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P-convexity of Musielak-Orlicz function spaces of Bochner type.

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Abstract

It is proved that the Musielak-Orlicz function space $L_\Phi(\mu, X)$ of Bochner type is P-convex if and only if both spaces $L_\Phi(\mu, R)$ and X are P-convex. In particular, the Lebesgue-Bochner space $L^p(\mu, X)$ is P-convex iff X is P-convex.

1 Introduction

Relationships between various kinds of convexities of Banach spaces and the reflexivity of them were developed by many authors. The earliest result concerning that problem was obtained by D. Milman in 1938 (see [17]). Milman proved that every uniformly convex Banach space is reflexive. D. Giesy [3] and R.C. James [9] raised the question whether Banach spaces which are uniformly non-l_n^1 with some positive integer $n \geq 2$ (i.e. B-convex spaces) are reflexive. Although there were some affirmative results in particular cases, the answer in general case was negative [10]. In 1970 C.A. Kottman [14] introduced a slightly stronger than B-convexity geometric property implying reflexivity and called it P-convexity. Ye Yining, He Miaohong and R. Płuciennik [22] proved that for Orlicz spaces reflexivity is equivalent to P-convexity. The same result for Musielak-Orlicz sequence and function spaces were obtained by Ye Yining and Huang Yafeng [23] and by P. Kolwicz and R. Płuciennik [11] respectively.

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In this paper we consider Musielak-Orlicz function spaces of Bochner type $L_\Phi(\mu, X)$. The question of whether or not a geometrical property is inherited from X into $L_\Phi(\mu, X)$ is one of the fundamental problems here. Considerations of that type for various kinds convexities for $L^p(\mu, X)$ were done by many authors (see for instance [4], [5], [15], [16], [19], [20], [21]). In [12] it is proved that the Orlicz-Bochner function space $L_\Phi(\mu, X)$ is P-convex iff both $L_\Phi(\mu)$ and X are P-convex. We showed that the same fact is true in the case of Musielak-Orlicz sequence spaces of Bochner type (see [13]). Although such result was expected, the proof turned out to be nontrivial and essentially different from the previous one in the case of Orlicz-Bochner spaces. Moreover, our result presented below and the main result from [13] have some interesting consequences. For example, using this result in the case $X = \mathcal{R}$, we get immediately the characterization of P-convexity of Musielak-Orlicz spaces of real valued functions and sequences. Such characterization was proved in a very long and complicated way in the paper [11] and [23]. It is worth to mention that some similar criteria for B-convexity of Musielak-Orlicz spaces of Bochner type were obtained in [1].

Denote by $\mathcal{N}$, $\mathcal{R}$ and $\mathcal{R}_+$ the sets of natural, real and positive real numbers, respectively. Let (T, Σ, μ) be a measure space with a σ-finite, complete and non-atomic measure μ. Denote by $L^0 = L^0(T)$ the set of all μ-equivalence classes of real valued measurable functions defined on T.

A function $\Phi : T \times \mathcal{R} \longrightarrow [0, \infty)$ is said to be a *Musielak-Orlicz function* if $\Phi(\cdot, u)$ is measurable for each $u \in \mathcal{R}$, $\Phi(t, u) = 0$ iff $u = 0$ and $\Phi(t, \cdot)$ is convex, even, not identically equal zero and $\frac{\Phi(t,u)}{u} \to 0$ as $u \to 0$ for μ-a.e. $t \in T$. Define on L^0 a convex modular I_Φ by

$$I_\Phi(x) = \int_T \Phi(t, x(t))\, d\mu$$

for every $x \in L^0$. By the *Musielak-Orlicz space* L_Φ we mean

$$L_\Phi = \{x \in L^0 : I_\Phi(cx) < \infty \text{ for some } c > 0\},$$

equipped with so called *Luxemburg norm* defined as follows

$$\|x\|_\Phi = \inf\left\{\epsilon > 0 : I_\Phi\left(\frac{x}{\epsilon}\right) \le 1\right\}.$$

For every Musielak-Orlicz-function Φ we define *complementary function in the sense of Young* $\Phi^* : T \times \mathcal{R} \longrightarrow [0, \infty)$ by the formula

$$\Phi^*(t, v) = \sup_{u>0}\{u\,|v| - \Phi(t, u)\}$$

for every $v \in \mathcal{R}$ and $t \in T$.

