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AN EXTENSION OF SIMONS' INEQUALITY AND APPLICATIONS*

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Abstract

This article is devoted to an extension of Simons' inequality. As a consequence, having a pointwise converging sequence of functions, we get criteria of uniform convergence of an associated sequence of functions.

I Introduction

Simons' inequality is a useful tool in Banach space geometry. Simons has observed in [S1] that this inequality allows to prove that if (f_n) is a uniformly bounded sequence of real valued continuous functions on a compact space which converges pointwise to a continuous function g, then there is a sequence of convex combinations of the f_n's that converges uniformly to g. Later, Godefroy ([G]) found other applications of this inequality (see also [FG] and [GZ]). And more recently, Acosta and Galán ([AG]) improved James theorem in the case of smooth Banach spaces. Our main result is the following extension of Simons' inequality [S1]. We believe that this extension may have applications in non linear analysis.

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II. Main result

Theorem 1. *Let B be a set and C be a non empty subset of a linear normed space that is stable with respect to taking infinite convex combinations. Let $f : C \times B \to \mathbb{R}$ be a bounded function such that the mappings $x \to f(x, b)$ are convex and Lipschitz continuous, with a Lipschitz constant independant of b. Let us also assume that*

$$(\star) \qquad \begin{cases} \text{for every } x \in C \text{ there is a } b \in B \text{ such that} \\ \qquad f(x, b) = \sup_{\beta \in B} f(x, \beta) \end{cases}$$

Then if $(x_n)_n$ is a sequence in C, we have

$$\inf_{x \in C} \sup_{\beta \in B} f(x, \beta) \leq \sup_{\beta \in B} \limsup_{n} f(x_n, \beta).$$

In particular, if we take as C a certain subset of $\ell^\infty(B)$, the Banach space of all bounded real functions on B, we get the "classical" Simons' inequality.

Corollary. (Simons' inequality). *Let B be a set and C be a non empty bounded subset of $\ell^\infty(B)$ that is stable with respect to taking infinite convex combinations. Let us assume that for every $x \in C$, there is a $b \in B$ such that*

$$x(b) = \sup_{\beta \in B} x(\beta)$$

Then if (x_n) is a sequence in C, we have

$$\inf_{x \in C} \sup_{\beta \in B} x(\beta) \leq \sup_{\beta \in B} \limsup_{n} x_n(\beta)$$

Let us now discuss the assumption "C is stable by taking infinite convex combinations". This assumption is clearly satisfied if C is a closed convex subset of a Banach space. On the other hand, it is always satisfied by bounded convex subsets of a finite dimensional vector space V. This can be proved by induction. Indeed, this is clear if the dimension of the space is equal to 1. We can assume that $0 \in C$. C satisfies one of the following conditions: either C is contained in a linear proper subspace of V or C has non empty interior. In the first case, the statement follows from our assumption. If C has non empty interior, let us assume that the result holds for vector spaces of dimension $\leq n$. Let C be a bounded convex subset of a vector space V of dimension $n+1$. Let us assume that there exist points x_n in C and scalars $\lambda_n > 0$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1$ and $\sum_{n=1}^{+\infty} \lambda_n x_n \notin C$. By Hahn-Banach Theorem, there exists $\varphi \in V^*$ such that

$$\varphi\left(\sum_{n=1}^{+\infty} \lambda_n x_n\right) = 1 \qquad \text{and} \qquad \varphi(x) \leq 1 \quad \text{for} \quad x \in C$$

There exists n_0 such that $\varphi(x_{n_0}) < 1$. Indeed, otherwise the induction hypothesis would not be satisfied for the convex $C \cap \{\varphi = 1\}$. But

$$\varphi\left(\sum_{n=1}^{+\infty} \lambda_n x_n\right) = \sum_{n=1}^{+\infty} \lambda_n \varphi(x_n) < 1$$

and this gives us a contradiction.

Looking at the extension of Simons' inequality we got, it is natural to recall a Min-Max Theorem (see [A], see also [S2]) and to compare both results. Recall that if C is a convex subset of a vector space V, the finite topology on C is the strongest topology for which, for each n and for each n-uple $K = (y_1, y_2, ..., y_n)$ of elements in C, the mappings $f_K : C_n^+ \to C$ defined by $f_K(\lambda_1, \lambda_2, ..., \lambda_n) = \sum_{i=1}^n \lambda_i y_i$ are continuous, where C_n^+ is the set of all $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n$ such that $\lambda_i \geq 0$ for all i and $\sum_{i=1}^n \lambda_i = 1$.

