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# SIMPLE SINGULARITIES OF MULTIGERMS OF CURVES

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**Abstract**

We classify stably simple reducible curve singularities in complex spaces of any dimension. This extends the same classification of irreducible curve singularities obtained by V. I. Arnold. The proof is essentially based on the method of complete transversals by J. Bruce et al.

## 1 Introduction

Classification of simple curve singularities has been discussed in a number of papers. J. W. Bruce and T. J. Gaffney have classified irreducible plane curves in [2]. In [5] C. G. Gibson and C. A. Hobbs gave the classification of irreducible simple curve singularities in the 3-dimensional complex space. M. Giusti in [6] classified simple complete intersection one-dimensional singularities. In [4] A. Frühbis-Krüger classified so called simple determinantal singularities. Classification of irreducible (stably) simple curve singularities in a linear complex space of any dimension was made by V. I. Arnold [1]. We consider reducible curve singularities in a linear complex space of any dimension and give the list of stably simple ones.

An irreducible curve singularity at the origin in $\mathbb{C}^n$ can be described by a germ of a complex analytic map $f : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$. Let $L$ (respectively $R$) be the group of coordinate changes in $(\mathbb{C}^n, 0)$ (respectively in $(\mathbb{C}, 0)$), i.e., the group of germs of non-degenerate analytic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ (respectively $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$), let $A = L \times R$. The group $L$ (respectively $R$) is called the group of left (respectively right)

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coordinate changes. The group $A$ acts on the space of germs of maps $f:(\mathbb{C},0)\to(\mathbb{C}^n,0)$ (or of curves) by

$$(g,h)\cdot f := g\circ f\circ h^{-1} \qquad g\in L, \quad h\in R.$$

Two germs of curves $f$ and $f'$ are equivalent if they lie in one orbit of the $A$-action.

A germ $f$ is simple if there exists a neighbourhood of $f$ in the space of germs which intersects only finite number of $A$-orbits. We consider the space of germs with standard Whitney's topology: the basis of the topology consists of coimages of open sets in the space of $k$-jets for any $k$. A germ $f$ is stably simple if it remains simple after the natural immersion $\mathbb{C}^n \hookrightarrow \mathbb{C}^N$.

A reducible curve singularity is determined by a collection of maps $(\mathbb{C},0)\to(\mathbb{C}^n,0)$.

**Definition.** *A multigerm in $\mathbb{C}^n$ is a set $F=(f_1,\ldots,f_k)$ of germs of analytic maps $f_i:(\mathbb{C},0)\to(\mathbb{C}^n,0)$, where $\operatorname{Im} f_i\cap\operatorname{Im} f_j=\{0\}$ for $i\neq j$ ($f_1$,... $f_k$ are called components of the multigerm).*

Let $A=L\times R_{(1)}\times\ldots\times R_{(k)}$, where $R_{(i)}$ is (the $i$-th copy of) the group of right equivalences. The group $A$ (of right-left equivalences) acts on the space of multigerms by the formula

$$(g,h_1,\ldots h_k)\cdot(f_1,\ldots f_k)=(g\circ f_1\circ h_1^{-1},\ldots,g\circ f_k\circ h_k^{-1}).$$

**Definition.** *A multigerm $F=(f_1,\ldots,f_k)$ is called simple if there exists a neighbourhood of $F$ in the space of multigerms which intersects only finite number of $A$-orbits. It is stably simple, if it remains simple after the immersion $\mathbb{C}^n\hookrightarrow\mathbb{C}^N$.*

**Definition.** *Two multigerms $F$ and $F'$ in $\mathbb{C}^n$ are equivalent if they lie in one orbit of the $A$-action.*

We shall classify stably simple multigerms with respect to the *stable* equivalence, see [1].

## 2 Statement of the classification

Denote by $G_n$ a multigerm consisting of coordinate axes in $\mathbb{C}^n$, more exactly

$$\begin{matrix} (t_1, & 0, & \dots & 0) \\ (0, & t_2, & \dots & 0) \\ \vdots & \vdots & \ddots & \vdots \\ (0, & 0, & \dots & t_n) \end{matrix}$$

We denote by $(t^m \times k)$ an irreducible curve of the form $(\underbrace{t^m, \dots, t^m}_{k})$.

The aim of the paper is to prove the following result.

**Theorem.** *Every stably simple multigerm up to permutation of curves is stably equivalent to one and only one multigerm from the following list ($m$, $n$, $k$ and $l$ are natural numbers):*

### 1 Pairs of curves with regular first component

The first component is equal to $(t, 0)$. We write only the second one.

#### 1.1 Multigerms with both regular components

1. $(0, t)$ 2. $(t, t^k)$

#### 1.2 Multigerms with the second component of multiplicity 2

1. $(t^2, t^{2m+1})$
2. $(t^2, t^{2m+1} + t^{2n})$ $m < n < 2m$
3. $(t^2, t^{2m+1}, t^{2n})$ $m < n \leq 2m$
4. $(t^2, t^{2m+1} + t^{2n}, t^{2s})$ $m < n < s \leq 2m$
5. $(t^2, t^{2n} + t^{2m+1})$ $n \leq m$
6. $(t^2, t^{2n}, t^{2m+1})$ $n \leq m$
7. $(t^2, t^{2n} + t^{2m+1}, t^{2s+1})$ $n \leq m < s < m + n$
8. $(t^{2r+1}, t^2)$
9. $(0, t^2, t^{2r+1})$

### 1.3 Multigerms with the 3-jet $((t,0),(0,t^3))$

1. $(t^{3m+1},t^3)$
2. $(t^{3m+1},t^3,t^{3n+2})$ $\quad m\le n\le 2m$
3. $(t^{3m+1}+t^{3n+2},t^3)$ $\quad m\le n<2m$
4. $(t^{3m+1}+t^{3n+2},t^3,t^{3\ell+2})$ $\quad m\le n<\ell\le 2m$
5. $(t^{3m+2},t^3)$
6. $(t^{3m+2},t^3,t^{3n+1})$ $\quad m<n\le 2m+1$
7. $(t^{3m+2}+t^{3n+1},t^3)$ $\quad m<n\le 2m$
8. $(t^{3m+2}+t^{3n+1},t^3,t^{3\ell+2})$ $\quad m<n<\ell\le 2m+1$
9. $(0,t^3,t^{3m+1})$
10. $(t^{3n+2},t^3,t^{3m+1})$ $\quad m\le n<2m$
11. $(t^{3\ell+2},t^3,t^{3m+1}+t^{3n+2})$ $\quad m\le n\le \ell<2m,$ besides $n=l=2m-1$
12. $(0,t^3,t^{3m+2})$
13. $(t^{3\ell+1},t^3,t^{3m+2}+t^{3n+1})$ $\quad m<n\le\ell\le 2m,$ besides $n=l=2m$
14. $(t^{3n+1},t^3,t^{3m+2})$ $\quad m<n\le 2m$
15. $(0,t^3,t^{3m+1},t^{3n+2})$ $\quad m\le n<2m$
16. $(0,t^3,t^{3m+1}+t^{3n+2})$ $\quad m\le n<2m-1$
17. $(0,t^3,t^{3m+1}+t^{3n+2},t^{3\ell+2})$ $\quad m\le n<\ell<2m$
18. $(0,t^3,t^{3m+2},t^{3n+1})$ $\quad m<n\le 2m$
19. $(0,t^3,t^{3m+2}+t^{3n+1})$ $\quad m<n<2m$
20. $(0,t^3,t^{3m+2}+t^{3n+1},t^{3\ell+1})$ $\quad m<n<\ell\le 2m$

### 1.4 Multigerms with the 3-jet $((t,0),(t^3,0))$

| | |
|---|---|
| 1. $(t^3,t^4)$ | 2. $(t^3,t^4,t^5)$ |
| 3. $(t^3,t^4,t^5,t^6)$ | 4. $(t^3,t^4+t^6)$ |
| 5. $(t^3,t^4+t^6,t^9)$ | 6. $(t^3,t^4,t^6)$ |
| 7. $(t^3,t^4,t^9)$ | 8. $(t^3,t^5,t^6)$ |
| 9. $(t^3,t^5,t^6+t^7)$ | 10. $(t^3,t^5,t^6,t^7)$ |
| 11. $(t^3,t^5+t^6,t^7)$ | 12. $(t^3,t^5+t^6,t^7,t^9)$ |
| 13. $(t^3,t^5+t^6,t^9)$ | 14. $(t^3,t^5,t^7)$ |
| 15. $(t^3,t^5,t^7,t^9)$ | 16. $(t^3,t^5,t^9)$ |
| 17. $(t^3,t^5+t^6,t^{12})$ | 18. $(t^3,t^5+t^6)$ |
| 19. $(t^3,t^5)$ | 20. $(t^3,t^5,t^{12})$ |
| 21. $(t^3,t^5+t^9)$ | 22. $(t^3,t^5+t^9,t^{12})$ |
| 23. $(t^3,t^6,t^7,t^8)$ | |

### 1.5 Multigerms with the 4-jet $((t,0),(0,t^4))$ or $((t,0),(t^4,0))$

| | |
|---|---|
| 1. $(t^5,t^4,t^6,t^7)$ | 2. $(t^6,t^4,t^5,t^7)$ |
| 3. $(0,t^4,t^5,t^7)$ | 4. $(0,t^4,t^5,t^6)$ |
| 5. $(0,t^4,t^5,t^6,t^7)$ | 6. $(t^7,t^4,t^5,t^6)$ |
| 7. $(0,t^4,t^6,t^7,t^9)$ | 8. $(0,t^4,t^6,t^7)$ |
| 9. $(t^9,t^4,t^6,t^7)$ | 10. $(t^4,t^5,t^6,t^7)$ |
| 11. $(t^4,t^5,t^6,t^7,t^8)$ | |

### 1.6 Multigerms with the 5-jet $((t,0),(0,t^5))$

| | |
|---|---|
| 1. $(t^9,t^5,t^6,t^7,t^8)$ | 2. $(0,t^5,t^6,t^7,t^8)$ |
| 3. $(0,t^5,t^6,t^7,t^8,t^9)$ | 4. $(t^8,t^5,t^6,t^7,t^9)$ |
| 5. $(0,t^5,t^6,t^7,t^9)$ | |

## 2 Pairs of curves with singular components

### 2.1 Infinite series

The first component is equal to $(t^2,t^{2m+1})$. We write only the second component (here $m \le n$).

| | |
|---|---|
| 1. $(t^{2n+1},t^2)$ | 2. $(t^{2n+1},t^2,t^{2n+3})$ |
| 3. $(0,t^2,t^{2n+1})$ | 4. $(t^{2n+1},0,t^2)$ |
| 5. $(0,t^{2n+1},t^2)$ | 6. $(0,0,t^2,t^{2n+1})$ |

### 2.2 Individual singularities

First component is equal to $(t^2,t^3)$.

| | |
|---|---|
| 1. $(t^2,0,t^3,t^4)$ | 2. $(t^2,0,t^3)$ |
| 3. $(0,0,t^3,t^4,t^5)$ | 4. $(0,t^5,t^3,t^4)$ |
| 5. $(t^5,0,t^3,t^4)$ | 6. $(0,0,t^3,t^4)$ |
| 7. $(0,t^4,t^3,t^5)$ | 8. $(t^4,0,t^3,t^5)$ |
| 9. $(0,0,t^3,t^5,t^7)$ | 10. $(0,t^7,t^3,t^5)$ |
| 11. $(t^7,0,t^3,t^5)$ | 12. $(0,0,t^3,t^5)$ |

## 3 Multigerms with regular components

1. $G_n$
2. $G_n,(t\times k,0,\dots,0)\quad 1<k\le n$
3. $G_n,(t,t^m\times k,0,\dots,0)\quad 1\le k<n,\ m>1$
4. $G_n,(t,0\times(n-1),t^m)\quad m>1$
5. $(t_1,0,0),(t_2,t_2^2,0),(t_3,0,t_3^2)$

# 4 Multigerms with one singular component

## 4.1 Series with any number of regular components

The regular part is equal to $G_n,\ n \geq 2$. We write only the singular component.

