Extending algebraic actions.

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Abstract

There is a well-known procedure -induction- for extending an action of a subgroup H of a Lie group G on a topological space X to an action of G on an associated space. Induction can also extend a smooth action of a subgroup H of a Lie group G on a manifold M to a smooth action of G on an associated manifold. In this paper elementary methods are used to show that induction also works in the category of (nonsingular) real algebraic varieties and regular or entire maps if G is a compact abelian Lie group.

Introduction

If H a closed subgroup of the Lie group G and $f : X \to Y$ is an $H-$equivariant function then f may be extended to a $G-$equivariant function $F : G \times_H X \to G \times_H Y$ by $F(g,x) = gf(x)$. Here $G \times_H X$ denotes the quotient space of $G \times X$ obtained by identifying (gh,x) with (g,hx), cf. [P], page 31 or [B], page 79. The set $G \times_H X$ is usually denoted by $ind_H^G(X)$ and the map F is usually denoted by $ind_H^G(f)$. If is a smooth $H-$equivariant map between $H-$manifolds X and Y then $ind_H^G(f) : ind_H^G(X) \to ind_H^G(Y)$ is a smooth $G-$equivariant map of $G-$manifolds. If X and Y are (nonsingular) real algebraic $H-$varieties and H has finite index in G and G is compact then it was shown in [DM] that $ind_H^G(X)$ and $ind_H^G(Y)$ can be given the structure of (nonsingular) algebraic varieties such that if f is a regular (resp. entire rational) $H-$equivariant map then $ind_H^G(f) : ind_H^G(X) \to ind_H^G(Y)$ is a regular (resp. entire rational) $G-$equivariant map. In this note we show that

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the same result is true if G is a compact abelian Lie group-the assumption that H has finite index in G is not necessary. The proof is entirely elementary.

A much stronger result has recently been obtained by G. Schwarz [S]; he shows, among other results, that if X is a (nonsingular) real algebraic $H-$variety, H is a closed subgroup of G and G is compact then $ind_H^G(X)$ can be given the structure of a (nonsingular) algebraic $G-$variety in a functorial way.

A good reference for definitions and background is [D].

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1   Definitions

Let G be a compact Lie group and V a representation space of G, that is, a real vector space on which G acts via linear maps. Let $p_i : V \to R, i = 1, 2, \cdots, n$ be polynomial maps. Then the set $M = \{v \in V \mid p_i(v) = 0$ for $i = 1, 2, \cdots, n\}$ is called a real algebraic $G-$ variety if it is invariant under the action of G. If the differentials $\{dp_i\}$ at $x \in M$ span a space of constant rank then M is said to be a nonsingular real algebraic $G-$variety.

If $X \subset V$ and $Y \subset W$ are real algebraic varieties a map $f : X \to Y$ said to be regular if it is the restriction of a polynomial map $F : V \to W$. We say that f is an entire rational map if there polynomial maps $h : V \to W, k : V \to R$, such that k does not vanish on X and $f = h/k$. If $X \subset V$ and $Y \subset W$ are real algebraic $G-$varieties and $f : X \to Y$ is equivariant as well as regular then f is a regular equivariant map. Is is shown in [DMS] that if f is a regular equivariant map then there is an **equivariant** polynomial map $F : V \to W$ such that f is the restriction of F to X; similarly, if f is an equivariant entire rational map, there exist **equivariant** polynomial maps $h : V \to W, k : V \to R$, such that k does not vanish on X and $f = h/k$.

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Theorem. *Let H be a closed subgroup of the compact abelian Lie group G. There is a functor $ind_H^G()$ from the category of (nonsingular) real algebraic H-varieties and regular H-equivariant maps (resp. H-equivariant entire rational maps) to the category of (nonsingular) real algebraic G-varieties and regular G-equivariant maps (resp. G-equivariant entire rational maps).*

The proof will follow from a series of lemmas.

The following lemma allows us to reduce the proof to a special case.

Lemma 1. *Let G be a compact abelian Lie group, let H be a closed subgroup and let $n = dimension\,G - dimension\,H$; $n \geq 0$. Then there is a sequence of groups $H = G_0 \subset G_1 \subset \ldots G_n \subset G$ such that $G_{j+1}/G_j \approx S^1$ for $j = 0, 1\, n-1$ and G_n has finite index in G.*

Proof of lemma. If $n = 0$ then H and G have the same components of the identity, $H_e = G_{e'}$ so G/H is a quotient of G/G_e which is finite so G_0 has finite index in G. If $n > 0$ we proceed by induction on n.

