ideal $I_x(p)$. Over $\mathbb{C}$ f is a product of two conjugated germs $f = f_1 f_2$ i.e.

$$f = (Re\ f_1)^2 + (Im\ f_1)^2.$$

If $\psi(t)$ is a local analytic parametrization of the complex branch $f_1 = 0$ of p then a mapping $g \to g(\psi(t))$ induces a homomorphism of fields of quotients

$$\Psi : Q(\mathcal{O}_x/I_x(p)) \to Q(\mathbb{C}\{\{t\}\}).$$

We show that Ψ is onto.
We have

$$0 = f_1(\psi(t)) = (Re\ f_1)(\psi(t)) + i(Im\ f_1)(\psi(t))$$

hence $\Psi((Re\ f_1)/(Im\ f_1)) = -i$. Furthermore for every Laurent series $B(t)$ there exists a germ of complex meromorphic function g such that $g(\psi(t)) = B(t)$. We remark that

$$\tilde{g} = Re\ g - \frac{Re\ f_1}{Im\ f_1} Im\ g$$

is a germ of a real meromorphic function and $\tilde{g}(\psi(t)) = B(t)$ too. Hence Ψ is an isomorphism.

The other way to prove this fact leads through the normalization of the divisor p. Indeed its normalization over $\mathbb{C}$ consists of two germs of conjugated smooth complex curves. Hence over $\mathbb{R}$ it is just one germ of a smooth complex curve. See also [10] s.6 for similar results concerning formal power series.

7 Proof of Theorem 1

It is well-known that for every Riemannian bundle E, every second residue homomorphism associated to a prime divisor is vanishing on $i^*(E) = \Gamma(E) \otimes \mathcal{M}$ (compare [13] §4.1 lemma 1.3). Therefore to finish the proof of theorem 1 we have to describe the intersection of kernels of second residue homomorphisms.

Let W be a linear space over the field $\mathcal{M}$ with a bilinear symmetric product $(.,.)$.

Proposition 1. *If for every prime divisor p the corresponding second residue homomorphism is vanishing on the equivalence class of W in $W(\mathcal{M})$ then there exists a Riemannian bundle E, such that $W = \Gamma(E) \otimes \mathcal{M}$.*

Proof. We generalize the method used in [13] §IV.3. Let $e_1, \dots, e_n$ be an orthogonal basis of W. We assume that $a_i = (e_i, e_i)$, $i = 1, \dots, n$, are analytic functions. Let L denote the sublattice of W spanned by $e_1, \dots, e_n$ over the ring $\mathcal{O}$ and L^* the dual sublattice spanned by $a_1^{-1}e_1, \dots, a_n^{-1}e_n$.

Lemma 5. *For every prime divisor p there is a sublattice W_p of W over the local ring $\mathcal{O}_{I(p)}$ such that:*
i. the bilinear form $(.,.)$ restricted to W_p is regular;
ii. $L \otimes \mathcal{O}_{I(p)} \subset W_p \subset L^* \otimes \mathcal{O}_{I(p)}$;
iii. if none of the a_i's is vanishing on p then $W_p = L \otimes \mathcal{O}_{I(p)} = L^* \otimes \mathcal{O}_{I(p)}$;
iv. for every finite set of prime divisors $p_1, \dots p_t$ there exists an analytic function f such that $(\frac{1}{f}L) \subset W_p$ for $p \neq p_i$ and $W_{p_i} \subset (\frac{1}{f}L) \otimes \mathcal{O}_{I(p_i)}$.

Proof. We apply [13] ch.4 theorem 3.1 for the local ring $\mathcal{O}_{I(p)}$ and construct a sublattice W_p which fulfills the condition from point i. The second point follows from the construction used in the proof of this theorem.

The third point is an obvious consequence of the second one. Indeed, if none of the a_i's is vanishing on p then L^* is a subset of $L \otimes \mathcal{O}_{I(p)}$.