We say that *Musielak-Orlicz-function* Φ *satisfies the* Δ_2*-condition* (write $\Phi \in \Delta_2$) if there exist a constant $k > 2$ and a measurable non-negative function f such that $I_\Phi(f) < \infty$ and

$$\Phi(t, 2u) \leq k\Phi(t, u) \tag{1}$$

for μ-a.e. $t \in T$ and every $u \geq f(t)$.

For more details we refer to [18].

Now let us define the type of spaces to be considered in this paper. For a real Banach space $\langle X, \|\cdot\|_X \rangle$, denote by $L^0(T, X)$, or just $L^0(X)$, the family of strongly measurable functions $f : T \longrightarrow X$ identifying functions which are equal μ-almost everywhere in T. Define a new modular $\widetilde{I}_\Phi : L^0(X) \to \langle 0, \infty)$ by the formula

$$\widetilde{I}_\Phi(f) = \int_T \Phi\left(t, \|f(t)\|_X\right) d\mu$$

for every $f \in L^0(X)$. Let

$$L_\Phi(\mu, X) = \left\{ f \in L^0(X) : \|f(\cdot)\|_X \in L_\Phi \right\}.$$

Then $L_\Phi(\mu, X)$ becomes a Banach space with the norm

$$\|f\| = \|\|f(\cdot)\|_X\|_\Phi$$

and it is called *a Musielak-Orlicz space of Bochner type.*

Linear normed space X is called P*-convex* if there exist $\epsilon > 0$ and $n \in N$ such that for all $x_1, x_2, \ldots x_n \in S(X)$

$$\min_{i \neq j;\ i,j \leq n} \|x_i - x_j\|_X \leq 2(1 - \epsilon),$$

where $S(X)$ denotes the unit sphere of X.

The notion of P-convexity can be characterized as follows.

Lemma 1. *A Banach space X is $P-$convex iff there exist $n_0 \in N$ and $\delta_0 > 0$ such that for any elements $x_1, x_2, ..., x_{n_0} \in X \setminus \{0\}$ integers i_0, j_0 can be found such that*

$$\left\| \frac{x_{i_0} - x_{j_0}}{2} \right\|_X \leq \frac{\|x_{i_0}\|_X + \|x_{j_0}\|_X}{2} \left(1 - \frac{2\delta_0 \min \{\|x_{i_0}\|_X, \|x_{j_0}\|_X\}}{\|x_{i_0}\|_X + \|x_{j_0}\|_X} \right).$$

For the proof see [12] or [13].

2 Auxiliary lemmas

To prove the main result, we need the following.

Lemma 2. *Assume that Φ and Φ^* satisfy the Δ_2-condition. Then for every $\epsilon \in (0,1)$ there are a measurable function $h_\epsilon : T \longrightarrow \mathcal{R}_+$ with $I_\Phi(h_\epsilon) < \epsilon$, numbers $a(\epsilon) \in (0,1)$ and $\gamma = \gamma(a(\epsilon)) \in (0,1)$ such that for μ-a.e. $t \in T$ the inequality*

$$\Phi\left(t, \frac{u+v}{2}\right) \leq \frac{1-\gamma}{2} [\Phi(t,u) + \Phi(t,v)] \tag{2}$$

holds true for every $u \geq h_\epsilon(t)$ and $\left|\frac{v}{u}\right| < a$.

Proof. Repeating the same argumentation as in the proof of Theorem 1.3 from [7] and Lemma 1.1 from [2] it can be proved that the conditions $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$ imply an existence of a number $\xi_1 > 1$ and a measurable function $z : T \longrightarrow \mathcal{R}_+$ such that $I_\Phi(z) < \infty$ and

$$\Phi\left(t, \frac{\xi_1}{2} u\right) \leq \frac{1}{2\xi_1} \Phi(t,u)$$

for μ-a.e. $t \in T$, for every $u \geq z(t)$.