There is a one to one correspondence between (closed) subgroups of G containing H and (closed) subgroups of G/H. If $n > 0$, let K be a circle subgroup of G/H, let $\pi : G \to G/H$ be the projection and let $G_1 = \pi^{-1}(K)$. Now dimension $G-$dimension $G_1 = n-1$ so we are done.

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Remark 1. Let H be a closed subgroup of the compact abelian Lie group G with dimension $G = 1+$ dimension H and $G/H \approx S^1$. Choose a circle subgroup $S^1 \subset G$ such that S^1 is not contained in H; then G is the quotient of $S^1 \times H$ by a cyclic group $Z_m = S^1 \cap H$. (If $S^1 \cap H = e$ then $m = 1$). We will henceforth assume that such a circle group has been chosen once and for all. We let $\zeta = e^{2\pi i/m}$; ζ is a generator for the group $Z_m = S^1 \cap H$.

We wish to define the induction functor $ind_H^G()$ from the category of (nonsingular) algebraic H-varieties and regular H-equivariant maps (resp. H-equivariant entire rational maps) to (nonsingular) algebraic G-varieties and regular G-equivariant maps (resp. G-equivariant entire rational maps). In view of Lemma 1 it suffices to define $ind_H^G()$ as

the composition of induction maps from G_j to G_{j+1} for $j = 0, 1, n - 1$ and from G_n to G. Since G_n has finite index in G, the functor $ind_{G_n}^G ()$ exists by the result of [DM1]. Thus, we need only consider the special case G_j to G_{j+1}, that is, we have proved:

Lemma 2. *It suffices to prove the theorem in the special case in which dimension $G = 1+$ dimension H and $G/H \approx S^1$.*

We next show that without loos of generality we may assume that our H−varieties are imbedded in complex representations on which G acts.

Lemma 3. *Let H be a closed subgroup of the compact abelian Lie group G with dimension $G = 1+$ dimension H and $G/H \approx S^1$. Let X be a (nonsingular) $H-$variety in the real $H-$representation space V. Then we may find a complex $G-$representation space $V' = V \otimes_R C$ such that X is an $H-$invariant (nonsingular) real algebraic variety in V'. Moreover, this procedure is functorial, that is, if $f : X \to Y$ is an $H-$equivariant regular (resp. entire rational) map between algebraic $H-$varieties $X \subset V$ and $Y \subset W$ where V and W are real $H-$representation spaces then there is an $H-$equivariant polynomial map $F' : V' \to W'$ between the complex $G-$representation spaces V' and W' such that $F' \mid X = f$ (resp. $f = h'/k'$ where $h' : V' \to W', k' : V' \to R$ are polynomial maps such that k' does not vanish on X).*

Proof. Let $p_i : V \to R, i = 1, 2, \ldots, n$, be polynomial maps such that $X = \{v \in V \mid p_i(v) = 0 \text{ for } i = 1, 2, \ldots, n\}$. (If X is nonsingular we assume that the differentials $\{dp_i\}$ at $x \in X$ span a space of constant rank). Then we have that $X = \{v \in V \otimes_R C \mid v = \bar{v}, p_i'(v) = 0 \text{ for } i = 1, 2, \ldots, n\}$ where $p_i'(v)$ denotes the same polynomials but with complex arguments rather than real arguments. We note that the equations $v = \bar{v}, p_i'(v) = 0$ may be regarded as real polynomial equations on the underlying real vector space of V'. Moreover, if $f : X \to Y$ is an $H-$equivariant regular map then the $H-$equivariant polynomial map $F' : V' \to W'$ between the complex $G-$representation spaces V' and W' given by extending the definition of F to complex arguments satisfies $F' \mid_X = f$. Similar remarks apply if f is an entire rational $H-$equivariant map.

We need only verify that the H action on V can be extended to a G action on V'. Since H is abelian every irreducible complex representa-

tion of H is one dimensional; in particular, we may write V' as a sum of one (complex) dimensional H-irreducible spaces, $V' = \sum_{s=1}^{a} V_s$. Thus, to prove that we may extend the action of H on V' to G it is sufficient to show that we may extend the action of H on each V_s to G. In view of remark 1 above, that means defining an action of the circle subgroup S^1 on each V_s that extends the action of Z_m on each V_s. The element ζ in H acts on V_s via complex multiplication by ζ^r for some r that depends on s. We let $t \in S^1$ act on V_s via complex multiplication by t^r. That extends the action to G. (Complex multiplication by t^r for $r' \equiv r \bmod m$ would also work).