Min-max theorem. *Let B be a compact space and let C be a convex subset of a vector space V, supplied with the finite topology.*
Assume that

i) for all $x \in C$, $b \to f(x,b)$ is upper semicontinuous on B,

ii) for all $b \in B$, $x \to f(x,b)$ is convex.

Then there exists $b_0 \in B$ such that

$$\inf_{x \in C} f(x, b_0) = \sup_{b \in B} \inf_{x \in C} f(x,b) = \sup_{c \in C(C,B)} \inf_{x \in C} f(x, c(x)).$$

where $\mathcal{C}(C, B)$ denotes the space of continuous functions from C to B.

The authors conjecture that this Min-Max Theorem should be deduced from Theorem 1.

Proof of theorem 1. Let us consider, for x in C, $\sigma(x) = \sup_{b\in B} f(x,b)$ and let us put,

$$\begin{aligned} m &= \inf\{\sigma(x), x \in C\} \\ M &= \sup\{\sigma(x), x \in C\}. \end{aligned}$$

Since f is bounded on $C \times B$, we have $-\infty < m \leq M < \infty$. Let $(x_n)_n$ be a sequence in C and put

$$C_p = \operatorname{conv}\{x_n, n \geq p\}.$$

We can assume $m > 0$. Let $0 < \delta < m$. Let (a_p) be a sequence such that $0 < a_p \leq 1$, $\sum_{p\geq 1} a_p = 1$ and $\sum_{p>n} a_p \leq \frac{\delta}{M} a_n$, and let (ϵ_n) be a sequence such that

$$0 < \epsilon_n \leq \frac{a_{n+1}(a_n + a_{n+1})}{2A_{n+1}}\delta.$$

where $A_n = \sum_{1\leq p\leq n} a_p$.
Let $y_1 \in C_1$ be such that $\sigma(y_1) \leq \inf_{y\in C_1} \sigma(y) + \epsilon_1$.

If $y_1, y_2, \ldots, y_{n-1}$ have been chosen, we write $z_{n-1} = \sum_{k=1}^{n-1} a_k y_k$ and take y_n in C_n such that

$$\sigma\left(\frac{z_n}{A_n}\right) \leq \inf_{y\in C_n} \sigma\left(\frac{z_{n-1} + a_n y}{A_n}\right) + \epsilon_n,$$

Now, put $z = \sum_{p\geq 1} a_p y_p$. Clearly, $z \in C$, so, by assumption, there exists b in B such that, $f(z,b) = \sigma(z)$. Since

$$z = A_{n-1}\frac{z_{n-1}}{A_{n-1}} + a_n y_n + \left(\sum_{p>n} a_p\right) \frac{\sum_{p>n} a_p y_p}{\sum_{p>n} a_p},$$

by convexity of f with respect to the first variable, we get :

$$f(z,b) \leq A_{n-1} f\left(\frac{z_{n-1}}{A_{n-1}}, b\right) + a_n f(y_n, b) + \left(\sum_{p>n} a_p\right) f\left(\frac{\sum\limits_{p>n} a_p y_p}{\sum\limits_{p>n} a_p}, b\right).$$

Therefore,

$$a_n f(y_n, b) \geq \sigma(z) - A_{n-1}\sigma\left(\frac{z_{n-1}}{A_{n-1}}\right) - \sum_{p>n} a_p M$$

Hence, by the choice of (a_n),

$$a_n f(y_n, b) \geq \sigma(z) - A_{n-1}\sigma\left(\frac{z_{n-1}}{A_{n-1}}\right) - \delta a_n. \tag{1}$$

Since f is Lipschitz continuous with respect to the first variable, with Lipschitz constant independent of the second variable, σ is Lipschitz continuous. Therefore, since $\lim\limits_n A_n = 1$, then $\lim\limits_p \sigma\left(\frac{z_p}{A_p}\right) - \sigma(z_p) = 0$, and so $\sigma(z) = \lim_p A_p \sigma\left(\frac{z_p}{A_p}\right)$, and

$$\begin{aligned}\sigma(z) - A_{n-1}\sigma\left(\frac{z_{n-1}}{A_{n-1}}\right) &= \sum_{p\geq n}\left[A_p\sigma\left(\frac{z_p}{A_p}\right) - A_{p-1}\sigma\left(\frac{z_{p-1}}{A_{p-1}}\right)\right] \\ &\geq \sum_{p\geq n}\left(\left[A_{p-1}\left(\sigma\left(\frac{z_p}{A_p}\right) - \sigma\left(\frac{z_{p-1}}{A_{p-1}}\right)\right)\right] + a_p m\right).\end{aligned}$$

Let us put $\Delta_p = \sigma\left(\frac{z_p}{A_p}\right) - \sigma\left(\frac{z_{p-1}}{A_{p-1}}\right)$. The following lemma will lead us to a good estimate of $\sum\limits_{p\geq n} A_{p-1}\Delta_p$. We will give the proof of this lemma after the end of the proof of the theorem.