1. $(0 \times n, t^2, t^{2m+1})$
2. $(t^{2m+1} \times k, 0 \times (n-k), t^2) \quad 1 \leq k \leq n$
3. $(t^2 \times k, 0 \times (n-k), t^3) \quad 1 < k \leq n$
4. $(t^2, 0 \times (n-1), t^3, t^4)$
5. $(t^2, t^4 \times k, 0 \times (n-k-1), t^3) \quad 0 \leq k < n$
6. $(0 \times n, t^3, t^4, t^5)$
7. $(t^5 \times k, 0 \times (n-k), t^3, t^4) \quad 0 \leq k \leq n$
8. $(t^4 \times k, 0 \times (n-k), t^3, t^5) \quad 0 \leq k \leq n$
9. $(0 \times n, t^3, t^5, t^7)$
10. $(t^7 \times k, 0 \times (n-k), t^3, t^5) \quad 1 \leq k \leq n$

## 4.2 Infinite series with two regular components

The regular part is equal to $G_2$. We write only the singular component.

1. $(t^2, t^2, t^{2m+1}) \quad m \geq 2$
2. $(t^2, t^2 + t^{2m+1}, t^{2m+3}) \quad m \geq 1$
3. $(t^2, t^2 + t^{2m+1}) \quad m \geq 1$
4. $(t^2, 0, t^{2m+1}, t^{2n}) \quad m < n \leq 2m$
5. $(t^2, t^{2n}, t^{2m+1} + t^{2n}) \quad m < n < 2m$
6. $(t^2, t^{2n}, t^{2m+1}) \quad m < n \leq 2m$
7. $(t^2, 0, t^{2m+1} + t^{2n}, t^{2\ell}) \quad m < n < \ell \leq 2m$
8. $(t^2, t^{2\ell}, t^{2m+1} + t^{2n}) \quad m < n < \ell \leq 2m$
9. $(t^2, 0, t^{2m+1} + t^{2n}) \quad m < n < 2m$
10. $(t^2, 0, t^{2m+1})$
11. $(t^2, t^{2m+1}, t^{2n}) \quad m < n \leq 2m+1$
12. $(t^2, t^{2m+1} + t^{2n}, t^{2\ell}) \quad m < n < \ell \leq 2m+1$
13. $(t^2, t^{2m+1} + t^{2n}) \quad m < n \leq 2m$
14. $(t^2, t^{2m+1})$
15. $(t^2, 0, t^{2m}, t^{2n+1}) \quad 1 < m \leq n$
16. $(t^2, t^{2n+1}, t^{2m}) \quad m \leq n$
17. $(t^2, 0, t^{2m} + t^{2n+1}, t^{2\ell+1}) \quad m \leq n < \ell < m+n$
18. $(t^2, t^{2\ell+1}, t^{2m} + t^{2n+1}) \quad m \leq n \leq \ell < m+n$
19. $(t^2, 0, t^{2m} + t^{2n+1}) \quad 1 < m \leq n$
20. $(t^2, t^{2m}, t^{2n+1}) \quad 1 < m \leq n$
21. $(t^2, t^{2m} + t^{2n+1}, t^{2\ell+1}) \quad m \leq n < \ell \leq m+n$
22. $(t^2, t^{2m} + t^{2n+1}) \quad 1 < m \leq n$

**4.3 Individual singularities**
The regular part is equal to $G_2$.

1. $(t^3, t^3, t^4, t^5)$ 2. $(t^3, 0, t^4, t^5, t^6)$
3. $(t^3, t^6, t^4, t^5)$ 4. $(t^3, 0, t^4, t^5)$
5. $(0, 0, t^4, t^5, t^6, t^7)$ 6. $(t^7, t^7, t^4, t^5, t^6)$
7. $(t^7, 0, t^4, t^5, t^6)$ 8. $(0, 0, t^4, t^5, t^6)$
9. $(t^6, t^6, t^4, t^5, t^7)$ 10. $(t^6, 0, t^4, t^5, t^7)$
11. $(0, 0, t^4, t^5, t^7)$

**4.4 Series with the regular part** $((t_1, 0), (t_2, t_2^2))$

1. $(0, 0, t^2, t^{2m+1})$
2. $(0, t^{2m+1}, t^2)$
3. $(t^{2m+1}, 0, t^2)$

**5 Multigerms with two singular components**

These multigerms contain three components. The first and the third components are $(t, 0, 0, 0)$ and $(0, 0, t_2^2, t_2^3)$ correspondingly. In the following list $m \geq 1$.

1. $(0, t_1^2, 0, 0, t_1^{2m+1})$
2. $(t_1^{2m+1}, t_1^2, 0, t_1^{2m+1})$
3. $(0, t_1^2, 0, t_1^{2m+1})$
4. $(t_1^{2m+1}, t_1^2, t_1^{2m+1}, 0)$
5. $(0, t_1^2, t_1^{2m+1}, 0)$
6. $(t_1^{2m+1}, t_1^2, 0, 0)$.

In what follows we denote by $i.j.k$ the $k$-th multigerm from part $i.j$ of the list.

## 3 The methods of classification

Let $f$ be a germ of a curve in $(\mathbb{C}^n, 0)$. Since $f$ is a germ of an analytic mapping it can be represented by power series.

**Definition.** *The power of the first monomial with a non-zero coefficient in the power decomposition of $f$ is called the multiplicity of $f$.*

Though this notion is applicable only for germs of irreducible curves we can speak about the multiplicity of each component of multigerm.

We shall also use the concept of the invariant semigroup of $f$ from [5, section 4].

By $M_n$ denote the ring of germs of analytic maps $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. $M_n$ is the maximal ideal in the ring of germs of analytic maps $(\mathbb{C}^n, 0) \to \mathbb{C}$.

**Definition.** *Let $f$ be a germ of a curve in $(\mathbb{C}^n, 0)$. For any $\phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ there is a natural valuation* ord : $\phi \mapsto \operatorname{ord}\phi$*, where* $\operatorname{ord}\phi$ *is the order of the power decomposition of $\phi$ at 0. Note that* ord *is a semigroup homomorphism. The subsemigroup $S_k(f) = \operatorname{ord} f^*(M_n^k) \subset \mathbb{Z}$ is an $A$-invariant of the curve $f$ and is called the invariant semigroup of $f$. $S_0(f)$ is called the (classical) value semigroup.*

**Definition.** *Let $p$ be the smallest positive integer in the value semigroup, and $q$ be the smallest integer in the semigroup which is greater than $p$ and is not a multiple of $p$. The integer $p$ is the* multiplicity *of the germ, and the pair $(p, q)$ is called the* invariant *pair of $f$ (see [5]).*

For multigerms we can define the invariant semigroup in the similar way. Let $F = (f_1, \dots, f_k)$ be a multigerm with $k$ components in $(\mathbb{C}^n, 0)$, $\phi \in M_n^1$. Then $F^*(\phi) := (f_1^*(\phi), \dots, f_k^*(\phi))$ and $\operatorname{ord} F^*(\phi) := (\operatorname{ord} f_1^*(\phi), \dots, \operatorname{ord} f_k^*(\phi))$. So the invariant semigroup of the multigerm $F$ is a subsemigroup of $\mathbb{N}^k$.

Denote by $A_r$ the subgroup of $A$ consisting of those $A$-changes whose $r$-jet is equal the identity, $A_r \triangleleft A$. We use the following statement (see [5, Proposition 4.2]).

**Lemma 1.** *Let $f$ be a germ of a curve in $(\mathbb{C}^n, 0)$ with $n \geq 2$. Then the largest integer $N_k$ which is not in the invariant semigroup $S_k(f)$ is at the same time the degree of $L_{(k-1)}$-determinacy of $f$.*

In the sequel we prove that our multigerms are finite determined in the formal case, but one can prove this in the analytic case using the Malgrange preparation theorem.

Now we shall formulate a very important theorem from [3].

**Lemma 2 (Mather; see [3]).** *Let $G$ be a Lie group acting smoothly on a finite dimensional manifold $V$. Let $X$ be a connected submanifold of $V$. Then $X$ is contained in a single orbit of $G$ if and only if*

1. *for each $x \in X$ the tangent space $T_x X \subset T_x(G \cdot x)$*
2. *dim $T_x(G \cdot x)$ is constant for all $x \in X$.*

We shall use this statement in the following situation: $V$ is a space of $m$-jets of multigerms with $k$ components, $G = A^{(m)}$ is the group of $m$-jets of $A$-changes.

**Lemma 3 (The method of complete transversals).** *Let $F$ be a multigerm in $(\mathbb{C}^n, 0)$ with $k$ components and let $j^m F$ be the $m$-jet of $F$. Let $W$ be a vector subspace of the space $H^{m+1}$ of homogeneous polynomial multigerms of degree $m+1$ with $k$ components in $\mathbb{C}^n$ such that*

$$T(A_1^{(m+1)} \cdot F) + W \supset H^{(m+1)}.$$

*Then any $(m+1)$-jet with the $m$-jet $j^m F$ is $A_1^{(m+1)}$-equivalent to $F + w$ for some $w \in W$.*

This statement can be derived from the Proposition 2.2 from [3]. The space $W$ is called a complete transversal. Sometimes the (affine) space $F + W$ is also called a complete transversal.

The general method is the following. We fix the 1-jet of a multigerm and move to higher jets using the method of complete transversals. When we obtain a complete transversal we try to simplify it using the Mather Lemma. If we obtain a finite number of jets we consider each of them separately. If our $m$-jet is $m$-determined we add it to our list. If we obtain a family which can be parameterised by a parameter (or by parameters) we conclude that this jet and all jets adjacent to it are not simple (the jet $g$ is adjacent to the jet $f$ if $g$ is contained in the closure of the orbit of $f$).