Take an arbitrary number $\epsilon > 0$. We can find a number $\lambda = \lambda_\epsilon > 0$ such that $I_\Phi(\lambda z) < \frac{\epsilon}{2}$. Define

$$A_k = \left\{ t \in T : \sup_{\lambda z(t) \leq u \leq z(t)} \frac{2(1+\frac{1}{k})\Phi\left(t, \frac{1}{2}(1+\frac{1}{k})u\right)}{\Phi(t,u)} \leq 1 \right\}.$$

It is easy to verify that $A_i \subset A_{i+1}$ for every $i \in \mathcal{N}$. Since $\frac{\Phi(t,u)}{u} \to 0$ as $u \to 0$ for μ-a.e. $t \in T$, $\Phi(t,\cdot)$ is not linear in a certain neighborhood of 0 for μ-a.e. $t \in T$. Hence $\mu(T \setminus \bigcup_{i=1}^{\infty} A_i) = 0$. Then the Beppo-Levi theorem implies that there exists a number $l = l_\epsilon \in \mathcal{N}$ such that

$$\int_{T\setminus A_l} \Phi(t, z(t))d\mu < \frac{\epsilon}{2}.$$

Define

$$h_\epsilon(t) = \lambda z(t)\chi_{A_l}(t) + z(t)\chi_{T\setminus A_l}(t).$$

Obviously $I_\Phi(h_\epsilon) < \epsilon$. Moreover, denoting $\xi = \xi(\epsilon) = \min\{\xi_1, 1 + \frac{1}{l}\}$, we have

$$\Phi\left(t, \frac{\xi}{2}u\right) \leq \frac{1}{2\xi}\Phi(t,u)$$

for μ-a.e. $t \in T$, and for all $u \geq h_\epsilon(t)$. Taking a number $a = a(\epsilon) \in (0,1)$ such that $1 + a \leq \xi$ and putting $\gamma = \gamma(a(\epsilon)) = \frac{a}{a+1} \in (0,1)$, we get

$$\Phi\left(t, \frac{1+a}{2}u\right) \leq \frac{1}{2(1+a)}\Phi(t,u) \leq \frac{1}{2(1+a)}\left(\Phi(t,u) + \Phi(t,au)\right) =$$

$$= \frac{1}{2}(1-\gamma)\left(\Phi(t,u) + \Phi(t,au)\right) \tag{3}$$

for μ-a.e. $t \in T$ and for all $u \geq h_\epsilon(t)$. Furthermore, the convexity of $\Phi(t,\cdot)$ implies that

$$\frac{2\Phi\left(t, \frac{1+a}{2}u\right)}{\Phi(t,u) + \Phi(t,au)}$$

is a non-decreasing function of a for μ-a.e. $t \in T$ and for all $u \geq h_\epsilon(t)$. Consequently, for every $a_0 < a$ inequality (3) holds with the same γ. Hence we obtain the thesis.

Moreover, the following two lemmas will be useful.

Lemma 3 (Lemma 3 in [11]). *If Φ satisfies the Δ_2-condition, then for every $\alpha \in (0,1)$ there exists a non-decreasing sequence (B_m^α) of measurable sets of finite measure such that*

$$\mu\left(T\setminus \bigcup_{m=1}^{\infty} B_m^\alpha\right) = 0$$

and for every $m \in \mathcal{N}$ *a number* $k_m^\alpha > 2$ *can be found such that*

$$\Phi(t, 2u) \le k_m^\alpha \Phi(t, u) \tag{4}$$

for μ*-a.e.* $t \in B_m^\alpha$ *and for every* $u \ge \alpha f(t)$*, where* f *is from the* Δ_2*-condition.*

Lemma 4 (Lemma 4 in [11]). *If* Φ *satisfies the* Δ_2*-condition, then for every* $\epsilon \in (0, 1)$ *there exist a measurable function* $g_\epsilon : T \longrightarrow \mathcal{R}_+$ *and* $k_\epsilon > 2$ *such that*

$$I_\Phi(g_\epsilon) < \epsilon \quad \text{and} \quad \Phi(t, 2u) \le k_\epsilon \, \Phi(t, u) \tag{5}$$

for μ*-a.e.* $t \in T$*, whenever* $u \ge g_\epsilon(t)$*.*

3 Main result

Theorem 1. *Let* Φ *be a Musielak-Orlicz function and let* X *be a Banach space. Then the following statements are equivalent:*

(*a*) $L_\Phi(\mu, X)$ *is* P*-convex.*

(*b*) *Both* L_Φ *and* X *are* P*-convex.*

(*c*) L_Φ *is reflexive and* X *is* P*-convex.*

(*d*) X *is* P*-convex,* $\Phi \in \Delta_2$ *and* $\Phi^* \in \Delta_2$*.*

Proof. (*a*) $\Rightarrow$ (*b*). Since the spaces L_Φ and X are embedded isometrically into $L_\Phi(\mu, X)$ and P-convexity is inherited by subspaces, $L_\Phi(\mu)$ and X are P-convex.