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Definition. *Let H be a closed subgroup of the compact abelian Lie group G with dimension $G = 1+$ dimension H and $G/H \approx S^1$. Let $\alpha : G \to S^1 = U(1)$ be the homomorphism with kernel H given by $\alpha(t) = t^m$ for $t \in S^1$, $\alpha(h) = 1$ for $h \in H$. We denote by $\bar{C}$ the complex numbers with the G action given by α.*

Notation. We denote the **action** of the complex number t on the vector v by $t * v$; tv will denote *multiplication* of the vector v by the complex number t.

The proof of the following lemma can be found in [P], page 31. It is given here for the sake of completeness. The assertion is that $V' \times 1$ is a slice in $V' \times S^1$ at $(0,1)$.

Lemma 4. *Let H be a closed subgroup of the compact abelian Lie group G with dimension $G = 1+$ dimension H and $G/H \approx S^1$. Let X be a (nonsingular) H-variety in the complex G-representation space V' and let $i : X \to V'$ be the inclusion. Let $j : G \times_H X \to V' \oplus \bar{C}$ be given by $j(g,x) = (gi(x), \alpha(g))$. Then j is a G-equivariant imbedding of $G \times_H X \to V' \oplus \bar{C}$.*

Proof. To see that j is G-equivariant we note that $j(g'(g,x)) = j(g'g, x) = (g'gi(x), \alpha(g'g)) = g'(gi(x), \alpha(g)) = g'j(g,x)$.

If $j(g,x) = j(g',y)$ then $\alpha(g) = \alpha(g')$ so $g' = hg$ for some h in H. Then $j(g,x) = (gi(x), \alpha(g))$ and $j(g',y) = j(gh,y) = (ghi(y), \alpha(gh)) = (g\, i(hy), \alpha(gh))$ so $i(x) = i(hy)$ and thus, since i is an imbedding,

$x = hy$. Hence j is one to one. The statement that j is an imbedding follows from the fact that i is a closed map and G is compact.

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We will now show that $j(G \times_H X)$ is a (nonsingular) algebraic $G-$variety in $V' \oplus \bar{C}$. Since every point in $j(G \times_H X)$ is of the form $g(x, 1)$ for some $x \in X$, and X is the zero set of some polynomials p'_i we must try to write (v, w) as $g(x, 1)$ for some g and some x and then verify that $p'_i(x) = 0$ for $i = 1, 2, \ldots, n$. If $w\bar{w} = 1$ then $w \in S^1$ and that is necessary and sufficient for w to equal $\alpha(g)$ for some $g \in G$ since α is onto. By remark 1, any g in G is of the form $t\,h$ for some $t \in S^1$ and some $h \in H$ so if $w = \alpha(g)$ then $w = t^m$ and $\bar{w} = t^{-m}$. (Note that t is only determined up to an mth root of unity). Then (v, w) can be written as $t * (\bar{t} * v, 1)$. Thus, we only need the equations $w\bar{w} = 1, p'_i(\bar{w}^{1/m} * v) = 0$ for $i = 1, 2, \ldots, n$. (Since g is not unique, the equations do not appear to be well defined no less polynomial in v and w and their conjugates).

We interpreted the equations as follows: $V' = \sum_{s=1}^{a} V_s$; the action of $t \in S^1 \subset G$ on $V_j \subset V'$ is given by $t * v_j = t^{r_j} v_j$ for some integer r_j. Now we extend the action of S^1 on V' to an "action" of C on V' by $q * v_j = q^{r_j} v_j$ for q in C. Finally, define $p'_i(\bar{w}^{1/m} * v)$ by $p'_i(\bar{w}^{1/m} * v) = p'_i(\bar{q} * v)$ where q is any m^{th} root of w, the ‾ indicates complex conjugate, and the $*$ indicates the "action" of C on V'. To put it more concretely, one computes $p'_i(v, w)$ by first choosing q an mth root of w (recall $Z_m = S^1 \cap H$) and then replacing every occurrence of the variable v_j in the polynomial F by $\bar{q}^{r_j} v_j$.