To prove that a given $k$-jet is not simple we often use the following observation. We consider a special submanifold $M \subset J^k$ that contains the jet. We prove that the tangent space to the $A^{(k)}$-orbit does not contain the tangent space to $M$ at any point of it. Then each point of $M$ is not simple. In particular, if the dimension of the tangent space to an $A^{(k)}$-orbit is less than the dimension of the submanifold $M$, then each point of $M$ is not simple.

Sometimes (as in [1]) we denote a germ $(t^m + t^n, t^k, \dots)$ by $((m, n), k, \dots)$.

# Part I
# Multigerms with two components

## 4 Pairs of curves with one regular component

### 4.1 Multigerms with both components regular

We refer to a curve as regular if it can be reduced to the form $(t, 0, ..., 0)$. In this part we assume that the first component of a pair is regular and has been reduced to the normal form.

**Lemma 4.** *A pair with two regular components is equivalent to one of the normal forms* 1.1.1 *and* 1.1.2*.*

**Proof.** The 1-jet of the second component is $(at, bt)$ which is equivalent to $(t, 0)$ or $(0, t)$. Consider the first case. Let $k$ be the minimal number such that the $k$-jet is not $(t, 0)$ (if k is infinity then the pair is not simple). Then the multigerm is $k$-determined and is equivalent to $((t, 0), (t, t^k))$. In the second case the multigerm is 1-determined and we obtain the normal form 1.1.1. ■

### 4.2 Multigerms with the second component of multiplicity 2

Now assume that the 1-jet of the second component is trivial. The non-trivial 2-jet is equivalent to $(t^2, 0)$ or $(0, t^2)$.

**Lemma 5.** *The second components with the* 2*-jet* $(t^2, 0)$ *or* $(0, t^2)$ *are equivalent to* 1.2.1–1.2.9*.*

**Proof.** Suppose the 2-jet of the second component is $(t^2, 0)$. Let $k$ be the minimal number such that the $k$-jet is not $(t^2, 0)$.

Suppose $k$ is odd and is equal to $2m+1$. Then the $k$-jet is equivalent to $(t^2, t^{2m+1})$. If there are no more non-trivial monomials then we obtain the normal form 1.2.1 since it is at most $4m$-determined. Note that a complete transversal in $J^n$ is trivial for odd $n$. Therefore a non-trivial complete transversal, for some $n \leq 2m$, is

$$(t^2, t^{2m+1} + bt^{2n}, ct^{2n}).$$

If $c \neq 0$ then we obtain the normal form 1.2.3. If, for all $k > 2n$, the $k$-jet equals the $2n$-jet then we obtain the normal form 1.2.2, otherwise for some $s$ a complete transversal in $J^{2s}$ lies in the family

$$(t^2, t^{2m+1} + t^{2n} + bt^{2s}, ct^{2s}), \quad n < s \leq 2m.$$

If $c \neq 0$ then we obtain the normal form 1.2.4. Let $c = 0$. The tangent space to the $A^{2s}$-orbit contains the vectors $(2t^{2+2s-2n}, 2nt^{2s} + (2m+1)t^{2(m+s-n)+1})$, $(t^{2+2s-2n}, 0)$, $(t^{2s}, 0)$ and $(0, t^{2(m+s-n)+1} + t^{2s})$. Therefore we obtain the normal form 1.2.2.

Consider the case when $k$ is even and is equal to $2n$, i.e., when the $k$-jet is equivalent to $(t^2, t^{2n})$. Let $2m+1$ be the minimal order of the jet which is different from $(t^2, t^{2n})$. In $J^{2m+1}$ it is equivalent to

$$(t^2, t^{2n} + bt^{2m+1}, ct^{2m+1}), \quad m \geq n$$

If $c \neq 0$ then the second component is equivalent to the normal form 1.2.6. Suppose $c = 0$. Then the $(2m+1)$-jet is equivalent to $(t^2, t^{2n} + t^{2m+1})$. If the higher jets are equal to the $(2m+1)$-jet then we obtain the normal form 1.2.5. Otherwise there exists $J^{2s+1}$ where a complete transversal is

$$(t^2, t^{2n} + t^{2m+1}, ct^{2s+1}).$$

Note that the tangent space to the $A^{2m+2n+1}$-orbit contains the vectors $(0, 0, t^{4n} + 2t^{2(m+n)+1})$, $(0, 0, t^{4n} + t^{2m+2n+1})$ and therefore the vector $(0, 0, t^{2m+2n+1})$. It gives the restriction $s < m + n$.

If $c \neq 0$ then we obtain the normal form 1.2.7. Otherwise the second component is equivalent to the normal form 1.2.5 since the tangent space to the $A^{2s+1}$-orbit contains the vectors $(t^{2s+1}, 0, 0)$ and $(2t^{2+2s-2m}, 2nt^{2n+2s-2m} + (2m+1)t^{2s+1})$, $(0, t^{2n+2s-2m} + t^{2s+1})$.

Suppose that the 2-jet is $(0, t^2)$. If, for all $k > 2$, the $k$-jet is equal to the 2-jet then the multigerm is not simple. Otherwise the second component is equivalent to the normal form 1.2.8 or 1.2.9. ∎

## 4.3 Pairs with the 3-jet $((t, 0), (0, t^3))$

**Lemma 6.** *Any simple pair of curves with the 3-jet* $((t, 0), (0, t^3))$ *is equivalent to one of the normal forms* 1.3.1–1.3.20

**Proof.** In $J^{3m}$ a complete transversal is trivial. In $J^{3m+1}$ it is equivalent to

$$(at^{3m+1}, t^3, bt^{3m+1}). \tag{1}$$

We can eliminate the monomials in the other coordinates by simple permutation of coordinates and left equivalences.

If $b \neq 0$ we obtain the $(3m+1)$-jet $(0, t^3, t^{3m+1})$. This jet is $(6m-1)$-determined. A complete transversal in $J^{3n+1}$ is trivial. If $n < 2m$ a complete transversal in $J^{3n+2}$ is equivalent to

$$(ct^{3n+2}, t^3, t^{3m+1} + dt^{3n+2}, et^{3n+2}). \tag{2}$$

If $e \neq 0$ we obtain the normal form 1.3.15. If $e = 0$ and $c \neq 0$ we obtain the normal form 1.3.10 if $d = 0$ and the normal form 1.3.11 $(n = l)$ if $d \neq 0$. In the last case, if $n = l = 2m - 1$ we obtain $(t^{6m-1}, t^3, t^{3m+1} + t^{6m-1})$. The tangent space to the $A^{(6m-1)}$-orbit contains the vectors $(0, 3t^{3m+1}, (3m+1)t^{6m-1})$ and $(0, t^{6m-1}, 0)$, so, using the Mather Lemma, we obtain the normal form 1.3.10. If $c = 0$ and $d \neq 0$ our jet is equivalent to $(0, t^3, t^{3m+1} + t^{3n+2})$. We have to move to higher jets. A complete transversal in $J^{3\ell+2}$, where $n < \ell < 2m$, is equivalent to

$$(at^{3\ell+2}, t^3, t^{3m+1} + t^{3n+2}, bt^{3\ell+2})$$

since the tangent space to the $A_1^{(3\ell+2)}$-orbit contains the vectors $(0, (3m+1)t^{3(m+\ell-n)+1} + (3n+2)t^{3\ell+2})$ and $(0, t^{3(m+\ell-n)+1} + t^{3\ell+2})$. If $b \neq 0$ we obtain the normal form 1.3.17. If $b = 0$ and $a \neq 0$ we obtain the normal form 1.3.11. If $a = b = 0$ for every $\ell$ such that $n < \ell < 2m$ we obtain the normal form 1.3.16. If in (2) $c = d = 0$, for every $n$ such that $m \leq n < 2m$, we obtain the normal form 1.3.9. Similar reasonings with the $(3m+2)$-jet $(0, t^3, t^{3m+2})$ produce the normal forms 1.3.12–1.3.14 and 1.3.18–1.3.20.

Let's now suppose that $b = 0$ and $a \neq 0$ in (1). We obtain the $(3m+1)$-jet $(t^{3m+1}, t^3)$. This jet is $(6m+2)$-determined. A complete transversal in $J^{3n+1}$ is trivial. If $m \leq n \leq 2m$ then a complete transversal in $J^{3n+2}$ is equivalent to

$$(t^{3m+1} + ct^{3n+2}, t^3, dt^{3n+2}). \tag{3}$$

If $d \neq 0$ we obtain the normal form 1.3.2. If $d = 0$ and $c \neq 0$ we obtain the $(3n+2)$-jet $(t^{3m+1} + t^{3n+2}, t^3)$. If $n = 2m$ we have the $(6m+2)$-jet

$(t^{3m+1}+t^{6m+2},t^3)$. Since the tangent space to the $A^{6m+2}$-orbit contains the vectors $((t^2,0),(t^{6m+2},0))$ and $((2t^2,0),(0,0))$ (here we write both components), we obtain $(t^{3m+1},t^3)$, i.e., the normal form 1.3.1. If $n < 2m$ we move to the higher jets. Consider a complete transversal in $J^{3\ell+2}$, where $n < \ell \leq 2m$. Since the tangent space to the $A_1^{3\ell+2}$-orbit contains the vectors $((3m+1)t^{3(m+\ell-n)+1}+(3n+2)t^{3\ell+2},3t^{3(\ell-n+1)})$, $(t^{3(m+l-n)+1}+t^{3\ell+2},0)$ and $(0,t^{3(\ell-n+1)})$, a complete transversal is equivalent to

$$(t^{3m+1}+t^{3n+2},t^3,ht^{3\ell+2}).$$

If $h \neq 0$ we obtain the normal form 1.3.4. If $h=0$, for any $\ell$ such that $n < \ell \leq 2m$, we obtain 1.3.3. If in (3) $c=d=0$, for any $n$ such that $m \leq n < 2m$, we obtain the normal form 1.3.1. Similar reasonings with the $(3m+2)$-jet $(3m+2,3)$ produce the normal forms 1.3.5–1.3.8. ■

## 4.4 Multigerms with the 3-jet $((t,0),(t^3,0))$

We begin with the following lemma.