To prove the last point notice that for every prime divisor p_i there exists an analytic function f_i which vanishes only on p_i and for all j the quotients $\frac{f_i}{a_j}$ belong to the local ring $\mathcal{O}_{I(p_i)}$. Thus

$$(\frac{1}{f_i}L) \otimes \mathcal{O}_{I(p_i)} \supset L^* \otimes \mathcal{O}_{I(p_i)} \supset W_{p_i}.$$

On the other hand, let p be any prime divisor different from p_i then $\frac{1}{f_i} \in \mathcal{O}_{I(p)}$ hence $\frac{1}{f_i}L \subset W_p$.

Obviously the product $f_1 \cdot \ldots \cdot f_t$ is the required function f.

Lemma 6. *The bilinear form* $(.,.)$ *restricted to the* $\mathcal{O}$ *module*

$$Q = \bigcap_{p \in \mathcal{P}} W_p$$

is regular and moreover Q *is a module of global sections of a Riemannian bundle.*

Proof. If V is compact then the ring of global analytic functions $\mathcal{O}$ is a Krull domain and we may apply [2] §VII.4 theorem 3 to show that Q is reflexive. Next since V is two dimensional hence Q is projective.

The proof of the general case is only a bit more complicated.

Step 1. *For every prime divisor* p_0 *we have* $Q_{p_0} = Q \otimes \mathcal{O}_{I(p_0)} = W_{p_0}$.

Indeed, there is an inclusion:

$$Q_{p_0} = (\bigcap_{p \in \mathcal{P}} W_p)_{p_0} \subset \bigcap_{p \in \mathcal{P}} (W_p)_{p_0} = W_{p_0} \cap \bigcap_{p \neq p_0} W = W_{p_0}.$$

On the other hand, there is a function f such that $W_{p_0} \subset (\frac{1}{f}L) \otimes \mathcal{O}_{I(p_0)}$ and $(\frac{1}{f}L) \subset W_p$ for $p \neq p_0$, hence $(\frac{1}{f}L) \cap W_{p_0} \subset Q$ and $W_{p_0} \subset Q_{p_0}$. Thus $Q_{p_0} = W_{p_0}$.

Step 2. *The bilinear form* $(.,.)$ *restricted to the* $\mathcal{O}$ *module* Q *is regular, i.e.* Q *is self-dual.*

Let α be any $\mathcal{O}$ functional on Q. Since α can be extended to a $\mathcal{M}$ functional on W, hence there exists $\eta \in W$ such that for every $x \in W$ $\alpha(x) = (\eta, x)$.

But on the other side α can be extended to a $\mathcal{O}_{I(p)}$ functional on $Q_p = W_p$, for every prime divisor p. W_p is regular hence η belongs to W_p. Since η belongs to each W_p hence it belongs to their intersection, i.e. to Q.

Step 3. *If m is a maximal ideal consisting of all analytic functions vanishing at certain point $x \in V$ then $Q_m = Q \otimes \mathcal{O}_m$ is a free $\mathcal{O}_m$ module of rank n.*

In the one-dimensional case the maximal ideal m is a prime divisor and step 3 follows directly from step 1.

Let us assume that V is two-dimensional. There is only a finite number of prime divisors containing point x and contained in the zero set of at least one a_j. Thus repeating the arguments from step 1 we obtain that

$$Q_m = \bigcap_{x \in p \in \mathcal{P}} W_p.$$

Indeed, there is an inclusion:

$$Q_m = (\bigcap_{p \in \mathcal{P}} W_p)_m \subset \bigcap_{p \in \mathcal{P}} (W_p)_m = \bigcap_{x \in p \in \mathcal{P}} W_p \cap \bigcap_{x \notin p} W = \bigcap_{x \in p \in \mathcal{P}} W_p.$$

On the other hand, there exists a function f such that $W_p \subset (\frac{1}{f}L) \otimes \mathcal{O}_{I(p)}$ for $x \in p \in \mathcal{P}$ and $(\frac{1}{f}L) \subset W_p$ otherwise. Hence $(\frac{1}{f}L) \cap \bigcap_{x \in p \in \mathcal{P}} W_p \subset Q$ and $\bigcap_{x \in p \in \mathcal{P}} W_p \subset Q_m$. Moreover Q_m is selfdual (repeat the arguments from step 2) hence reflexive (compare [2] ch.VII §4.3). But a reflexive module over the regular local ring of dimension 2 is free (see for example [16]).