(*b*) $\Rightarrow$ (*c*). Every P-convex Banach space is reflexive (see Theorem 3.2 in [14]). Hence L_Φ is reflexive.

(*c*) $\Rightarrow$ (*d*). The reflexivity of Musielak-Orlicz function space L_Φ is equivalent to the fact that $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$ (see [8]).

(*d*) $\Rightarrow$ (*a*). Suppose that $\Phi \in \Delta_2$ and $\Phi^* \in \Delta_2$ and X is P-convex. Let n_0 be a natural number from Lemma 1. For every $t \in T$ define

$$f(t) = \max\left\{h_{\frac{1}{4n_0}}(t), g_{\frac{1}{4n_0}}(t)\right\},$$

where functions $h_{\frac{1}{4n_0}}$ and $g_{\frac{1}{4n_0}}$ are respectively from Lemma 2 and Lemma 4 with $\epsilon = \frac{1}{4n_0}$. Hence $I_\Phi(f) < \frac{1}{2n_0}$. Put in Lemma 3 $\alpha = a$, where a is from Lemma 2. We have $I_\Phi\left(\frac{1}{a}f\right) < \infty$, because $\Phi \in \Delta_2$. Take a set $B^a_{m_0}$ from Lemma 3 with m_0 large enough to satisfy

$$\int\limits_{T\setminus B^a_{m_0}} \Phi\left(t, \frac{f(t)}{a}\right) d\mu < \frac{1}{2n_0}. \tag{6}$$

Let $l \in \mathcal{N}$ be such that $\frac{1}{a} \leq 2^l$. Then, by Lemma 3, there exists a number $k^a_{m_0} > 2$ such that

$$\Phi\left(t, \frac{1}{a}v\right) \leq (k^a_{m_0})^l \Phi(t, v)$$

for μ-a.e. $t \in B^a_{m_0}$, whenever $v \geq af(t)$. Putting

$$\frac{v}{a} = u \text{ and } \frac{1}{(k^a_{m_0})^l} = \beta(a, m_0) = \beta_{m_0},$$

we get

$$\Phi(t, au) \geq \beta_{m_0}\Phi(t, u) \tag{7}$$

for μ-a.e. $t \in B^a_{m_0}$ and for every $u \geq f(t)$. Moreover, by Lemma 4, we have

$$\Phi\left(t, \frac{1}{a}v\right) \leq (k_\epsilon)^l \Phi(t, v)$$

for μ-a.e. $t \in T$, and $v \geq f(t)$, where $k_\epsilon = k_{\frac{1}{4n_0}}$. Analogously, we can obtain

$$\Phi(t, au) \geq \beta\Phi(t, u) \tag{8}$$

for μ-a.e. $t \in T$, and for every $u \geq \frac{f(t)}{a}$.

Now, we will show that there exists a number $r_1 \in (0, 1)$ such that for any elements $x_1, x_2, ..., x_{n_0}$ of Banach space X and for μ-a.e. $t \in T_M$ we have

$$\sum_{i=1}^{n_0}\sum_{j=i}^{n_0} \Phi\left(t, \left\|\frac{x_i - x_j}{2}\right\|_X\right) \leq \frac{n_0 - 1}{2} r_1 \sum_{i=1}^{n_0} \Phi\left(t, \|x_i\|_X\right), \tag{9}$$

where $T_M = \left\{t \in T : \max_{1\leq i\leq n_0}\{\|x_i\|_X\} \geq \frac{f(t)}{a}\right\}$.

Take $x_1, x_2, ..., x_{n_0} \in X$. Let k be an index such that $\|x_k\|_X = \max_{1\le i\le n_0}\{\|x_i\|_X\}$. For the clarity of the proof, we will divide it into two parts.