**Lemma 7.** *A pair, the second component of which has the 8-jet* $(t^3,t^6,t^7)$, *is not simple.*

**Proof.** Let $X$ be the 11-dimensional space

$$(a_1t^3+a_2t^4+a_3t^5+a_4t^6+a_5t^7+a_5t^8,b_1t^6+b_2t^7+b_3t^8,c_1t^7+c_2t^8).$$

Let $x$ be a point of $X$ with nonzero coordinates: $a_i \neq 0, b_i \neq 0, c_i \neq 0$. For a group $G$ acting on $J^8$ denote by $X_G$ the intersection $T_xX \cap T_x(Gx)$, where $T_xX$ is the tangent space to $X$ and $T_x(Gx)$ is the tangent space to the $G$-orbit of the point $x$. Let's evaluate the dimension of $X_{A^8}$. The group $A^8$ is the product of $L^8$ and $R^8$. If $l \circ x \notin X$, where $l$ is a $L^8$-change and $x$ is a point of $X$ then, for any $R^8$-change $r$, the point $l \circ x \circ r^{-1}$ lies outside X. Moreover if $l \circ x \circ r^{-1}$ is a point of $X$ then the points $x \circ r^{-1}$ and $l \circ x$ lie in $X$ too. It means that a vector $v$ from $X_{A^8}$ is the sum of some vectors $v_1 \in X_{R^8}$ and $v_2 \in X_{L^8}$. Therefore $X_{A^8}$ is the sum of $X_{R^8}$ and $X_{L^8}$. $X_{R^8}$ is a 6-dimensional space and includes the vectors $(t^6,0,0),(t^7,0,0)$, and $(t^8,0,0)$, since these vectors are the linear combinations of $(3a_1t^6+4a_2t^7+5a_3t^8,0,0),(3a_1t^7+4a_2t^8,0,0)$ and $(3a_1t^8,0,0)$. The preimage of $X_{L^8}$ with respect to the tangent mapping $g: T_eL^8 \to T_xJ^8$ is

$$<(x,0,0),(y,0,0),(z,0,0),(0,y,0),(0,0,z),(0,z,0),(x^2,0,0)>.$$

Therefore the intersection $X_{R^8} \cap X_{L^8}$ is at least 3-dimensional (it contains the vectors $g(x^2,0,0)$, $g(y,0,0)$, $g(z,0,0)$) and the dimension of $X_{A^8}$ is not more than 10. This implies that any open neighbourhood of the point $x$ intersects infinitely many different orbits. ■

**Lemma 8.** *Second components of simple multigerms with the* 4*-jet* $(t^3, t^4)$ *are classified by the normal forms* 1.4.1–1.4.7.

**Proof.** First note that the second component is at most 9-determined and a complete transversal in $J^7$ and $J^8$ is trivial. A complete transversal in $J^5$ is $(t^3, t^4+bt^5, ct^5)$. If $c \neq 0$ then the 5-jet is equivalent to $(t^3, t^4, t^5)$. Therefore we obtain the normal forms 1.4.2 and 1.4.3. Suppose $c = 0$. Then the 5-jet is equivalent to $(t^3, t^4)$. In $J^6$ a complete transversal is $(t^3, t^4+bt^6, ct^6)$. If $c \neq 0$ then we obtain the normal form 1.4.6. Suppose $c = 0$ and $b \neq 0$, then the second component is equivalent to the normal forms 1.4.4 or 1.4.5. If $c = 0$ and $b = 0$ then there are two possibilities: the normal form 1.4.1 or 1.4.7. ■

**Lemma 9.** *Second components of simple multigerms with the* 5*-jet* $(t^3, t^5)$ *are classified by the normal forms* 1.4.8–1.4.22.

**Proof.** A complete transversal in $J^6$ is

$$(t^3, t^5 + bt^6, ct^6).$$

Consider $c \neq 0$. Then the multigerm is 7-determined and we obtain the normal forms 1.4.8–1.4.10 Suppose $c = 0$ and $b \neq 0$. A complete transversal in $J^7$ is $(t^3, t^5 + t^6 + et^7, ft^7)$ which is equivalent to $(t^3, t^5+t^6, ft^7)$ since the tangent space to the $A^7$-orbit contains the vectors $(t^7,0,0)$, $(3t^5, 5t^7, 0)$, $(t^5+t^6,0,0)$ and $(t^6,0,0)$. If $f \neq 0$ then the multigerm is 9-determined and we obtain the normal forms 1.4.11 and 1.4.12. Suppose $f = 0$. A complete transversal is non-trivial only in $J^9$ and $J^{12}$. Since $(t^3, t^5+t^6+\alpha t^9)$ is equivalent to $(t^3, t^5+t^6)$ (the tangent space to $A^9$ includes the vectors $(0, 5t^8+6t^9)$ and $(0,t^8)$) we obtain the normal forms 1.4.13, 1.4.17 and 1.4.18.

Consider the case $c = 0$ and $b = 0$. As above a complete transversal in $J^7$ is equivalent to $(t^3, t^5+ht^7, gt^7)$. If $g \neq 0$ then we obtain the normal forms 1.4.14 and 1.4.15. Suppose $g = 0$, then the 7-jet is equivalent to $(t^3, t^5)$. Note that a complete transversal is non-trivial only in $J^9$ and $J^{12}$. In $J^9$ it is equivalent to $(t^3, t^5+ht^9, lt^9)$. If $l \neq 0$ we obtain the normal form 1.4.16. Consider $l = 0$. There are two cases: $h = 0$ and $h \neq 0$. In the first case a complete transversal in $J^{12}$ is equivalent to $(t^3, t^5, lt^{12})$ and we obtain two normal forms: 1.4.19 and 1.4.20. In the second case we obtain the 9-jet $(t^3, t^5 + t^9)$ which produces the normal forms 1.4.21 and 1.4.22. ■

Note that Lemma 7 implies that if the 6-jet of the second component is equal to $(t^3, t^6)$ then the component is equivalent to $(t^3, t^6, t^7, t^8)$. Therefore one has

**Lemma 10.** *The only multigerm with the second component's* 6*-jet* $(t^3, t^6)$ *is equivalent to* $((t, 0), (t^3, t^6, t^7, t^8))$*, i.e., to the normal form* 1.4.23*.*

## 4.5 Multigerms with the 4-jet $((t, 0), (t^4, 0))$ or $((t, 0), (0, t^4))$

The list is restricted by the lemma.

**Lemma 11.** *Multigerms with the 7-jet*

$$(a_1t^4 + ... + a_4t^7, b_1t^4 + ... + b_4t^7, c_1t^4 + ... + c_4t^7).$$

*of the second component are not simple.*

**Proof.** As in the Lemma 7 consider the 12-dimensional space $X$:

$$(a_1t^4 + ... + a_4t^7, b_1t^4 + ... + b_4t^7, c_1t^4 + ... + c_4t^7).$$

At a point $x$ with nonzero coordinates $a_i \neq 0, b_i \neq 0, c_i \neq 0$ the dimension of $X_{L^7}$ is 7 and the dimension of $X_{R^7}$ is 4. Therefore the dimension of $X_{A^8}$ is not greater than 11. It implies that, for any neibourhood $U(x)$ of $x$, the number of orbits which intersects $U(x)$ is infinite. ■

**Lemma 12.** *The second components of simple multigerms with the* 4*-jet* $(0, t^4)$ *are classified by the normal forms* 1.5.1–1.5.9*.*

**Proof.** In $J^5$ a complete transversal is

$$(at^5, t^4, bt^5).$$

Let $b \neq 0$. A complete transversal in $J^6$ is $(ct^6, t^4, t^5 + dt^6, ft^6)$. If $f = 0$, then from Lemma 11 it follows that there are two simple components

$(t^6, t^4, t^5, t^7)$ and $(0, t^4, t^5, t^7)$. Consider $f \neq 0$. Then the 6-jet is equivalent to $(0, t^4, t^5, t^6)$. It produces the normal forms 1.5.4, 1.5.5 and 1.5.6.

Let $a \neq 0$ and $b = 0$. Then the 5-jet is equivalent to $(t^5, t^4)$ and a complete transversal in $J^6$ lies in the family

$$(t^5 + dt^6, t^4, ft^6).$$

Note that if $f = 0$, then the pair is adjacent to the pair of Lemma 11 and therefore is not simple. So the 6-jet is equivalent to $(t^5, t^4, t^6)$. Lemma 11 implies that a simple component with such a 6-jet is equivalent to $(t^5, t^4, t^6, t^7)$.

Let $a = b = 0$. A complete transversal in $J^6$ lies in the family $(ct^6, t^4, dt^6)$. Suppose $d = 0$, then the component is adjacent to the component of the Lemma 11. So $d \neq 0$ and the 6-jet is equivalent to $(0, t^4, t^6)_6$. Lemma 11 implies that the 7-jet is equivalent to $(0, t^4, t^6, t^7)$, what produces the last three normal forms. ■

Now consider multigerms with the second component $(t^4, 0)_4$. Lemma 11 implies that its 7-jet is equivalent to $(t^4, t^5, t^6, t^7)$. It is 8-determined, therefore we obtain two multigerms more:

**Lemma 13.** *The second components of simple multigerms with the* 4*-jet* $(t^4, 0)$ *has one of two forms* 1.5.10 *or* 1.5.11.

### 4.6 Curves with the 5-jet $(0, t^5)$ or $(t^5, 0)$

First consider the multigerms with the 5-jet $((t, 0), (0, t^5))$.

**Lemma 14.** *A pair of curves with the* 9*-jet* $((t, 0), (t^8, t^5, t^6, t^7))$ *is not simple.*

**Proof.** The dimension of the space $X$:

$$(a_1t^8 + a_2t^9, b_1t^5 + b_2t^6 + \ldots + b_5t^9, c_1t^6 + \ldots + c_4t^9, d_1t^7 + \ldots + d_3t^9)$$

is 14. At the point with nonzero coordinates the dimensions of $X_{R^9}$ is not greater than 5 and that of $X_{L^9}$ is 7. Therefore the dimension of $X_{A^9}$ is less than 14 and the pair is not simple. ■

**Lemma 15.** *The second components of a simple multigerm with the* 5*-jet* $(0, t^5)$ *is equivalent to one of the normal forms* 1.6.1–1.6.5.

**Proof.** Lemma 11 implies that the 7-jet of the second component is equivalent to $(0, t^5, t^6, t^7)$. A complete transversal in $J^8$ lies in the family

$$(at^8, t^5, t^6 + bt^8, t^7 + ct^8, dt^8).$$

If $d \neq 0$ then we obtain the first three normal forms. Otherwise the 8-jet is equivalent to $(at^8, t^5, t^6, t^7)$. Lemma 14 implies that the 9-jet is equivalent to $(at^8, t^5, t^6, t^7, t^9)$ and we obtain the normal forms 1.6.4 and 1.6.5. ∎

Multigerms with the 5-jet $((t, 0), (t^5, 0))$ are not simple since they are adjacent to the multigerm of Lemma 11. By the same reason there are no more simple pairs with regular first component.

# 5 Pairs of curves with singular components

Here we denote parameters on both components by $t$.

