AN EXTENSION OF SIMONS’ INEQUALITY
AND APPLICATIONS

Robert DEVILLE and Catherine FINET

Abstract

This article is devoted to an extension of Simons’ inequality. As a consequence, having a pointwise converging sequence of functions, we get criteria of uniform convergence of an associated sequence of functions.

I Introduction

Simons’ inequality is a useful tool in Banach space geometry. Simons has observed in [S1] that this inequality allows to prove that if \((f_n)\) is a uniformly bounded sequence of real valued continuous functions on a compact space which converges pointwise to a continuous function \(g\), then there is a sequence of convex combinations of the \(f_n\)’s that converges uniformly to \(g\). Later, Godefroy ([G]) found other applications of this inequality (see also [FG] and [GZ]). And more recently, Acosta and Galán ([AG]) improved James theorem in the case of smooth Banach spaces. Our main result is the following extension of Simons’ inequality [S1]. We believe that this extension may have applications in non linear analysis.

Acknowledgement. The authors thank the referee for the remarks he made on this article.

*The research presented in this paper was supported by a FNRS grant and La Banque Nationale de Belgique. The paper was written when the first author visited the Department of Mathematics of the University of Mons-Hainaut (Belgium).

2000 Mathematics Subject Classification: 46B20.
Servicio de Publicaciones. Universidad Complutense. Madrid, 2001
II. Main result

Theorem 1. Let $B$ be a set and $C$ be a non empty subset of a linear normed space that is stable with respect to taking infinite convex combinations. Let $f : C \times B \to \mathbb{R}$ be a bounded function such that the mappings $x \to f(x, b)$ are convex and Lipschitz continuous, with a Lipschitz constant independent of $b$. Let us also assume that

(*) \quad \left\{ \begin{array}{l} \text{for every } x \in C \text{ there is a } b \in B \text{ such that } \\ f(x, b) = \sup_{\beta \in B} f(x, \beta) \end{array} \right.

Then if $(x_n)_n$ is a sequence in $C$, we have

$$\inf_{x \in C} \sup_{\beta \in B} f(x, \beta) \leq \sup_{\beta \in B} \limsup_{n} f(x_n, \beta).$$

In particular, if we take as $C$ a certain subset of $\ell^\infty(B)$, the Banach space of all bounded real functions on $B$, we get the “classical” Simons’ inequality.

Corollary. (Simons’ inequality). Let $B$ be a set and $C$ be a non empty bounded subset of $\ell^\infty(B)$ that is stable with respect to taking infinite convex combinations. Let us assume that for every $x \in C$, there is a $b \in B$ such that

$$x(b) = \sup_{\beta \in B} x(\beta)$$

Then if $(x_n)$ is a sequence in $C$, we have

$$\inf_{x \in C} \sup_{\beta \in B} x(\beta) \leq \sup_{\beta \in B} \limsup_{n} x_n(\beta)$$

Let us now discuss the assumption “$C$ is stable by taking infinite convex combinations”. This assumption is clearly satisfied if $C$ is a closed convex subset of a Banach space. On the other hand, it is always satisfied by bounded convex subsets of a finite dimensional vector space $V$. This can be proved by induction. Indeed, this is clear if the dimension of the space is equal to 1. We can assume that $0 \in C$. $C$ satisfies one of the following conditions: either $C$ is contained in a linear proper subspace of $V$ or $C$ has non empty interior. In the first case, the statement follows.
from our assumption. If $C$ has non empty interior, let us assume that the result holds for vector spaces of dimension $\leq n$. Let $C$ be a bounded convex subset of a vector space $V$ of dimension $n + 1$. Let us assume that there exist points $x_n$ in $C$ and scalars $\lambda_n > 0$ such that

$$\sum_{n=1}^{+\infty} \lambda_n = 1$$

and

$$\sum_{n=1}^{+\infty} \lambda_n x_n \notin C.$$  

By Hahn-Banach Theorem, there exists $\varphi \in V^*$ such that

$$\varphi \left( \sum_{n=1}^{+\infty} \lambda_n x_n \right) = 1$$

and

$$\varphi(x) \leq 1 \text{ for } x \in C.$$  

There exists $n_0$ such that $\varphi(x_{n_0}) < 1$. Indeed, otherwise the induction hypothesis would not be satisfied for the convex $C \cap \{\varphi = 1\}$. But

$$\varphi \left( \sum_{n=1}^{+\infty} \lambda_n x_n \right) = \sum_{n=1}^{+\infty} \lambda_n \varphi(x_n) < 1$$

and this gives us a contradiction.