I. Suppose that there is $i_1 \in \{1, 2, ..., n_0\} \setminus \{k\}$ such that

$$\frac{\|x_{i_1}\|_X}{\|x_k\|_X} < a.$$

Since $\|x_k\| \ge \frac{f(t)}{a} > f(t)$ for μ-a.e. $t \in T_M$, by inequality (2), we obtain

$$\Phi\left(t, \left\|\frac{x_{i_1} - x_k}{2}\right\|_X\right) \le \Phi\left(t, \frac{\|x_{i_1}\|_X + \|x_k\|_X}{2}\right)$$

$$\le \frac{1}{2}(1-\gamma)\left(\Phi\left(t, \|x_{i_1}\|_X\right) + \Phi\left(t, \|x_k\|_X\right)\right).$$

Hence, by the convexity of $\Phi(t, \cdot)$ for μ-a.e. $t \in T$, we get

$$\sum_{i=1}^{n_0}\sum_{j=i}^{n_0}\Phi\left(t, \left\|\frac{x_i - x_j}{2}\right\|_X\right) \le$$

$$\le \frac{n_0-1}{2}\sum_{i=1}^{n_0}\Phi\left(t, \|x_i\|_X\right) - \frac{\gamma}{2}\left(\Phi\left(t, \|x_{i_1}\|_X\right) + \Phi\left(t, \|x_k\|_X\right)\right)$$

$$\le \frac{n_0-1}{2}\sum_{i=1}^{n_0}\Phi\left(t, \|x_i\|_X\right) - \frac{\gamma}{2n_0}\left(n_0\Phi\left(t, \|x_k\|_X\right)\right)$$

$$\le \frac{n_0-1}{2}\sum_{i=1}^{n_0}\Phi\left(t, \|x_i\|_X\right) - \frac{\gamma}{2n_0}\sum_{i=1}^{n_0}\Phi\left(t, \|x_i\|_X\right)$$

$$= \frac{n_0-1}{2}\left(1 - \frac{\gamma}{n_0(n_0-1)}\right)\sum_{i=1}^{n_0}\Phi\left(t, \|x_i\|_X\right) \tag{10}$$

for μ-a.e. $t \in T_M$.

II. Assume that for all $i \ne k$ we have

$$\frac{\|x_i\|_X}{\|x_k\|_X} \ge a. \tag{11}$$

Then $\|x_i\| > 0$ for every $i \neq k$. Let i_0, j_0 be from Lemma 1. We may assume that

$$a < \frac{\|x_{i_0}\|_X}{\|x_{j_0}\|_X} \leq \frac{1}{a}. \tag{12}$$

Really, otherwise we have

$$a > \frac{\|x_{i_0}\|_X}{\|x_{j_0}\|_X} \geq \frac{\min\left\{\|x_{i_0}\|_X, \|x_{j_0}\|_X\right\}}{\max\left\{\|x_{i_0}\|_X, \|x_{j_0}\|_X\right\}} \geq \frac{\min\left\{\|x_{i_0}\|_X, \|x_{j_0}\|_X\right\}}{\|x_k\|_X},$$

which contradicts to inequality (11). Hence, applying Lemma 1 and inequality (12), we get

$$\left\|\frac{x_{i_0} - x_{j_0}}{2}\right\|_X \leq \left(1 - \frac{2\delta a}{1+a}\right) \frac{\|x_{i_0}\|_X + \|x_{j_0}\|_X}{2}.$$

Therefore, by the convexity of $\Phi(t, \cdot)$ for μ-a.e. $t \in T$, we obtain

$$\Phi\left(t, \left\|\frac{x_{i_0} - x_{j_0}}{2}\right\|_X\right) \leq \frac{1}{2}(1-\alpha)\left(\Phi\left(t, \|x_{i_0}\|_X\right) + \Phi\left(t, \|x_{j_0}\|_X\right)\right), \tag{13}$$

where $\alpha = \frac{2\delta a}{1+a} \in (0,1)$. Consequently, by inequalities (13) and (8), we have