**Lemma 16.** *A pair of curves with the* 3*-jet* $((t^2, t^3), (t^2, \alpha t^3))$*, where* $\alpha \neq 1$*, is not simple.*

**Proof.** Let 's consider the tangent space to the $A^{(3)}$-orbit. It is generated by the following 8 vectors: $((t^2, 0), (t^2, 0))$, $((0, t^2), (0, t^2))$, $((t^3, 0), (\alpha t^3, 0))$, $((0, t^3), (0, \alpha t^3))$, $((2t^2, 3t^3), (0, 0))$, $((t^3, 0), (0, 0))$, $((0, 0), (2t^2, 3\alpha t^3))$, $((0, 0), (2t^3, 0))$. Note, that

$$((0, 0), (2t^2, 3\alpha t^3)) - 2((t^2, 0), (t^2, 0)) + ((2t^2, 3t^3), (0, 0)) = 3((0, t^3), (0, \alpha t^3)).$$

So, these vectors are linearly dependent and we can remove $((0, t^3), (0, \alpha t^3))$ from the list above. Hence it is obvious that the tangent space to the $A^{(3)}$-orbit does not contain the vector $((0, 0), (0, t^3))$ from the tangent space to our 1-dimensional submanifold. But if the tangent space to the $A^{(3)}$-orbit does not contain the tangent space to the submanifold, for each point of it, the multigerm fails to be simple. ∎

Let us consider the 2-jet $((t^2, 0), (0, t^2))$ first.

**Lemma 17.** *Any simple multigerm with the 2-jet of the second component* $(0, t^2)$*, such that the first component equals* $(2, 2m + 1)$ *and the invariant pair of the second component is greater than or equals* $(2, 2m+1)$*, is equivalent to one of the normal forms* 2.1.1–2.1.3.

**Proof.** We shall not calculate tangent spaces in obvious cases, but in other cases we have to do it. In $J^{2m}$ a complete transversal is trivial for any $m$. In $J^{2m+1}$ it is equivalent to

$$((t^2, at^{2m+1}, 0), (bt^{2m+1}, t^2, ct^{2m+1})).$$

There exists $m$ such that $a \neq 0$ (we suppose that the multiplicity of the first component is less than or equals that of the second one). If $c \neq 0$ we obtain the normal form $((t^2, t^{2m+1}), (0, t^2, t^{2m+1}))$. It is clear, that this multigerm is $(2m+1)$-determined, i.e., we obtain the normal form 2.1.3, $n = m$. Now suppose $c = 0$. If $b \neq 0$ then using multiplications of coordinates in the target and of parameters in the sources by constants we obtain the normal form

$$((t^2, t^{2m+1}), (t^{2m+1}, t^2)). \tag{4}$$

If $b = 0$ we obtain the normal form

$$((t^2, t^{2m+1}), (0, t^2)). \qquad (5).$$

We shall discuss (4) later. Now consider (5). In $J^{2n}$ a complete transversal is trivial for any $n$. In $J^{2n+1}$ a complete transversal is given by

$$((t^2, t^{2m+1}, 0), (bt^{2n+1}, t^2, ct^{2n+1})).$$

If $c \neq 0$ we obtain the normal form 2.1.3. If $c = 0$ and $b \neq 0$ we obtain the $(2n+1)$-jet

$$((t^2, t^{2m+1}), (t^{2n+1}, t^2)).$$

Starting from this moment, it is not important for our consideration that $n > m$, so we can also consider (4). A complete transversal in $J^{2n+2}$ is trivial. Let 's consider it in $J^{2n+3}$. We can simply obtain the following vectors from the tangent space to the A-orbit of our $(2n+2)$-jet: $((2t^{2n+3}, 0), (0, 0))$ and $((0, 0), (0, 2t^{2n+3}))$. Note that the tangent space contains the vector $((0, t^{2n+3}), ((0, t^{2+(2n+1)(n-m+1)}))$, the power of the last monomial is greater than or equals $2n+3$. So, the tangent space contains $((0, t^{2n+3}), (0, 0))$. Now, we want to obtain the vector $((0, 0), (t^{2n+3}, 0))$. It is obvious, if we note that the tangent space contains $((t^{2m+3}, 0), (t^{2n+3}, 0))$, $((2t^{2m+3}, (2m+1)t^{4m+2}), (0, 0))$ and $((0, t^{4m+2}), (0, t^{(2m+1)(2n+1)}))$, the power of the last monomial is greater than $2n+3$. After this we can conclude that a complete transversal is equivalent to

$$((t^2, t^{2m+1}), (t^{2n+1}, t^2, at^{2n+3})). \tag{6}$$

Now we shall prove that this $(2n+3)$-jet is $L$ -determined for every $a$. If, using left equivalence, we eliminate a monomial with even degree in one component, then we obtain a monomial of higher degree in the other component, so we have to eliminate monomials of odd degree.

Let 's consider the $(2n+2k+1)$-jet ($k \geq 2$) of our multigerm. We shall eliminate the monomial $t^{2n+2k+1}$ in the second component. Our construction does not depend on $a$ in (6). We can obtain this monomial as $x_1 x_2^k$. In the first component we obtain $t^{(2m+1)k+2}$. Now we have two possibilities: $k = 2i$ or $k = 2i+1$.

In the first case we consider our multigerm in $J^{2n+4i+1}, i \geq 1$ and our monomial in the first component is $t^{2i(2m+1)+2}$. It can be obtained as $x_1^{(2m+1)i+1}$. So, in the second component we obtain a monomial of degree $(2n+1)((2m+1)i+1) = 2n+1+(2n+2m+1+2nm)i > 2n+1+4i$.

In the second case we consider our multigerm in $J^{2n+4i+3}, i \geq 1$. The monomial in the first component is $t^{2i(2m+1)+2+2m+1}$, it can be obtained as $x_1^{(2m+1)i+1} x_2$. So, in the second component we obtain a monomial of degree $(2n+1)((2m+1)i+1)+2 = (2n+2m+2mn+1)i+2n+3 > 2n+4i+3$.

In this construction we did not use that $n \geq m$, so it works for eliminating the monomials in the first component as well.

Therefore, in (6), if $a \neq 0$ we obtain the normal form 2.1.2, if $a = 0$ we obtain the normal form 2.1.1. ∎

Now we suppose that the first component is $(t^2, t^{2m+1})$ and the invariant pair of second component is greater than or equals $(2, 2m+1)$.

**Lemma 18.** *Any pair of curves with the* $(2m+1)$*-jet* $((t^2, t^{2m+1}), (0, 0, t^2))$ *is equivalent to one of the normal forms* 2.1.4–2.1.6.

**Proof.** In $J^{2n+1}$ a complete transversal is equivalent to

$$((t^2, t^{2m+1}, 0, 0), (at^{2n+1}, bt^{2n+1}, t^2, ct^{2n+1})).$$

If $c \neq 0$ then we obtain the normal form 2.1.6 which is $(2n+1)$-determined. If $c = 0$ and $b \neq 0$ we have

$$((t^2, t^{2m+1}, 0), (at^{2n+1}, t^{2n+1}, t^2)).$$

One can obtain the vector $((0,0,0),(t^{2n+1},0,0))$ as a linear combination of the vectors $((t^{2m+1},0,0),(t^{2n+1},0,0))$, $((2t^{2m+1},(2m+1)t^{4m},0),(0,0,0))$ and $((0,t^{4m},0),(0,a^{2m}t^{2m(2n+1)},0))$. Then we can use the Mather Lemma and obtain the normal form 2.1.5, which is $(2n+1)$-determined. If $b=c=0$, $a\neq 0$ we obtain the normal form 2.1.4 which is also $(2n+1)$-determined. ∎

Now we shall consider the 2-jet $((t^2,0),(t^2,0))$.

**Lemma 19.** *There exist two simple multigerms with the 2-jet* $((t^2,0),(t^2,0))$. *They are classified by the normal forms* 2.2.1 *and* 2.2.2.

**Proof.** Using Lemma 16, we have only one possibility for the 3-jet: $((t^2,t^3,0),(t^2,0,t^3))$. The 4-jet of this multigerm is equivalent to

$$((t^2,t^3,0,0),(t^2+at^4,bt^4,t^3+ct^4,dt^4).$$

The tangent space to the $A^{(4)}$-orbit contains the vectors $((0,0),(2t^4,0))$ (so we can suppose $a=0$), $((0,t^4),(0,t^4))$, $((2t^3,3t^4),(0,0))$, $((t^3,0),\ (bt^4,0))$, $((0,0),\ (2t^4,0))$ (so we can suppose $b=0$), $((0,0,0),(2t^3,0,3t^4))$, $((0,0,0),(t^3+ct^4,0,0))$ and $((0,0,0),(2t^4,0,0))$ (so we can suppose $c=0$). Hence, if $d\neq 0$ we obtain the first curve, if $d=0$ we obtain the second curve. Note, that both these 4-jets are 4-determined. ∎

**Lemma 20.** *Multigerms with the 5-jet*

$$((t^2,t^3,0),(t^5,t^4+\alpha t^5,t^3))$$

*are not simple.*

**Proof.** Simple calculations in $J^5$ show that the tangent space to the $A^{(5)}$-orbit does not contain the vector $((0,0,0),(0,t^5,0))$, so $\alpha$ is a modulus in our family. ∎

Now we have to consider the case when the multiplicity of the second curve equals 3 and the first curve is $(t^2,t^3)$. Curves from the other cases are adjacent to the family from Lemma 16. More exactly, we have to consider the 3-jet $((t^2,t^3,0),(0,0,t^3))$.

**Lemma 21.** *Every simple multigerm with the 3-jet* $((t^2,t^3,0),(0,0,t^3))$ *is equivalent to one of the normal forms* 2.2.3–2.2.12.

**Proof.** A complete transversal in $J^4$ is equivalent to

$$((t^2, t^3), (at^4, bt^4, t^3, ct^4))$$

Let 's suppose that $c \neq 0$. Then our 4-jet is equivalent to $((t^2, t^3, 0, 0), (0, 0, t^3, t^4))$. This jet is 5-determined. The tangent space to its $A^{(5)}-$orbit contains the vectors $((0, 0, 0, 0), (0, 0, t^5, 0))$, $((0, 0, 0, 0), (0, 0, 3t^4, 4t^5))$ and $((0, 0, 0, 0), (0, 0, t^4, 0))$, so we obtain the curves 2.2.3–2.2.6.