Looking at the extension of Simons’ inequality we got, it is natural to recall a Min-Max Theorem (see [A], see also [S2]) and to compare both results. Recall that if $C$ is a convex subset of a vector space $V$, the finite topology on $C$ is the strongest topology for which, for each $n$ and for each $n$-uple $K = (y_1, y_2, ..., y_n)$ of elements in $C$, the mappings $f_K : C_n^+ \to C$ defined by $f_K(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i=1}^{n} \lambda_i y_i$ are continuous, where $C_n^+$ is the set of all $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n$ such that $\lambda_i \geq 0$ for all $i$ and $\sum_{i=1}^{n} \lambda_i = 1$.

**Min-max theorem.** Let $B$ be a compact space and let $C$ be a convex subset of a vector space $V$, supplied with the finite topology.

Assume that

i) for all $x \in C$, $b \to f(x, b)$ is upper semicontinuous on $B$,

ii) for all $b \in B$, $x \to f(x, b)$ is convex.

Then there exists $b_0 \in B$ such that

$$\inf_{x \in C} f(x, b_0) = \sup_{b \in B} \inf_{x \in C} f(x, b) = \sup_{c \in C \cap \{C, B\}} \inf_{x \in C} f(x, c(x)).$$
where $C(C, B)$ denotes the space of continuous functions from $C$ to $B$.

The authors conjecture that this Min-Max Theorem should be deduced from Theorem 1.

**Proof of theorem 1.** Let us consider, for $x$ in $C$, $\sigma(x) = \sup_{b \in B} f(x, b)$ and let us put,

\[
    m = \inf \{ \sigma(x), x \in C \},
    \quad M = \sup \{ \sigma(x), x \in C \}.
\]

Since $f$ is bounded on $C \times B$, we have $-\infty < m \leq M < \infty$. Let $(x_n)_n$ be a sequence in $C$ and put

\[ C_p = \text{conv} \{ x_n, n \geq p \}.
\]

We can assume $m > 0$. Let $0 < \delta < m$. Let $(a_p)$ be a sequence such that $0 < a_p \leq 1$, $\sum_{p \geq 1} a_p = 1$ and $\sum_{p > n} a_p \leq \frac{\epsilon}{M} a_n$, and let $(\epsilon_n)$ be a sequence such that

\[ 0 < \epsilon_n \leq \frac{a_{n+1}(a_n + a_{n+1})}{2A_{n+1}} \delta.
\]

where $A_n = \sum_{1 \leq p \leq n} a_p$.

Let $y_1 \in C_1$ be such that $\sigma(y_1) \leq \inf_{y \in C_1} \sigma(y) + \epsilon_1$.

If $y_1, y_2, \ldots, y_{n-1}$ have been chosen, we write $z_{n-1} = \sum_{k=1}^{n-1} a_k y_k$ and take $y_n$ in $C_n$ such that

\[ \sigma \left( \frac{z_n}{A_n} \right) \leq \inf_{y \in C_n} \sigma \left( \frac{z_{n-1} + a_n y}{A_n} \right) + \epsilon_n,
\]