$$\begin{aligned}
&\sum_{i=1}^{n_0}\sum_{j=i}^{n_0} \Phi\left(t, \left\|\frac{x_i - x_j}{2}\right\|_X\right) \\
\leq\ & \frac{n_0 - 1}{2}\sum_{i=1}^{n_0} \Phi\left(t, \|x_i\|_X\right) - \frac{\alpha}{2}\left(\Phi\left(t, \|x_{i_0}\|_X\right) + \Phi\left(t, \|x_{j_0}\|_X\right)\right) \\
&\leq \frac{n_0 - 1}{2}\sum_{i=1}^{n_0} \Phi\left(t, \|x_i\|_X\right) - \alpha\Phi\left(t, a\|x_k\|_X\right) \\
&\leq \frac{n_0 - 1}{2}\sum_{i=1}^{n_0} \Phi\left(\|x_i\|_X\right) - \frac{\alpha\beta}{n_0}\left(n_0\Phi\left(\|x_k\|_X\right)\right) \\
&\leq \frac{n_0 - 1}{2}\sum_{i=1}^{n_0} \Phi\left(t, \|x_i\|_X\right) - \frac{\alpha\beta}{n_0}\sum_{i=1}^{n_0} \Phi\left(t, \|x_i\|_X\right) \\
&= \frac{n_0 - 1}{2}\left(1 - \frac{2\alpha\beta}{n_0(n_0-1)}\right)\sum_{i=1}^{n_0} \Phi\left(t, \|x_i\|_X\right)
\end{aligned} \tag{14}$$

for μ-a.e. $t \in T_M$. Define

$$r_1 = \max\left\{1 - \frac{\gamma}{n_0(n_0-1)}, 1 - \frac{2\alpha\beta}{n_0(n_0-1)}\right\}.$$

Combining inequalities (10) and (14), we get inequality (9). Repeating the same argumentation as in the proof of inequality (9), a number $r_2 \in (0,1)$ can be found such that the inequality

$$\sum_{i=1}^{n_0}\sum_{j=i}^{n_0}\Phi\left(t, \left\|\frac{x_i - x_j}{2}\right\|_X\right) \le \frac{n_0 - 1}{2} r_2 \sum_{i=1}^{n_0}\Phi\left(t, \|x_i\|_X\right) \tag{15}$$

holds for every $x_1, x_2, \ldots, x_{n_0}$ elements from Banach space X and for μ-a.e. $t \in B_{m_0}^a$ satisfying $\max_{1\le i\le n_0}\{\|x_i\|_X\} \ge f(t)$. Using (7) in place of (8) one can find that (15) is true with

$$r_2 = \max\left\{1 - \frac{\gamma}{n_0(n_0-1)}, 1 - \frac{2\alpha\beta_{m_0}}{n_0(n_0-1)}\right\}.$$

Let $f_1, \ldots, f_{n_0} \in S\left(L^\Phi(\mu, X)\right)$. Define

$$E = \left\{t \in T : \sum_{i=1}^{n_0}\Phi(t, \|f_i(t)\|_X) \ge n_0\Phi(t, f(t))\right\}.$$

Obviously $\max\limits_{1\le i\le n_0}\{\|f_i(t)\|_X\} \ge f(t)$ for every $t \in E$. Divide the set E into two following subsets:

$$E_1 = \left\{t \in E : \max_{1\le i\le n_0}\{\|f_i(t)\|_X\} \ge \frac{f(t)}{a}\right\},$$

$$E_2 = \left\{t \in E : f(t) \le \max_{1\le i\le n_0}\{\|f_i(t)\|_X\} < \frac{f(t)}{a}\right\}.$$

Next decompose the set E_2 into two subsets E_{21} and E_{22} defined by

$$E_{21} = E_2 \cap B_{m_0}^a,$$

$$E_{22} = E_2 \setminus B_{m_0}^a.$$

By inequalities (9) and (15), we have

$$\sum_{i=1}^{n_0}\sum_{j=i}^{n_0}\Phi\left(t,\left\|\frac{f_i(t)-f_j(t)}{2}\right\|_X\right)\leq\frac{1}{n_0}\binom{n_0}{2}\sum_{i=1}^{n_0}\Phi\left(t,\|f_i(t)\|_X\right) \tag{16}$$

for μ-a.e. $t\in E_1\cup E_{21}$, where $r=\max\{r_1,r_2\}$. Obviously $r\in(0,1)$. Moreover, by the definitions of the set E and the function f, we have

$$\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{T\setminus E}\right)<\frac{1}{2}. \tag{17}$$