Now let us consider the case $c = 0$. If $b \neq 0$ this 4-jet is equivalent to

$$((t^2, t^3, 0), (0, t^4, t^3)). \tag{7}$$

If $b = 0$, $a \neq 0$ this 4-jet is equivalent to

$$((t^2, t^3, 0), (t^4, 0, t^3)). \tag{8}$$

If $a = b = 0$ we obtain

$$((t^2, t^3, 0), (0, 0, t^3)). \tag{9}$$

Now we have to consider the 5-jet. Using Lemma 20 we see that in the second component the fourth coordinate equals $t^5$. In (7) and (8) we obtain the normal forms 2.2.7 and 2.2.8, which are 5-determined. In (9) we see that a complete transversal in $J^6$ is trivial and this curve is 7-determined. So we obtain the normal forms 2.2.9, 2.2.10, 2.2.11 and 2.2.12. ∎

## Part II

# Multigerms with three and more components

**Lemma 22.** *Simple multigerm can not contain three nonregular components.*

**Proof.** A non-regular curve is adjacent to the curve $(2, 3)$. That is why the first curve is adjacent to $(2, 3)$, the second one to $(0, 0, 2, 3)$ and the third one to $(0,0,0,0,2,3)$. Note that the third curve in the triple is adjacent to $(0,0,1,0,2,3)$ therefore the triple is adjacent to $((2,3,0,\ldots), (0,0,2,3,4,0), (0,0,1,0,0,3))$. By changing indexes of axis and curves we can obtain the triple $((1,0,0,0,0,3),(0,2,3,0,0,0),(2,0,0,3,4,0))$. The second and the third curves are adjacent to curves with the 2-jet $(1,2)$. So it remains to prove that a triple with the 2-jet $((1,0),(1,2),(1,2))$ is not simple. Therefore we have to prove the following statement. ■

**Lemma 23.** *A triple with the 2-jet* $((1,0),(1,2),(1,2))$ *is not simple.*

**Proof.** Consider the 4-dimensional subspace of the space of 2-jets:

$$((t,0),(\alpha_1 t_1, \alpha_2 t_1^2),(\alpha_3 t_2, \alpha_4 t_2^2)), \alpha_i \neq 0$$

Using $R^{(2)}$-changes each such triple can be reduced to the same form with $\alpha_1 = \alpha_2 = 1$. Note, that if one can reduce the 2-jet to the form, where $\alpha_3 = \beta_1, \alpha_4 = \beta_2$, then one can do it using only the following changes: $\tilde{x} = ax$, $\tilde{y} = by$, $t_1 = c\tilde{t_1}$ and $t_2 = d\tilde{t_2}$. Hence we have the following equations for $a, b, c, d$:

$$ac = ad = 1, \quad \alpha_3 bc^2 = \beta_1, \quad \alpha_4 bd^2 = \beta_2.$$

Therefore we have $c = d = a^{-1}$. Then $\alpha_3 ba^{-2} = \beta_1$ and $\alpha_4 ba^{-2} = \beta_2$. It implies that $\alpha_3/\beta_1 = a^2 b^{-1} = \alpha_4/\beta_2$. So the 2-jet under consideration has a continuous invariant (a modulus) $\alpha_3/\alpha_4$, i.e., it is not simple. ■

# 6 Multigerms with regular components

**Lemma 24.** *Any simple multigerm with all components regular is equivalent to one of the normal forms* 3.1–3.5.

**Proof.** Consider the 1-jet of our multigerm. Suppose that first $n$ curves can be (and are) reduced to the form $G_n$ and the 1-jets of all other curves have zero coordinates with numbers greater than $n$ (otherwise one can reduce $n+1$ curves to the form $G_{n+1}$). If the total number of curves is equal to $n$, we obtain the normal form 3.1.

Suppose that $n \geq 2$. Then our multigerm can not contain more than $(n+1)$ components. The reason is that $(n+2)$-lines in $\mathbb{C}^n$ is not a simple multigerm. For $n = 2$ they have a continuous invariant — double ratio. For $n > 2$ consider the space of 1-jets of $k$ lines in $\mathbb{C}^n$. It is $nk$-dimensional. The dimension of the tangent space to the $L^{(1)}$-orbit is less than or equals $n^2$. The dimension of the tangent space to the $R^{(1)}$-orbit is less than or equals $k$. For our multigerm to be simple it is necessary to have $nk \le n^2 + k$ i.e., $k \le n + 1 + \frac{1}{n-1}$. Hence $k \le n + 1$.

So, we have $G_n$ and a non-singular curve with zero coordinates with numbers greater than $n$. By $R^{(1)}$-transformations and permutations of curves and coordinates we can reduce the 1-jet to $G_n$ and $(\underbrace{t, \dots, t}_{k}, 0, \dots, 0)$, where $1 \le k \le n$. If $k \ge 2$ we obtain the normal form 3.2 (since $x_1 x_2^m$ equals $t_{n+1}^{m+1}$ for the last curve and zero for the first $n$ curves, $m \ge 1$). If $k = 1$ we move to higher jets. A complete transversal in $J^m$ is equivalent to

$$G_n, (t, a_1 t^m, \dots, a_n t^m).$$

If $a_n \neq 0$ we obtain the normal form 3.4. If $a_n = 0$ and there exists $j$ such that $a_j \neq 0$ we obtain the normal form 3.3 (since it is $m$-determined).

Now suppose $n = 1$. There are at least 3 components in our multigerm. If the multigerm has at least 4 components, then it fails to be simple, since its 1-jet is adjacent to 4 lines in $\mathbb{C}^2$. So, the multigerm contains 3 components. Note, that by lemma 23 the family $((t_1, 0), (t_2, t_2^2), (t_3, \alpha t_3^2))$ is not simple. So we have only one possibility for the 2-jet: $((t_1, 0, 0), (t_2, t_2^2, 0), (t_3, 0, t_3^2))$.This jet is 2-determined and we obtain the normal form 3.5. ∎

# 7 Multigerms with one non–regular component

We can suppose that the regular part of the multigerm is equivalent to one of 5 forms from the previous section (see Lemma 24 ) or to $((t_1, 0), (t_2, t_2^m))$. The forms 3.2–3.4 are not suitable, since if we add to it one non–regular curve, the 1-jet of the multigerm will be adjacent to $(n+2)$ lines in $\mathbb{C}^n$, which is not simple. If we add one non-regular curve to the normal form 3.5, the 1-jet of the multigerm would be adjacent to 4 lines in $\mathbb{C}^2$. So we have 2 possibilities for the regular part: $G_n$ or $((t_1, 0), (t_2, t_2^m))$. First consider the case, when the multiplicity of the singular component equals 2 and the regular part of the multigerm is $G_n$.

**Lemma 25.** *Any simple multigerm with the 2-jet* $(G_n, (\underbrace{0, \dots, 0}_{n}, t^2))$ *is equivalent to one of the normal forms* 4.1.1 *and* 4.1.2.

**Proof.** A complete transversal in $J^{2m}$ is trivial for any $m$. A complete transversal in $J^{2m+1}$ is equivalent to

$$G_n, \quad (a_1 t^{2m+1}, \dots, a_n t^{2m+1}, t^2, a_{n+1} t^{2m+1}).$$

If $a_{n+1} \neq 0$ we obtain the normal form 4.1.1. If $a_{n+1} = 0$ but there exists $j$ such that $a_j \neq 0$ we obtain the normal form 4.1.2 ($k$ depends on the number of the coefficients $a_j$ different from zero). ■

Now, we shall consider the other 2-jets: $G_n, (\underbrace{t^2, \dots, t^2}_{k}) \quad 1 \leq k \leq n$. This 2-jet is not sufficient, so we move to 3-jets. A complete transversal in $J^3$ is equivalent to

$$(G_n, (t^2 + a_1 t^3, \dots, t^2 + a_k t^3, a_{k+1} t^3, \dots, a_{n+1} t^3)). \tag{10}$$

First, consider the case $a_{n+1} \neq 0$.

**Lemma 26.** *Any simple multigerm with the 3-jet equals to* $G_n, (t^2, \dots, t^2, \underbrace{0, \dots, 0}_{n-k}, t^3)$ *is equivalent to one of the normal forms* 4.1.3–4.1.5.

**Proof.** The normal form 4.1.3 is 3-determined since $x_1 x_2 = t^4$. Consider the 3-jet $(G_n, (t^2, \underbrace{0, \dots, 0}_{n-1}, t^3)$. It is 5-determined so we have to consider the 4-jet. A complete transversal in $J^4$ is equivalent to

$$(G_n, (t^2, b_2 t^4, \dots, b_n t^4, t^3 + b_{n+1} t^4, b_{n+2} t^4)).$$

If $b_{n+2} \neq 0$ we obtain the normal form 4.1.4 which is 4-determined. If $b_{n+2} = 0$ we can eliminate $t^4$ in the first and the $(n+1)$-st coordinates, since the tangent space to the $A^{(4)}$-orbit contains the vectors: $(2t^4, 0, \dots, 0)$, $(2t^3, \underbrace{0 \dots, 0}_{n-1}, 3t^4)$ and $(t^3, 0, \dots, 0)$. Therefore we obtain the normal form 4.1.5 ($k$ depends on the number of the coefficients $b_j$ different from zero). ■

Now, suppose that $a_{n+1} = 0$ in (10). We shall prove that $n$ can not be greater than 2.

**Lemma 27.** *The 3-jet* $G_n, (a_1t^2 + b_1t^3, \ldots, a_nt^2 + b_nt^3)$ *is not simple if* $n > 2$.

**Proof.** The dimension of the submanifold of such jets (in $J^{(3)}$) equals $2n$. Now we shall estimate the dimension of the stabilizer of the regular part, i.e. of the subgroup in $A^{(3)}$ which preserves the regular part. The dimension of the tangent space to the $L^{(3)}$-orbit is less than or equals $n$, since it is generated by the images of the vectors $(\underbrace{0, \ldots, 0}_{j-1}, x_j, 0, \ldots, 0)$ with $1 \leq j \leq n$. The dimension of the tangent space to the $R^{(3)}$-orbit is less than or equals 2, since it is generated by the images of two vectors: $t$ and $t^2$. Hence the dimension of the tangent space to the $A^{(3)}$-orbit is less than or equals $(n + 2)$. If the dimension of the tangent space is less than the dimension of the submanifold then this 3-jet fails to be simple. Thus $n + 2 \geq 2n$, i.e., $n \leq 2$. ∎

So, if in (10) $a_{n+1} = 0$ then $n = 2$. Consider this case in detail. First suppose that $k = n = 2$ in (10).

**Lemma 28.** *Any simple multigerm with the 3-jet* (10)*, where* $k = n = 2$*, is equivalent to one of the normal forms* 4.2.1–4.2.3.

**Proof.** We shall describe all simple multigerms with the 2-jet $(G_2, (t^2, t^2))$. A complete transversal in $J^{2m}$ $(m > 1)$ is trivial for any $m$. In $J^{2m+1}$ it is equivalent to $(G_2, (t^2, t^2 + at^{2m+1}, bt^{2m+1}))$ since the tangent space to the $A_1^{(2m+1)}$-orbit contains the vector $(2t^{2m+1}, 2t^{2m+1})$. If $b \neq 0$ we obtain the normal form 4.2.1 since it is $(2m+1)$-determined. If $b = 0$ and $a \neq 0$ we obtain the $(2m + 1)$-jet $(G_2, (t^2, t^2 + t^{2m+1}))$ which is $(2m + 3)$-determined, since $x_1x_2^2 - x_1^2x_2$ is equal to $t^{2m+5}$ + *higher order terms* for the singular component and to zero for both regular components. Hence we have to move to the $(2m + 3)$-jet. A complete transversal in $J^{(2m+3)}$ is equivalent to

$$(G_2, (t^2, t^2 + t^{2m+1}, ct^{2m+3})).$$

This is so since the tangent space to the $A_1^{(2m+3)}$-orbit contains the vectors $v_1 = (2t^{2m+3}, 2t^{2m+3})$, $v_2 = (2t^4, 2t^4 + (2m + 1)t^{2m+3})$, $v_3 = (t^4 + t^{2m+3}, 0)$ and $v_4 = (0, t^4 + t^{2m+3})$. $v_1 + v_2 - 2v_3 - 2v_4$ is equal to $(0,(2m+1)t^{2m+3})$ for the singular component and to zero for both regular components. If $c \neq 0$ we obtain the normal form 4.2.2, if $c = 0$ we obtain the normal form 4.2.3. ■

**Remark.** If $m = 1$ in the normal form 4.2.1 we obtain the normal form 4.1.3 with $k = n = 2$.

Now we shall consider the case $n = 2$, $k = 1$ in (10).