Furthermore, the rank n lattice L is contained in Q_m hence its rank is equal to n too.

Step 4. *The construction of the vector bundle E.*

Q is a submodule of the free module L^*. Thus every element of Q gives us a section of a free sheaf $\mathcal{O}_V^n$. Let $\mathcal{F}$ be a subsheaf generated by these sections. From step 3 we have that $\mathcal{F}$ is locally free hence it is a sheaf of sections of certain vector bundle E on V of rank n. From step 2 it follows that E is Riemannian.

8 Proof of Theorem 2

Let (E,b) be a Riemannian bundle such that $i \circ b = \langle\langle f_1, \ldots, f_k\rangle\rangle$. We have two possibilities to consider either the signature of b is equal to 0 or to 2^k. Indeed, the signature of b is equal to the signature of $\langle\langle f_1(x), \ldots f_k(x)\rangle\rangle$ at any point $x \in V$, at which no f_i is vanishing.

We start with the first case. The signature of E is 0, thus

$$dim\ E^+ = dim\ E^- = \frac{1}{2} dim\ E = 2^{k-1} > hdim\ V.$$

Hence both vector bundles E^+ and E^- have nonvanishing global sections, say η^+ and η^-. We have $b(\eta^+, \eta^-) = 0$, $b(\eta^+, \eta^+)$ is positive in every point, $b(\eta^-, \eta^-)$ is negative in every point. Hence we may put

$$b(\eta^+, \eta^+) = g_1^2, \quad b(\eta^-, \eta^-) = -g_2^2,$$

where g_1 and g_2 are analytic functions.

$\eta = g_2\eta^+ + g_1\eta^-$ is a nonzero section of E which is self-orthogonal, $b(\eta, \eta) = 0$. But a Pfister form which has a nontrivial zero is hyperbolic.

In the second case we may restrict ourselves to the case $hdim\ V = 1$ or 2, since the case $hdim\ V = 0$ is obvious. We have

$$E = E_0 \perp E_1,$$

where both E_0 and E_1 are positively defined Riemannian vector bundles and moreover the first one is trivial while the dimension of the second one is not greater than $hdim\ V$.
Therefore in the Witt ring $W(\mathcal{M})$ we have

$$2^k\langle 1\rangle - \langle\langle f_1, \ldots, f_k\rangle\rangle = 2^k\langle 1\rangle - i^*(E_0) - i^*(E_1) = rank(E_1)\cdot\langle 1\rangle - i^*(E_1).$$

The dimension of the anisotropic part of the form $rank(E_1) \cdot \langle 1\rangle - i^*(E_1)$ is not greater than $2rank(E_1) \leq 2hdim\ V$. But it belongs to the k power of the fundamental ideal of $W(\mathcal{M})$, where $k > hdim\ V$ hence due to the Arason-Pfister theorem (see [12] th.3.1 ch.10) it must be hyperbolic. Thus

$$\langle\langle f_1, \ldots, f_k\rangle\rangle = 2^k\langle 1\rangle.$$

Now let $\alpha = \langle\langle f_1, \ldots, f_k \rangle\rangle$ be a torsion element. Then for every prime divisor p $\partial_p\alpha$ also is torsion.

If $dim\ V = 1$ then for each p the residue field is isomorphic to the field of real numbers hence its Witt ring is torsion free. Thus α belongs to the image of i^* and we may apply the first part of the theorem to show that it is hyperbolic.

If $dim V = 2$ and $k > 2$ then we base on the fact that $\partial_p\alpha$ is equivalent to $k-1$-fold Pfister form multiplied by a rank one form.
If p is an elliptic point then the residue field is a field of Laurent series in one variable and every two- and more-fold Pfister form is hyperbolic.
If p is a real curve then the field of meromorphic functions on p is the field of meromorphic functions on its normalization i.e. on $\mathbb{R}$ or on S^1.
But we have already proved that any two- and more-fold Pfister form which is torsion in the Witt ring of the field of meromorphic functions of one-dimensional manifold then it is hyperbolic.
Therefore α belongs to the image of i^* and we may once more apply the first part of the theorem to show that α is hyperbolic.