Then we have:

Lemma 5. *Under the assumptions of Lemma 4 let X be the $H-$variety in the complex $G-$representation space V' given by $\{v \in V' \mid p'_i(v) = 0$ for $i = 1, 2, \ldots, n\}$; then $j(G \times_H X) = \{(v, w) \in V' \oplus \bar{C} \mid w\bar{w} = 1, p'_i(\bar{w}^{1/m} * v) = 0$ for $i = 1, 2, \ldots, n\}$. Moreover, $j(G \times_H X)$ si a (nonsingular) $G-$variety in $V' \oplus \bar{C}$.*

Proof. We will show that

i) the zeroes of the $p''_i s$ do not depend on which m^{th} root q is chosen to compute the value of p_i,

ii) the "equations" vanish exactly on $j(Gx_H X)$,

iii) the equations are equivalent to (have the same zero set as) poly-nomial equations in the variables, v, w and their conjugates, and

iv) if X is nonsingular then $j(Gx_H X)$ is nonsingular.

i) If $p_i'(\bar{w}^{1/m} * v) = 0$, that is, if $p_i'(\bar{q} * v) = 0$ for some choice of q then for any choice of q, $p_i'(\bar{q} * v) = 0$ because the p_i' are H–invariant, and two $q's$ differ by an mth root of unity any m^{th} root of unity is a power of $\zeta \in H$.

ii) We note that a point of the form $(x, 1)$ satisfies $w\bar{w} = 1$ and $p_i'(\bar{w}^{1/m} * v) = 0$ for $i = 1, 2, \ldots, n$, if and only if x in X since we may take $q = 1$. Since the equations are invariant under G they vanish precisely on $j(G \times_H X)$.

iii) As interpreted, $p_i'(v, w)$ is a polynomial in v and q; we must show that it is equivalent to polynomials in v and w.

Fix i and write

$\Diamond$ $p_i'(v, w) = P_0(v, w) + P_1(v, w)q + P_2(v, w)q^2 \ldots + P_{m-1}(v, w)q^{m-1}$
by grouping together monomials that have the same power of q mod m and replacing each occurrence of q^m by w. Thus $P_0(v, w)$ consists of all those monomials in which q appears to a power divisible by m. Each P_i is a polynomial in v and w.

Now let (v, w) be a point in $j(Gx_H X)$, that is, $p_i'(v, w) = 0$. Consider the system of m linear equations obtained from $\Diamond$) by substituting $q\zeta^j$ for q. By i) the result in each cases is $p_i'(v, w) = 0$ so we get:

$$0 = P_0(v, w) + P_1(v, w)q + P_2(v, w)q^2 \ldots + P_{m-1}(v, w)q^{m-1}$$
$$0 = P_0(v, w) + P_1(v, w)q\zeta + P_2(v, w)q^2\zeta^2$$
$$\ldots + P_{m-1}(v, w)q^{m-1}\zeta^{m-1}$$

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$$0 = P_0(v, w) + P_1(v, w)q\zeta^{m-1} + P_2(v, w)q^2\zeta^{2(m-1)}$$
$$\ldots + P_{m-1}(v, w)q^{m-1}\zeta^{(m-1)(m-1)}$$

These equations have the form:

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{m-1} \\ 1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(m-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \zeta^{m-1} & \zeta^{2(m-1)} & \cdots & \zeta^{(m-1)^2} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 q \\ P_2 q^2 \\ \cdots \\ P_{m-1} q^{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

Since the coefficient matrix is a Vandermonde matrix with nonzero determinant $\left(\prod_{0<i<j<m} (\zeta^j - \zeta^i) \right)$ there is a unique solution to the system of equations which is $P_j(v, w)q^j = 0$ for all j. Thus, $p_i'(v\,w) = 0$ if and only if the polynomials $P_j(v\,w) = 0$ for all $j = 1, 2, \ldots m - 1$. (There is a different set of $P_j's$ for each i; we write them as P_j^i if we need to be precise). Hence, the set $j(G \times_H X)$ is the zero set of a family of polynomials, i.e., $j(G \times_H X)$ is a real algebraic G–variety.

iv) If X is a nonsingular variety then the differentials $\{dp_i'\}$ at $x \in X$ span a space of constant rank = dimension V'–dimension X. To see that the differentials dP_j^i and $d(w\bar{w})$ span a space of constant rank = dimension $V'\oplus\bar{C}$–dimension $j(Gx_H X) = 2+$dimension $V' - (1+$ dimension $X) = 1+$ dimension V'–dimension X at points $(x, 1)$ in dimension $j(Gx_H X)$ we note that the dp_i' are linear combinations of the dP_j^i because the p_i' are linear combinations of the P_j^i and that $w\bar{w}$ does not involve the v_j coordinates. Hence, the dimension of the space spanned is at least $1+$ dimension V'–dimension X; it cannot be more since dimension $j(Gx_H X) = 1+$ dimension X. To check at other points (v, w) in $j(Gx_H X)$ one translates the problem to $V' \times 1$ by multiplying by a suitable element of G.