Now, put $z = \sum_{p \geq 1} a_p y_p$. Clearly, $z \in C$, so, by assumption, there exists $b$ in $B$ such that, $f(z, b) = \sigma(z)$. Since

\[ z = A_{n-1} \frac{z_{n-1}}{A_{n-1}} + a_n y_n + \left( \sum_{p > n} a_p \right) \frac{\sum_{p > n} a_p y_p}{\sum_{p > n} a_p},
\]
by convexity of $f$ with respect to the first variable, we get:

$$f(z, b) \leq A_{n-1} f\left(\frac{z_{n-1}}{A_{n-1}}, b\right) + a_n f(y_n, b) + \left(\sum_{p>n} a_p\right) f\left(\frac{\sum_{p>n} a_p y_p}{\sum_{p>n} a_p}, b\right).$$

Therefore,

$$a_n f(y_n, b) \geq \sigma(z) - A_{n-1} \sigma\left(\frac{z_{n-1}}{A_{n-1}}\right) - \sum_{p>n} a_p M.$$

Hence, by the choice of $(a_n)$,

$$a_n f(y_n, b) \geq \sigma(z) - A_{n-1} \sigma\left(\frac{z_{n-1}}{A_{n-1}}\right) - \delta a_n.$$

Since $f$ is Lipschitz continuous with respect to the first variable, with Lipschitz constant independent of the second variable, $\sigma$ is Lipschitz continuous. Therefore, since $\lim A_n = 1$, then $\lim_p \sigma\left(\frac{z_p}{A_p}\right) - \sigma(z_p) = 0$, and so $\sigma(z) = \lim_p A_p \sigma\left(\frac{z_p}{A_p}\right)$, and

$$\sigma(z) - A_{n-1} \sigma\left(\frac{z_{n-1}}{A_{n-1}}\right) = \sum_{p\geq n} A_p \sigma\left(\frac{z_p}{A_p}\right) - A_{p-1} \sigma\left(\frac{z_{p-1}}{A_{p-1}}\right)$$

$$\geq \sum_{p\geq n} \left[ A_{p-1} \left( \sigma\left(\frac{z_p}{A_p}\right) - \sigma\left(\frac{z_{p-1}}{A_{p-1}}\right) \right) + a_p m \right].$$

Let us put $\Delta_p = \sigma\left(\frac{z_p}{A_p}\right) - \sigma\left(\frac{z_{p-1}}{A_{p-1}}\right)$. The following lemma will lead us to a good estimate of $\sum_{p\geq n} A_{p-1} \Delta_p$. We will give the proof of this lemma after the end of the proof of the theorem.

**Lemma.** We have, for every $n \geq 2$, $\Delta_n \geq -a_n \delta$.

It follows from the lemma that

$$\sum_{p\geq n} A_{p-1} \Delta_p \geq -\delta \sum_{p\geq n} A_{p-1} a_p \geq -\delta \sum_{p\geq n} a_p.$$
Therefore,
\[ \sigma(z) - A_{n-1} \sigma\left( \frac{z_{n-1}}{A_{n-1}} \right) \geq \sum_{p \geq n} a_p (m - \delta) \geq a_n (m - \delta). \]

This estimate and (1) yield
\[ f(y_n, b) \geq m - 2\delta. \]

As \( y_n \in C_n \), by convexity of \( f \) in the first variable, for each \( n \) there exists \( k(n) \geq n \) such that \( f(x_{k(n)}, b) \geq m - 2\delta \). So, \( \limsup_{n} f(x_n, b) \geq m - 2\delta \), for all \( \delta > 0 \). Thus the theorem is proved.

We now give the proof of the lemma:

**Proof of the lemma.** We first claim that : \( \Delta_2 \geq -\epsilon_1 \) and for \( n > 2 \), \( \Delta_{n+1} \geq \gamma_n \Delta_n - 2\epsilon_n \), where \( \gamma_n = \frac{a_{n+1}}{a_n} \frac{A_n - 1}{A_{n+1}}. \)

Indeed, \( \Delta_2 = \sigma\left( \frac{z_2}{A_2} \right) - \sigma\left( \frac{z_1}{A_1} \right) \), As \( \frac{z_2}{A_2} \in C_1 \), \( \frac{z_1}{A_1} = y_1 \), by definition of \( y_1 \), \( \Delta_2 \geq -\epsilon_1 \).