**Lemma 29.** *Any simple multigerm with the 2-jet* $(G_2,(t^2,0))$ *is equivalent to one of the normal forms* 4.2.4–4.2.22.

**Proof.** Consider a complete transversal in $J^{2m+1}$. It is equivalent to

$$(G_2,(t^2, at^{2m+1}, bt^{2m+1})). \tag{11}$$

Suppose $b \neq 0$. We obtain the $(2m+1)$-jet $(t^2, 0, t^{2m+1})$. This jet is $4m$-determined since $x_3^2$ is equal to $t^{4m+2}$ for the singular curve and to zero for the regular ones. This jet is not sufficient, so we have to move to higher jets. A complete transversal in $J^{2n+1}$ is trivial for any $n$. Consider it in $J^{2n}$ $(n \leq 2m)$. It is equivalent to

$$(G_2,(t^2, ct^{2n}, t^{2m+1} + dt^{2n}, et^{2n})). \tag{12}$$

If $e \neq 0$ we obtain the normal form 4.2.4. If $e = 0$ and $c \neq 0 \neq d$ we obtain the normal form 4.2.5 if $n < 2m$. If $n = 2m$ we can eliminate $t^{2n}$ in the third coordinate, since the tangent space to the $A^{(2n)}$-orbit contains the vectors $(2t^{2m+1}, 0, (2m+1)t^{4m})$ and $(t^{2m+1}, 0, 0)$. Thus we obtain the normal form 4.2.6 $(n = 2m)$. If $d = 0$ and $c \neq 0$ we also obtain the normal form 4.2.6. If $c = 0$ and $d \neq 0$ we obtain the $(2n)$-jet $(t^2, 0, t^{2m+1} + t^{2n})$, where $n < 2m$. We have to consider a complete transversal in $J^{2\ell}$ $(n < \ell \leq 2m)$. It is equivalent to $(t^2, pt^{2\ell}, t^{2m+1} + t^{2n}, qt^{2\ell})$ since the tangent space to the $A^{(2\ell)}$-orbit contains the vectors $(2t^{2(1+\ell-n)}, (2m+1)t^{2(m+\ell-n)+1} + 2nt^{2\ell})$ and $(0, t^{2(m+\ell-n)+1} + t^{2\ell})$. If $q \neq 0$ we obtain the normal form 4.2.7. If $q = 0$ and $p \neq 0$ we obtain the normal form 4.2.8. If $p = 0$ and $q = 0$, for every $\ell$ such that $n < \ell \leq 2m$, we obtain the normal form 4.2.9. If $c = d = e = 0$ in (12) we obtain the normal form 4.2.10.

Now suppose $b = 0$ and $a \neq 0$ in (11). We have the $(2m+1)$-jet $(t^2, t^{2m+1})$. It is $(4m+2)$-determined, since $x_1 x_2^2$ is equal to $t^{4m+4}$ for the singular curve and to zero for the regular ones. A complete transversal in $J^{2n+1}$ is trivial for any $n$. In $J^{2n}$, where $m < n \leq 2m+1$, it is equivalent to

$$G_2, (t^2, t^{2m+1} + ct^{2n}, dt^{2n}). \tag{13}$$

If $d \neq 0$ we obtain the normal form 4.2.11. If $d = 0$ and $c \neq 0$ we move to higher jets. We suppose $n < 2m+1$, since if $n = 2m+1$ the tangent space to the $A^{(4m+2)}$-orbit contains the vectors: $(2t^{2m+3}, (2m+1)t^{4m+2})$ and $(t^{2m+3}, 0)$. A complete transversal in $J^{2\ell}$, where $n < \ell \leq 2m+1$, is equivalent to $(t^2, t^{2m+1} + t^{2n}, et^{2\ell})$. If $e \neq 0$ we obtain the normal form 4.2.12. If $e = 0$ for every $\ell$ such that $n < \ell \leq 2m+1$, we obtain the normal form 4.2.13. If in (13) $c = d = 0$ for every $n$ such that $m < n \leq 2m$, we obtain the normal form 4.2.14.

Now we shall consider a complete transversal in $J^{2m}$. It is equivalent to

$$(G_2, (t^2, at^{2m}, bt^{2m})). \tag{14}$$

Suppose $a \neq 0$ or $b \neq 0$. This jet is not finite determined, so we need to move to higher jets. In $J^{2n}$ $(m < n)$ a complete transversal is trivial for any $n$.

First, suppose $b \neq 0$. Then this $(2m)$-jet is equivalent to $(G_2, (t^2, 0, t^{2m}))$. In $J^{2n+1}$ $(m \leq n)$ a complete transversal is equivalent to

$$(t^2, ct^{2n+1}, t^{2m} + dt^{2n+1}, et^{2n+1}).$$

If $e \neq 0$ we obtain the normal form 4.2.15, which is $(2n+1)$-determined. Now suppose $e = 0$. If $c \neq 0$ and $d \neq 0$ we obtain the normal form 4.2.18, where $l = n$. If $c \neq 0$ and $d = 0$ we obtain the normal form 4.2.16. These both forms are $(2n+1)$-determined. Now suppose $c = e = 0$ and $d \neq 0$. Then we have the $(2n+1)$-jet $(t^2, 0, t^{2m} + t^{2n+1})$. This jet is $(2(m+n)-1)$-determined, since $x_3^2 - x_1^m x_3$ is equal to $t^{2(m+n)+1}$ for the singular component and to zero for the regular ones. A complete transversal in $J^{2\ell}$ is trivial for any $l$. In $J^{2\ell+1}$ $(n < \ell < m+n)$ it is equivalent to

$$G_2, (t^2, pt^{2\ell+1}, t^{2m} + t^{2n+1}, qt^{2\ell+1})$$

since the tangent space to the $A^{(2\ell+1)}$-orbit contains the vectors $(2t^{2(1+\ell-n)}, 0, 2mt^{2(m+\ell-n)} + (2n+1)t^{2\ell+1})$ and $(0, 0, t^{2(m+\ell-n)} + t^{2\ell+1})$. If $q \neq 0$ we obtain the normal form 4.2.17. If $q = 0$ and $p \neq 0$ we obtain the normal form 4.2.18. If $p = q = 0$ for every $l$ such that $n < l < m+n$, we obtain the normal form 4.2.19.

Now, suppose $b = 0$ and $a \neq 0$ in (14). We obtain the $2m$-jet $(G_2, (t^2, t^{2m}))$. A complete transversal in $J^{2n}$ is trivial for any $n$. In $J^{2n+1}$, $m \leq n$, it is equivalent to $(G_2, (t^2, t^{2m} + ct^{2n+1}, dt^{2n+1}))$. Suppose $d \neq 0$. We obtain the normal form 4.2.20 which is $(2n+1)$-determined. If $d = 0$ and $c \neq 0$ we have the $(2n+1)$-jet $(t^2, t^{2m} + t^{2n+1})$ which is $(2(m+n)+1)$-determined since $x_1 x_2^2$ is equal to $t^{2(m+n)+3}$ for the singular component and to zero for the regular ones. A complete transversal in $J^{2l+1}$ $(n < l \leq m+n)$ is equivalent to $(t^2, t^{2m} + t^{2n+1}, et^{2l+1})$. If $e \neq 0$ we obtain the normal form 4.2.21. If $e = 0$ for each $l$ such that $n < l \leq m+n$, we obtain the normal form 4.2.22. ■

Now we shall consider multigerms with the regular part $G_n$ and the multiplicity of the singular component greater than or equals 3. First we shall consider the 3-jet $(G_n, (\underbrace{0, \dots, 0}_{n}, t^3))$.

**Lemma 30.** *Any multigerm with the 5-jet*

$$(G_n, (a_1 t^3 + b_1 t^4 + c_1 t^5, \dots, a_{n+1} t^3 + b_{n+1} t^4 + c_{n+1} t^5))$$

$(n > 1)$ *is not simple.*

**Proof.** The dimension of the submanifold of such jets in $J^5$ is equal to $3n+3$. Let us estimate the dimension of the orbit at a point of the submanifold under the action of the stabilizer of the regular part. It is generated by the images of the following vectors from $L^{(5)}$: $(\underbrace{0, \dots, 0}_{k-1}, x_k, 0, \dots, 0)$, $1 \leq k \leq n+1$, and $(\underbrace{0, \dots, 0}_{k-1}, x_{n+1}, 0, \dots, 0)$, $1 \leq k \leq n$. Therefore the dimension of the orbit under the action of elements from $L^{(5)}$ is less than or equals $(n+1) + n = 2n+1$. Only $t$, $t^2$ and $t^3$ can give rise to non-zero vectors. Hence the dimension of the tangent space to the $R^{(5)}$-orbit is less than or equals 3. Now one can see that the dimension of the tangent space to the $A^{(5)}$-orbit at a point of the submanifold is less than or equals $2n+4$. By similar reasoning as in Lemma 27 we conclude that if the multigerm is simple then $3n+3 \leq 2n+4$, i.e., $n \leq 1$. We have a contradiction with the condition of the Lemma.

**Lemma 31.** *Any simple multigerm with the 3-jet* $(G_n, (\underbrace{0, \dots, 0}_{n}, t^3))$, $n > 1$, *is equivalent to one of the normal forms* 4.1.6–4.1.10.