Now, let $t\in E_{22}$. Then, by inequality (6), we have

$$\begin{aligned}\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{E_{22}}\right)&=\sum_{i=1}^{n_0}\int\limits_{E_2\setminus B^a_{m_0}}\Phi\left(t,\|f_i(t)\|_X\right)d\mu\\&\leq\int\limits_{E_2\setminus B^a_{m_0}}n_0\Phi\left(t,\max_{1\leq i\leq n_0}\{\|f_i(t)\|_X\}\right)d\mu\\&<\int\limits_{E_2\setminus B^a_{m_0}}n_0\Phi\left(t,\frac{f(t)}{a}\right)d\mu\leq\int\limits_{T\setminus B^a_{m_0}}n_0\Phi\left(t,\frac{f(t)}{a}\right)d\mu<\frac{1}{2}.\end{aligned} \tag{18}$$

Hence, by inequalities (17) and (18), we get

$$\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{T\setminus(E_1\cup E_{21})}\right)=\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{T\setminus E}\right)+\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{E_{22}}\right)<1.$$

Since $\|f_i\|=1$ for $i=1,2,...n_0$ and $\Phi\in\Delta_2$, we have $\widetilde{I}_\Phi(f_i)=1$ for $i=1,2,...,n_0$. Consequently,

$$\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{E_1\cup E_{21}}\right)\geq n_0-1. \tag{19}$$

Therefore, by inequalities (16) and (19), we have

$$\sum_{i=1}^{n_0}\sum_{j=i}^{n_0}\widetilde{I}_\Phi\left(\frac{1}{2}(f_i-f_j)\right)=$$

$$= \sum_{i=1}^{n_0}\sum_{j=i}^{n_0} \widetilde{I}_\Phi\left(\frac{1}{2}(f_i - f_j)\chi_{T\setminus(E_1\cup E_{21})}\right) + \sum_{i=1}^{n_0}\sum_{j=i}^{n_0} \widetilde{I}_\Phi\left(\frac{1}{2}(f_i - f_j)\chi_{E_1\cup E_{21}}\right)$$

$$\leq \frac{1}{n_0}\binom{n_0}{2}\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{T\setminus(E_1\cup E_{21})}\right) + \frac{r}{n_0}\binom{n_0}{2}\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{E_1\cup E_{21}}\right)$$

$$= \frac{1}{n_0}\binom{n_0}{2}\left(\sum_{i=1}^{n_0}\widetilde{I}_\Phi(f_i) - \sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{E_1\cup E_{21}}\right)\right)$$

$$+\frac{r}{n_0}\binom{n_0}{2}\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{E_1\cup E_{21}}\right)$$

$$= \binom{n_0}{2}\left(1 - \frac{1-r}{n_0}\sum_{i=1}^{n_0}\widetilde{I}_\Phi\left(f_i\chi_{E_1\cup E_{21}}\right)\right)$$

$$\leq \binom{n_0}{2}\left(1 - \frac{(1-r)(n_0-1)}{n_0}\right) \leq \binom{n_0}{2}(1-p),$$

where $p = \frac{(1-r)}{2}$. So, there exist $i_1, j_1 \in \{1, 2, ..., n_0\}$ such that

$$\widetilde{I}_\Phi\left(\frac{1}{2}(f_{i_1} - f_{j_1})\right) \leq 1 - p.$$

Finally, by the Δ_2 -condition for Φ, we obtain that

$$\left\|\frac{1}{2}(f_{i_1} - f_{j_1})\right\| \leq 1 - q(p), \qquad 0 < q(p) < 1 (\text{cf. } [6]).$$

Thus, by Lemma 1, the space $L_\Phi(\mu, X)$ is P-convex. This completes the proof.

Theorem 1 is a generalization of Theorem 1 from [12]. Moreover, the following characterization of P-convex Musielak Orlicz spaces of real functions L_Φ, proved directly in [11] in a very complicated way, is an immediate consequence of Theorem 1.

Corollary 1. *The following statements are equivalent:*

(a) *L_Φ is P-convex.*

(b) *L_Φ is reflexive.*

(c) *$\Phi \in \delta_2$ and $\Phi^* \in \Delta_2$.*

Proof. It is enough to apply Theorem 1 with $X = \mathcal{R}$.

Corollary 2. *The Lebesgue-Bochner function space* $L^p(u, X)$ $(1 < p < \infty)$ *is* P*-convex iff* X *is* P*-convex.*

Proof. The Lebesgue space L^p is a Musielak-Orlicz space generated by the Orlicz function $\Phi(t, u) = |u|^p$ for every $t \in T$ satisfying all the assumptions of Theorem 1.

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