Let \( r_n = \frac{a_{n+1}}{a_n} \) and \( y = \frac{y_n + r_n y_{n+1}}{1 + r_n} \). Since \( y \in C_n \), by the choice of \( z_n \) it holds
\[ \sigma\left( \frac{z_n}{A_n} \right) \leq \sigma\left( \frac{\frac{y_{n+1} + r_n y_n}{A_n (1 + r_n)}}{A_{n+1} + r_n A_n} \right) + \epsilon_n. \]

We have \( A_n (1 + r_n) = A_{n+1} + A_{n-1} r_n \), so
\[ \sigma\left( \frac{z_n}{A_n} \right) \leq \sigma\left( \frac{A_{n+1} + r_n A_n - 1 \frac{z_{n-1}}{A_{n-1}}}{A_{n+1} + r_n A_n} \right) + \epsilon_n. \]

And, by convexity,
\[ (A_{n+1} + r_n A_n - 1) \sigma\left( \frac{z_n}{A_n} \right) \leq A_{n+1} \sigma\left( \frac{z_{n+1}}{A_{n+1}} \right) + r_n A_n - 1 \sigma\left( \frac{z_{n-1}}{A_{n-1}} \right) + (A_{n+1} + r_n A_n - 1) \epsilon_n. \]

This inequality can be rewritten as follows :
\[ A_{n+1} \left[ \sigma\left( \frac{z_{n+1}}{A_{n+1}} \right) - \sigma\left( \frac{z_n}{A_n} \right) \right] \geq r_n A_n - 1 \left[ \sigma\left( \frac{z_n}{A_n} \right) - \sigma\left( \frac{z_{n-1}}{A_{n-1}} \right) \right] - (A_{n+1} + r_n A_n - 1) \epsilon_n. \]
finally we get that
\[ \Delta_{n+1} \geq r_n \frac{A_{n-1}}{A_{n+1}} \Delta_n - \left( 1 + r_n \frac{A_{n-1}}{A_{n+1}} \right) \epsilon_n \geq \gamma_n \Delta_n - 2\epsilon_n \]
this proves the claim.

The lemma then follows easily by induction from the claim and from the choice of the sequence \((\epsilon_n)\).

III Applications

In this section, we present some applications of Theorem 1 which cannot be deduced from Simons’ inequality. Recall that the convex hull of a sequence \((x_n)\) is the set of finite combinations \(\sum_{n=1}^{N} \lambda_n x_n\) with \(\lambda_n \geq 0\) for all \(n\) and \(\sum_{n=1}^{N} \lambda_n = 1\).

**Theorem 2.** Let \(B\) be a set, \(X\) be a Banach space, \(C\) be a closed convex subset of \(X\) and \(f : C \times B \to \mathbb{R}\) be a bounded function such that the mappings \(x \mapsto f(x, b)\) are convex and Lipschitz continuous, with a Lipschitz constant independent of \(b\). Let us assume that for every \(x \in C\) there exists \(b \in B\) such that
\[ f(x, b) = \sup_{\beta \in B} f(x, \beta) \]
If \((x_n)\) is a sequence in \(C\) such that for every \(\beta \in B\), \(f(x_n, \beta) \geq 0\) and \(\lim_{n \to \infty} f(x_n, \beta) = 0\), then, for all \(\epsilon > 0\), there exists \(x\) in the convex hull of the sequence \((x_n)\) such that
\[ \sup_{\beta \in B} f(x, \beta) \leq \epsilon \]
Of course, when $f$ takes values in the positive real numbers, if for every $\beta \in B$, $f(x_n, \beta)$ converges pointwise to 0, the conclusion of Theorem 2 is that there exists a sequence $(y_n)$ of convex combinations of $(x_n)$ such that $f(y_n, \beta)$ converges to 0 uniformly with respect to $\beta$.