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In view of lemma 5 we may make the following definition.

Definition. *Let H be a closed subgroup of the compact abelian Lie group G with dimension $G = 1+$ dimension H and $G/H \approx S^1$. Let X be a (nonsingular) H–variety in the complex G–representation space V' given by $\{v \in V' \mid p_i'(v) = 0$ for $i = 1, 2, \cdots, n\}$. Then $ind_H^G(X)$ is*

the (nonsingular) G−variety in $V' \oplus \bar{C}$ given by

$$\{(v, w) \in V' \oplus \bar{C} \mid w\bar{w} = 1, P_j^i(v, w) = 0$$

for

$$i = 1, 2, \cdots, n, j = 0, 1, 2, \cdots, m - 1\}.$$

Lemma 6. *Let H be a closed subgroup of the compact abelian Lie group G with dimension $G = 1+$ dimension H and $G/H \approx S^1$. Let $F : V' \to W'$ be an H−equivariant polynomial map between the complex G−representation spaces V' and W'; then there is a G−equivariant polynomial map $F' : V' \oplus \bar{C} \to W' \oplus \bar{C}$ between the complex G−representation spaces $V' \oplus \bar{C}$ and $W' \oplus \bar{C}$ such that $F'(v, 1) = (F(v), 1)$.*

Proof. Since G is abelian we have that $W' = \sum_{j=1}^b W_j$ where the W_j are one dimensional G−invariant subspaces. Since the map F' will have the desired properties if and only if the projections of F' onto each of the factors has the desired properties it suffices to prove the lemma in the case that W' is one dimensional. That is, we consider a polynomial $F(v_1, v_2, \cdots, v_s)$ from V' to W' where the action of $t \in S^1 \subset G$ on V' is given by $t * v_j = t^{r_j} v_j$ for some integer r_j and where the action of $t \in S^1 \subset G$ on W' is given by $t * w = t^d w$ for some integer d. As in lemma 5 we extend the action of S^1 on V' and W' to an "action" of C on V' and W' by $q * v_j = q^{r_j} v_j$ and $q * w = q^d w$ for q in C. Finally, define $F'(v, z) = (q * F(\bar{q} * v), z)$ where q is any m^{th} root of z, the $\bar{}$ indicates complex conjugate, and the $*$ indicates the "action" of C on V' and W'. To put it more concretely, one computes $F'(v, z)$ by first choosing q, an mth root of z (recall $Z_m = S^1 \cap H$) and then replacing every occurrence of the variable v_j in the polynomial F by $\bar{q}^{r_j} v_j$ and then multiplying the result by q^d.

We will now verify that.

i) F' is well defined, that is, that $F'(v, z)$ is independent on the choice of q.

ii) F' is G−equivariant.

iii) $F'(v, 1) = (F(v), 1)$ and

iv) F' is a polynomial in v and z.

i) Note that q is only determined up to an mth root of unity and any m^{th} root of unity is a power of $\zeta \in H$. Then $F'(v, z) = (q * F(\bar{q} * v), z) = (q\zeta\bar{\zeta} * F(\bar{q} * v), z)$, which, by the H−equivariance of F is equal to $(\zeta q * F(\bar{\zeta}\bar{q} * v), z)$ so F' is well defined.

ii) If $h \in H$ we have by definition that $F(h * (v, z)) = F'(hv, z) = (q * F(\bar{q}h * v), z) = (qh * F(\bar{q} * v), z) = hF(v, z)$ since H acts trivially on $\bar{C}$. If $t \in S^1$ we have

$$
\begin{aligned}
F(t * (v, z)) &= F'(tv, t^m z) = (qt * F(\bar{q}\bar{t}t * v), t^m z) \\
&= t * (F(\bar{q} * v), z) = t * F(v, z).
\end{aligned}
$$

iii) $F'(v, 1) = (q * F(\bar{q} * v), 1)$ and the result follows by taking $q = 1$.

iv) $F'(v, z)$ is a polynomial in v and q; we must show that it is a polynomial in v and z.