9 Proof of Corollary 1

Let α denotes the equivalence class of the Pfister form $\langle\langle f_1, \ldots, f_k \rangle\rangle$ in the Witt ring $W(\mathcal{M})$. Then the class 2α contains $\langle 1, 1\rangle \otimes \langle\langle f_1, \ldots, f_k \rangle\rangle = \langle\langle 1, f_1, \ldots, f_k \rangle\rangle$.

Basing on theorems 1 and 2, and the fact that the signature of α is 0 (in every point x where it is defined), it is enough to show that every second residue homomorphism associated to a prime divisor maps 2α to zero.

If p is an elliptic point then the corresponding residue field is a field of convergent Laurent series over the field of complex numbers. Thus every element of the Witt ring has order 2, and

$$\partial_p(2\alpha) = 2\partial_p(\alpha) = 0.$$

In the following we shall consider only the "real" case; p is either a point or an analytic curve. In general we have a decomposition for $i = 1, \ldots, k$

$$f_i = \pi^{t_i} \cdot g_i,$$

where π is a uniformizer of the ring $\mathcal{O}_{I(p)}$ and g_i has neither a pole nor a zero at p. We have to consider three cases:

- all t_i are even,
- exactly one t_i is odd, say t_1,
- more than one t_i are odd, say $t_1, \ldots, t_m$.

In the first case we obtain directly from definition that $\partial_p(\alpha) = 0$.
In the second one

$$\partial_p(\alpha) = \langle g_1 \rangle \langle\langle g_2, \ldots, g_k \rangle\rangle.$$

Since π being an uniformizer is changing the sign at p hence in very point of p at least one of g_i, $i = 2, \ldots k$, is nonpositive (if p is a curve then they may have also a pole).
In one-dimensional case this finishes the proof since p consists of just one point and g_i are real numbers.
In the two-dimensional case p is a real curve and we take the normalization of p. Then we multiply the coefficients of the form by squares of denominators and apply the corollary for dimension 1.
In the third case we replace the Pfister form $\langle\langle f_1, \ldots, f_k \rangle\rangle$ by an equivalent one
$\langle\langle f_1, f_1 f_2, \ldots, f_1 f_m, f_{m+1} \ldots, f_k \rangle\rangle$ which belongs to the case 2.

The second estimate follows directly from the second part of theorem 2 since α is a torsion element.

10 The structure of $W(\mathcal{O}(V))$

Corollary 2 follows directly from lemma 2.
Corollary 3 from lemma 3 and the fact the determinant of any bilinear form $i \diamond b$ is locally a square or minus square.

To prove the last corollary we need the following lemma.

Lemma 7. *If the variety V is two-dimensional and compact then for every Riemannian bundle E over M with signature 0 there exist hyperbolic bundles H_1 and H_2 such that*

$$E \perp H_1 = H_2 \perp T \perp L,$$

where T is a trivial bundle with the Riemannian product given by the identity matrix, and L is a negatively defined one or two-dimensional Riemannian bundle.

Proof. We decompose the bundle E into the direct sum of a positive and negative definite parts. We consider the positive definite part E^+. The diagonal form of $i^*(E^+)$ consists only of positive definite functions $g_1, \ldots, g_m$

$$i^*(E^+) = \langle g_1, \ldots, g_m \rangle.$$

Thus due to corollary 1 we have in $W(\mathcal{M}(V))$

$$4m \cdot \langle 1 \rangle - 4 \cdot i^*(E^+) = \langle\langle 1, 1, -g_1 \rangle\rangle + \ldots \langle\langle 1, 1, -g_m \rangle\rangle = 0.$$

Thus having added the proper hyperbolic bundle to E we may assume that the positive part is trivial. The dimension of V is two hence the negative definite part is trivial too except may be some one or two dimensional component.

To finish the proof of corollary 4 it is enough to notice that if *det* of rank one bundle is equal to minus square then the bundle is trivial and that if *det* of rank two negative definite bundle is equal to square then it corresponds to the form $\langle -g, -g \rangle$ where g is positive definite.

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