**Proof.** Indeed, we have $\sup_{\beta \in B} \limsup_n f(x_n, \beta) = 0$. Let us denote $\tilde{C}$ the closed convex hull of the sequence $(x_n)$. Theorem 1 shows that

$$\inf_{x \in \tilde{C}} \sup_{\beta \in B} f(x, \beta) \leq 0$$

Since the convex hull of the sequence $(x_n)$ is dense in $\tilde{C}$, the above inequality and the Lipschitz continuity of $f$ with respect to the first variable imply Theorem 2.

If we take $B = \mathbb{N}$ in the above theorem, we get the following result.

**Corollary.** Let $C$ be a closed convex subset of a Banach space $X$ and $(f_n)$ be a sequence of convex continuous functions from $C$ to $\mathbb{R}$, which is uniformly bounded and uniformly Lipschitz on $C$. Let us assume that for every $x \in C$ there exists an $n_0 \in \mathbb{N}$ such that

$$f_{n_0}(x) = \sup_n f_n(x)$$

If $(x_n)$ is a sequence in $C$ such that for every $p \in \mathbb{N}$, $f_p(x_n) \geq 0$ and $\lim_n f_p(x_n) = 0$; then, for all $\epsilon > 0$, there exists $x$ in the convex hull of the sequence $(x_n)$ such that

$$\sup_{p \in \mathbb{N}} f_p(x) \leq \epsilon$$

**Remark.** The hypothesis of the convexity of $(f_n)$ cannot be dropped. Indeed, consider $X = \mathbb{R}$, $f_n(x) = \inf \{(x + n)^+, 1\}$ and a sequence $(x_n)$ tending to $-\infty$. On the other hand, if you take $f_n(x) = (x + n)^+$, you see that the hypothesis of the uniform boundedness of the sequence $(f_n)$ also cannot be dropped.

Let us recall the following result (see [S1, Corollary 10]). Let $K$ be a compact space and $(f_n)_n$ be a uniformly bounded sequence of continuous functions on $K$. If the sequence $(f_n)$ converges pointwise to zero on $K$ then it converges weakly to zero.
We now give a vector-valued extension of this result.

**Proposition.** Let $K$ be a compact space, $X$ be a Banach space and $(f_n)_n$ be a uniformly bounded sequence of continuous functions from $K$ to $X$. If the sequence $(f_n)$ converges pointwise to 0 on $K$ then there exists a sequence of linear convex combinations of $(f_n)$ which is uniformly convergent on $K$.

**Proof.** Let us denote $C$ the closed convex hull of the functions $f_n$ in the Banach space $C(K,X)$ of continuous functions from $K$ into $X$. The function $F : C \times K \to \mathbb{R}$ defined by $F(f,x) := \|f(x)\|_X$ is bounded, convex and continuous with respect to the first variable, and, for every $f \in C$, there exists $x \in K$ such that $\|f(x)\|_X = \sup_{y \in K} \|f(y)\|_X$. By assumption, $\sup_{x \in K} \limsup_n \|f_n(x)\|_X = 0$. According to Theorem 1,

$$\inf_{f \in C} \sup_{x \in K} \|f(x)\|_X \leq 0$$

This proves the proposition.

Let us mention that, by a remark of the referee, this result is also a consequence of Simon’s inequality. The subset $K \times B_{X^*}$ is a boundary of $C(K,X)$. If $(f_n)$ converges pointwise to zero on $K$, it converges pointwise on the boundary and so, it converges weakly to zero.

**References**


Robert Deville
Université de Bordeaux I
Mathématiques Pures de Bordeaux
Cours de la Libération, 351,
F 33405 Talence Cedex (France)
E-mail: deville@math.u-bordeaux.fr

Catherine Finet
Université de Mons-Hainaut
Institut de Mathématique et d’Informatique
“Le Pentagone”
Avenue du Champ de Mars, 6
B 7000 Mons (Belgique)
E-mail: catherine.finet@umh.ac.be

Recibido: 17 de Noviembre de 1999
Revisado: 3 de Julio de 2000