Lemma. *We have, for every $n \geq 2$, $\Delta_n \geq -a_n\delta$.*

It follows from the lemma that

$$\sum_{p\geq n} A_{p-1}\Delta_p \geq -\delta \sum_{p\geq n} A_{p-1} a_p \geq -\delta \sum_{p\geq n} a_p$$

Therefore,

$$\sigma(z) - A_{n-1}\sigma\left(\frac{z_{n-1}}{A_{n-1}}\right) \geq \sum_{p\geq n} a_p(m-\delta) \geq a_n(m-\delta).$$

This estimate and (1) yield

$$f(y_n, b) \geq m - 2\delta.$$

As $y_n \in C_n$, by convexity of f in the first variable, for each n there exists $k(n) \geq n$ such that $f\left(x_{k(n)}, b\right) \geq m - 2\delta$. So, $\sup_b \limsup_n f(x_n, b) \geq m - 2\delta$, for all $\delta > 0$. Thus the theorem is proved.

■

We now give the proof of the lemma:

Proof of the lemma. We first claim that : $\Delta_2 \geq -\epsilon_1$ and for $n > 2$, $\Delta_{n+1} \geq \gamma_n \Delta_n - 2\epsilon_n$, where $\gamma_n = \frac{a_n+1}{a_n}\frac{A_{n-1}}{A_{n+1}}$.

Indeed, $\Delta_2 = \sigma\left(\frac{z_2}{A_2}\right) - \sigma\left(\frac{z_1}{A_1}\right)$. As $\frac{z_2}{A_2} \in C_1$, $\frac{z_1}{A_1} = y_1$, by definition of y_1, $\Delta_2 \geq -\epsilon_1$.
Let $r_n = \dfrac{a_{n+1}}{a_n}$ and $y = \dfrac{y_n + r_n y_{n+1}}{1+r_n}$. Since $y \in C_n$, by the choice of z_n it holds

$$\sigma\left(\frac{z_n}{A_n}\right) \leq \sigma\left(\frac{z_{n+1} + r_n z_{n-1}}{A_n\left(1+r_n\right)}\right) + \epsilon_n.$$

We have $A_n\left(1+r_n\right) = A_{n+1} + A_{n-1} r_n$, so

$$\sigma\left(\frac{z_n}{A_n}\right) \leq \sigma\left(\frac{A_{n+1}\frac{z_{n+1}}{A_{n+1}} + r_n A_{n-1}\frac{z_{n-1}}{A_{n-1}}}{A_{n+1} + r_n A_{n-1}}\right) + \epsilon_n.$$

And, by convexity,

$$\begin{aligned}(A_{n+1} + r_n A_{n-1})\sigma\left(\frac{z_n}{A_n}\right) \leq \; & A_{n+1}\sigma\left(\tfrac{z_{n+1}}{A_{n+1}}\right) + r_n A_{n-1}\sigma\left(\tfrac{z_{n-1}}{A_{n-1}}\right) \\ & +(A_{n+1} + r_n A_{n-1})\epsilon_n.\end{aligned}$$

This inequality can be rewritten as follows :

$$\begin{aligned}A_{n+1}\left[\sigma\left(\frac{z_{n+1}}{A_{n+1}}\right) - \sigma\left(\frac{z_n}{A_n}\right)\right] \;\geq\; & r_n A_{n-1}\left[\sigma\left(\frac{z_n}{A_n}\right) - \sigma\left(\frac{z_{n-1}}{A_{n-1}}\right)\right] \\ & -(A_{n+1} + r_n A_{n-1})\epsilon_n\end{aligned}$$

finally we get that

$$\Delta_{n+1} \geq r_n \frac{A_{n-1}}{A_{n+1}} \Delta_n - \left(1 + r_n \frac{A_{n-1}}{A_{n+1}}\right) \epsilon_n \geq \gamma_n \Delta_n - 2\epsilon_n$$

this proves the claim.

■

The lemma then follows easily by induction from the claim and from the choice of the sequence (ϵ_n).