Write

$$\Diamond\Diamond) F'(v, z) = P_0(v, z) + P_1(v, z)q + P_2(v, z)q^2 \ldots + P_{m-1}(v, z)q^{m-1}$$

by grouping together monomials that have the same power of q mod m and replacing each occurrence of q^m by z. Thus $P_0(v, z)$ consists of all those monomials in which q appears to a power divisible by m. Each P_i is a polynomial in v and z.

Now consider the system of m linear equations obtained from $\Diamond\Diamond)$ by substituting $q\zeta^j$ for q. By i) the result is $F'(v, z)$ in each case.

$$F'(v, z) = P_0(v, z) + P_1(v, z)q + P_2(v, z) \ldots + P_{m-1}(v, z)q^{m-1}$$
$$
\begin{aligned}
F'(v, z) = P_0(v, z) &+ P_1(v, z)q\zeta + P_2(v, z)q^2\zeta^2 \\
&\ldots + P_{m-1}(v, z)q^{m-1}\zeta^{m-1}
\end{aligned}
$$

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$$
\begin{aligned}
F'(v, z) = P_0(v, z) &+ P_1(v, z)q\zeta^{m-1} + P_2(v, z)q^2\zeta^{2(m-1)} \\
&\ldots + P_{m-1}(v, z)q^{m-1}\zeta^{(m-1)(m-1)}
\end{aligned}
$$

These equations have the form

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & \zeta & \zeta^2 & \cdots & \zeta^{m-1} \\
1 & \zeta^2 & \zeta^4 & \cdots & \zeta^{2(m-1)} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \zeta^{m-1} & \zeta^{2(m-1)} & \cdots & \zeta^{(m-1)^2}
\end{bmatrix}
\begin{bmatrix}
P_0 \\
P_1 q \\
P_2 q^2 \\
\cdots \\
P_{m-1} q^{m-1}
\end{bmatrix}
=
\begin{bmatrix}
F'(v, z) \\
F'(v; z) \\
F'(v, z) \\
\cdots \\
F'(v, z)
\end{bmatrix}
$$

Since the coefficient matrix is a Vandermonde matrix with nonzero determinant there is a unique solution to the system of equations which is $P_0(v, z) = F'(v, z)$, $P_i = 0$ for $i > 0$. Thus, $F'(v, z)$ is a polynomial in v and z. ∎

An immediate consequence of lemma 6 is .

Lemma 7. *Let H be a closed subgroup of the compact abelian Lie group G with dimension $G = 1+$ dimension H and $G/H \approx S^1$. There is a functor $ind_H^G()$ from the category of (nonsingular) real algebraic $H-$varieties and regular $H-$equivariant maps to the category of (nonsingular) real algebraic $G-$varieties and regular $G-$equivariant maps.*

Remark 2. Lemma 6 also applies to the case in which F is an $H-$invariant real or complex function. Hence we have another corollary of lemma 6.

Lemma 8. *Let H be a closed subgroup of the compact abelian Lie group G with dimension $G = 1+$ dimension $H, G/H \approx S^1$. There is a functor $ind_H^G()$ from the category of (nonsingular) real algebraic $H-$varieties and $H-$equivariant entire rational maps to the category of (nonsingular) real algebraic $G-$varieties and $G-$equivariant entire rational maps.*

Remark 3. Choices were made in defining $ind_H^G(X)$; there was a choice as to the extension to G of a linear action of H on the vector space V' in lemma 3 and there was a choice made in the "composition series" $H = G_0 \subset G_1 \subset \ldots G_n \subset G$ in lemma 1. For any $H-$variety X the identity map $i : X \to X$ is an $H-$equivariant regular map and if $ind_H^G(X)$ and $ind_H^G(X)'$ are the varieties that result from two different choices of extension in lemma 3 there are regular $G-$equivariant maps $ind_H^G(i) : ind_H^G(X) \to ind_H^G(X)', ind_H^G(i)' : ind_H^G(X)' \to ind_H^G(X)$ by lemma 6 with $ind_H^G(i)'^0 \, ind_H^G(i)$ and $ind_H^G(i)^0 \, ind_H^G(i)'$ the identity so that the result is, in fact, unique. It is a priori possible that different choices in lemma 1 might lead to different functors.

The proof of the theorems is now complete.

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