**Proof.** A complete transversal in $J^4$ is equivalent to $(a_1t^4, \dots, a_nt^4, t^3, bt^4)$ If $b \neq 0$ we have the 4-jet $(0, \dots, 0, t^3, t^4)$. A complete transversal in $J^5$ is equivalent to $(c_1t^5, \dots, c_nt^5, t^3, t^4, dt^5)$. For any $c$ and $d$ this jet is 5-determined. If $d \neq 0$ we obtain the normal form 4.1.6. If $d = 0$ we obtain the normal form 4.1.7. If $b = 0$ the 4-jet is equivalent to $(\underbrace{t^4, \dots, t^4}_{k}, \underbrace{0, \dots, 0}_{n-k}, t^3)$. Using Lemma 30 we note that there is only one possibility for the 5-jet:

$$(\underbrace{t^4, \dots, t^4}_{k}, \underbrace{0, \dots, 0}_{n-k}, t^3, dt^5), \quad d \neq 0.$$

If $k \neq 0$ this jet is 5-determined and we obtain the normal form 4.1.8. If $k = 0$ we move to higher jets. A complete transversal in $J^6$ is trivial. A complete transversal in $J^7$ is equivalent to $(p_1t^7, \dots, p_nt^7, t^3, t^5, qt^7)$. If $q \neq 0$ this 7-jet is 7-determined and we obtain the normal form 4.1.9. If $q = 0$ and there exists $i$ such that $p_i \neq 0$ we obtain the normal form 4.1.10. If $q = 0$ and $p_i = 0$ for each $i$ we obtain the normal form 4.1.8 with $k = 0$ which is 7-determined. ∎

Now we shall complete the classification of multigerms with a non-zero 3-jet of the singular component and the regular part $G_n$. At first, note that $n = 2$ since the 3-jet satisfies the conditions of Lemma 27. It is obvious, that we have the following possibilities for the 3-jet of the singular component: $(0, 0, t^3)$, $(t^3, t^3, 0)$ and $(t^3, 0, 0)$. $(0, t^3, 0)$ we don' t consider because it can be obtained from the last 3-jet by permutations of coordinates and components.

**Lemma 32.** *Any simple multigerm with the 3-jet* $(t^3, t^3, 0)$ *and* $(t^3, 0, 0)$ *is equivalent to one of the normal forms* 4.3.1–4.3.4.

**Proof.** Using Lemma 30 we conclude that the 5-jet is equivalent to $(a_1t^3, a_2t^3, t^4, t^5)$. If $a_1 \neq 0$ and $a_2 \neq 0$ we obtain the normal form 4.3.1 which is 5-determined. Let $a_2 = 0$ and $a_1 \neq 0$. We have the 5-jet $(t^3, 0, t^4, t^5)$. This jet is 6-determined, so we have to consider a complete transversal in $J^6$. It is equivalent to $(t^3, bt^6, t^4, t^5, ct^6)$ since the tangent space to the $A^{(6)}$-orbit contains the vectors: $(3t^4, 0, 4t^5, 5t^6, 0)$, $(t^4, 0, 0, 0, 0)$, $(0, 0, t^5, 0, 0)$ and $(3t^5, 0, 4t^6, 0)$. If $c \neq 0$ we obtain the normal form 4.3.2. If $c = 0$ and $b \neq 0$ we obtain the normal form 4.3.3. If $b = c = 0$ we obtain the normal form 4.3.4. ∎

Now we shall consider multigerms with the zero 3-jet of the singular component. Using Lemma 27 we conclude that $n = 2$, moreover by Lemma 30 there is only one possibility for the 5-jet: $(G_2, (0, 0, t^4, t^5))$.

**Lemma 33.** *Any multigerm with the 7-jet*

$$(G_2, (a_1t^4 + b_1t^5 + c_1t^6 + d_1t^7, \dots, a_4t^4 + b_4t^5 + c_4t^6 + d_4t^7))$$

*is not simple.*

**Proof.** The dimension of the submanifold of such jets in $J^7$ is equal to 16. Let us estimate the dimension of the orbit at a point of the submanifold under the action of the stabilizer of the regular part. The tangent space to the orbit at this point is generated by the images of the vectors: $(\underbrace{0, \dots, 0}_{i-1}, x_i)\quad 1 \le i \le 4$, $(0, 0, 0, x_3)$, $(0, 0, x_4, 0)$; $(x_j, 0, 0, 0)$, $(0, x_j, 0, 0)$, $\quad j = 1, 2$. Hence the dimension of the orbit under the action of elements from $L^{(7)}$ is less than or equals $4 + 2 + 4 = 10$. Only $t$, $t^2$, $t^3$ and $t^4$ can give rise to non-zero vectors. Hence the dimension of the tangent space to the orbit under the action of elements from $R^{(7)}$ is less than or equals 4. Now one can see that the dimension of the tangent space to the orbit at a point of the submanifold is less than or equals $10 + 4 = 14$. Since $16 \ge 14$, the multigerm is not simple. ∎

**Lemma 34.** *Any simple multigerm with the 5-jet* $(G_2, (0, 0, t^4, t^5))$ *is equivalent to one of the normal forms* 4.3.5–4.3.11.

**Proof.** A complete transversal in $J^6$ is equivalent to

$$(at^6, bt^6, t^4, t^5 + dt^6, ct^6). \tag{15}$$

Since the tangent space to the $A^{(6)}$-orbit contains the vectors $(0, 0, 4t^5, 5t^6)$, $(0, 0, t^5 + dt^6, 0)$ and $(0, 0, 4t^6, 0)$, we can suppose that $d = 0$. If $c \ne 0$ we have the 6-jet $(0, 0, t^4, t^5, t^6)$. This jet is 7-determined, so we move to $J^7$. A complete transversal is equivalent to $(pt^7, qt^7, t^4, t^5, t^6, rt^7)$. If $r \ne 0$ we obtain the normal form 4.3.5. If $r = 0$ but $p \ne 0$ and $q \ne 0$ we obtain the normal form 4.3.6. If $p = r = 0$ and $q \ne 0$ or $q = r = 0$ and $p \ne 0$ we obtain the normal form 4.3.7 (these both cases are equivalent since we can permutate coordinates and regular components). If $p = q = r = 0$ we obtain the normal form 4.3.8.

If $c = 0$ in (15), we have (by similar reasonings) three possibilities for the 6-jet: $(t^6, t^6, t^4, t^5)$, $(t^6, 0, t^4, t^5)$ and $(0, 0, t^4, t^5)$. Using Lemma 33 we see that there is only one multigerm for each of these three 6-jets — the normal forms 4.3.9–4.3.11. All of them are 7-determined. ■

We have completely analysed simple multigerms with one singular component and the regular part $G_n$. Now we have to consider multigerms with one singular component and the regular part

$$((t_1, 0), (t_2, t_2^m)). \tag{16}$$

Lemma 23 states that a multigerm with the 2-jet

$$((t_1, 0), (t_2, t_2^2), (t_3, \alpha t_3^2)) \tag{17}$$

is not simple. Since the third component of our multigerm is singular we obtain $m = 2$ in (16) (if $m > 2$ the 2-jet of our multigerm is adjacent to $((t_1, 0), (t_2, 0), (0, at^2))$ which is adjacent to the family (17)). Moreover there is only one possibility for the 2-jet:

$$((t_1, 0, 0), (t_2, t_2^2, 0), (0, 0, t^2)). \tag{18}$$

**Lemma 35.** *Any simple multigerm with the 2-jet* (18) *is equivalent to one of the normal forms* 4.4.1–4.4.3*.*

**Proof.** The 2-jet (18) is not finite determined. A complete transversal in $J^{2m}$ is trivial for any $m$. In $J^{2m+1}$ it is equivalent to $((t_1, 0), (t_2, t_2^2), (at^{2m+1}, bt^{2m+1}, t^2, ct^{2m+1}))$. If $c \neq 0$ we obtain the normal form 4.4.1, which is $(2m + 1)$-determined. If $c = 0$ and $b \neq 0$ then the jet is equivalent to

$$(t_1, 0), (t_2 + at_2^2, t_2^2), (0, t^{2m+1}, t^2).$$

Since the tangent space to the $A^{(2m+1)}$-orbit contains the vectors (we write only the second (non-zero) component): $(t_2^2 + 2at_2^3, 2t_2^3)$, $(t_2^3 + at_2^4, 0)$, $(0, t_2^3 + at_2^4)$, $(t_2^4, 0)$ and $(0, t_2^4)$, then the tangent space also contains $(t_2^2, 0)$. Thus by the Mather Lemma we conclude that the $(2m + 1)$-jet is equivalent to the normal form 4.4.2. If $b = c = 0$ and $a \neq 0$ we obtain the normal form 4.4.3. ■

# 8 Multigerms with two singular components

Lemma 23 states that each element of the family of 2-jets

$$((t,0),(t_1,t_1^2),(t_2,\alpha t_2^2)) \tag{19}$$

is not simple.

Suppose that the multigerm contains $n$ regular components and two singular ones. Then the 1-jet of the multigerm is adjacent to $(n+2)$ lines in $\mathbb{C}^n$. In the proof of Lemma 24 we saw that if $n>1$ then this 1-jet is not simple. Hence the multigerm contains three components: one regular and two singular.

The 2-jet of the multigerm is equivalent to

$$((t,0,0),(0,t_1^2,0),(0,0,t_2^2)) \tag{20}$$

since it may not be adjacent to (19).

**Lemma 36.** *The multigerm with the 3-jet*

$$\begin{array}{l}((t,0,0),(a_1t_1^2+b_1t_1^3,a_2t_1^2+b_2t_1^3,a_3t_1^2+b_3t_1^3),\\(a_4t_2^2+b_4t_2^3,a_5t_2^2+b_5t_2^3,a_6t_2^2+b_6t_2^3))\end{array}$$

*is not simple.*

**Proof.** The dimension of the submanifold of such jets in $J^3$ equals 12. Let us estimate the dimension of the orbit at a point of the submanifold under the action of the stabilizer of the regular part. The dimension of the orbit under the action of elements from $L^{(3)}$ is less than or equals $3+2+2=7$. The dimension of the orbit under the action of elements from $R^{(3)}$ is less than or equals $2+2=4$. Now one can see that the dimension of the tangent space to the orbit at a point of the submanifold is less than or equals $7+4=11<12$. ∎

We shall not write the regular component (we suppose it to be equal to $(t,0,0)$). Using Lemma 36 we can suppose that the second singular component equals $(0,0,t_2^2,t_2^3)$.

**Lemma 37.** *Any simple multigerm with the 2-jet* (20) *is equivalent to one of* 5.1–5.6.

**Proof.** A complete transversal in $J^{2m}$ is trivial for any $m$. In $J^{2m+1}$ a complete transversal is equivalent to

$$(at_1^{2m+1}, t_1^2, bt_1^{2m+1}, ct_1^{2m+1}, dt_1^{2m+1})$$

(we write only the second component, the first and the third components are not changed). If $d \neq 0$ we obtain the normal form 5.1. If $d = 0$ and $c \neq 0$ we can suppose $c = 1$. The tangent space to the $A^{(2m+1)}$-orbit contains the vectors (we write only the singular components): $((0,0,t_1^{2m+1},0),(0,0,t_2^3,0))$, $((0,0,0,0),(0,0,2t_2^3,3t_2^4))$ and $((0,0,0,0),(0,0,0,t_2^4))$. So, using the Mather Lemma we can suppose $b = 0$. If $a \neq 0$ we obtain the normal form 5.2. If $a = 0$ we obtain the normal form 5.3.

Now suppose $c = d = 0$. If $a \neq 0$ and $b \neq 0$ we obtain the normal form 5.4. If $b \neq 0$ and $a = 0$ we obtain the normal form 5.5. If $b = 0$ and $a \neq 0$ we obtain the normal form 5.6. All these normal forms are $(2m+1)$-determined. ■

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