■

III Applications

In this section, we present some applications of Theorem 1 which cannot be deduced from Simons' inequality. Recall that the convex hull of a sequence (x_n) is the set of finite combinations $\sum_{n=1}^{N} \lambda_n x_n$ with $\lambda_n \geq 0$ for all n and $\sum_{n=1}^{N} \lambda_n = 1$.

Theorem 2. *Let B be a set, X be a Banach space, C be a closed convex subset of X and $f : C \times B \to \mathbb{R}$ be a bounded function such that the mappings $x \to f(x,b)$ are convex and Lipschitz continuous, with a Lipschitz constant independent of b. Let us assume that for every $x \in C$ there exists a $b \in B$ such that*

$$f(x,b) = \sup_{\beta \in B} f(x,\beta)$$

If (x_n) is a sequence in C such that for every $\beta \in B$, $f(x_n,\beta) \geq 0$ and $\lim_n f(x_n,\beta) = 0$, then, for all $\epsilon > 0$, there exists x in the convex hull of the sequence (x_n) such that

$$\sup_{\beta \in B} f(x,\beta) \leq \epsilon$$

Of course, when f takes values in the positive real numbers, if for every $\beta \in B$, $f(x_n, \beta)$ converges pointwise to 0, the conclusion of Theorem 2 is that there exists a sequence (y_n) of convex combinations of (x_n) such that $f(y_n, \beta)$ converges to 0 uniformly with respect to β.

Proof. Indeed, we have $\sup_{\beta \in B} \limsup_n f(x_n, \beta) = 0$. Let us denote $\tilde{C}$ the closed convex hull of the sequence (x_n). Theorem 1 shows that

$$\inf_{x \in \tilde{C}} \sup_{\beta \in B} f(x, \beta) \leq 0$$

Since the convex hull of the sequence (x_n) is dense in $\tilde{C}$, the above inequality and the Lipschitz continuity of f with respect to the first variable imply Theorem 2.

If we take $B = \mathbb{N}$ in the above theorem, we get the following result.

Corollary. *Let C be a closed convex subset of a Banach space X and (f_n) be a sequence of convex continuous functions from C to $\mathbb{R}$, which is uniformly bounded and uniformly Lipschitz on C. Let us assume that for every $x \in C$ there exists a $n_0 \in \mathbb{N}$ such that*

$$f_{n_0}(x) = \sup_n f_n(x)$$

If (x_n) is a sequence in C such that for every $p \in \mathbb{N}$, $f_p(x_n) \geq 0$ and $\lim_n f_p(x_n) = 0$; then, for all $\epsilon > 0$, there exists x in the convex hull of the sequence (x_n) such that

$$\sup_{p \in \mathbb{N}} f_p(x) \leq \epsilon$$

Remark. The hypothesis of the convexity of (f_n) cannot be dropped. Indeed, consider $X = \mathbb{R}$, $f_n(x) = \inf\{(x+n)^+, 1\}$ and a sequence (x_n) tending to $-\infty$. On the other hand, if you take $f_n(x) = (x+n)^+$, you see that the hypothesis of the uniform boundedness of the sequence (f_n) also cannot be dropped.

Let us recall the following result (see [S1, Corollary 10]). Let K be a compact space and $(f_n)_n$ be a uniformly bounded sequence of continuous functions on K. If the sequence (f_n) converges pointwise to zero on K then it converges weakly to zero.

We now give a vector-valued extension of this result.

Proposition. *Let K be a compact space, X be a Banach space and $(f_n)_n$ be a uniformly bounded sequence of continuous functions from K to X. If the sequence (f_n) converges pointwise to 0 on K then there exists a sequence of linear convex combinations of (f_n) which is uniformly convergent on K.*

Proof. Let us denote C the closed convex hull of the functions f_n in the Banach space $C(K, X)$ of continuous functions from K into X. The function $F : C \times K \to \mathbb{R}$ defined by $F(f, x) := \|f(x)\|_X$ is bounded, convex and continuous with respect to the first variable, and, for every $f \in C$, there exists $x \in K$ such that $\|f(x)\|_X = \sup_{y \in K} \|f(y)\|_X$. By assumption, $\sup\limits_{x \in K} \limsup\limits_{n} \|f_n(x)\|_X = 0$. According to Theorem 1,

$$\inf_{f \in C} \sup_{x \in K} \|f(x)\|_X \leq 0$$

This proves the proposition.

$\blacksquare$

Let us mention that, by a remark of the referee, this result is also a consequence of Simon's inequality. The subset $K \times B_{X^*}$ is a boundary of $C(K, X)$. If (f_n) converges pointwise to zero on K, it converges pointwise on the boundary and so, it converges weakly to zero.

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