

# The starlikeness and convexity of multivalent functions involving certain inequalities

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## ABSTRACT

In the present paper, a theorem for the starlikeness and convexity of multivalent functions involving certain inequalities is given. Some interesting consequences of the main result are also mentioned.

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## 1. Introduction and definitions

Let  $\mathcal{T}(p)$  denote the class of functions  $f(z)$  of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathcal{N} = \{1, 2, 3, \dots\}), \quad (1.1)$$

which are *analytic* and *multivalent* in the open unit disc  $\mathcal{U} = \{z : z \in \mathcal{C} \text{ and } |z| < 1\}$ . A function  $f(z)$  belonging to  $\mathcal{T}(p)$  is said to be *multivalently starlike of order  $\alpha$*  in  $\mathcal{U}$  if it satisfies the inequality:

$$\Re \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \quad (1.2)$$

and, a function  $f(z) \in \mathcal{T}(p)$  is said to be *multivalently convex of order  $\alpha$*  in  $\mathcal{U}$  if it satisfies the inequality:

$$\Re \left\{ \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}). \quad (1.3)$$

For the aforementioned definitions, one may refer to [1] and [2] (see also [11]). Further, a function  $f(z) \in \mathcal{T}(p)$  is said to be in the subclass  $\mathcal{T}_\lambda(p; \alpha)$  if it satisfies the inequality

$$\Re \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right\} > \alpha, \quad (1.4)$$

$$(z \in \mathcal{U}; 0 \leq \lambda \leq 1; 0 \leq \alpha < p; p \in \mathcal{N}).$$

From the above definitions, the following subclasses of the classes  $\mathcal{T}(p)$  and  $\mathcal{T} \equiv \mathcal{T}(1)$  emerge from the families of functions  $\mathcal{T}_\lambda(p; \alpha)$ :

$$\mathcal{T}_\lambda(1; \alpha) = \mathcal{T}_\lambda(\alpha), \quad (0 \leq \lambda \leq 1; 0 \leq \alpha < 1), \quad (1.5)$$

$$\mathcal{T}_0(p; \alpha) = \mathcal{S}_p(\alpha), \quad (0 \leq \alpha < p; p \in \mathcal{N}), \quad (1.6)$$

$$\mathcal{T}_1(p; \alpha) = \mathcal{K}_p(\alpha), \quad (0 \leq \alpha < p; p \in \mathcal{N}), \quad (1.7)$$

$$\mathcal{S}_p(\alpha) \subseteq \mathcal{S}_p(0) = \mathcal{S}_p, \quad (0 \leq \alpha < p; p \in \mathcal{N}) \quad (1.8)$$

$$\mathcal{K}_p(\alpha) \subseteq \mathcal{K}_p(0) = \mathcal{K}_p, \quad (0 \leq \alpha < p; p \in \mathcal{N}), \quad (1.9)$$

$$\mathcal{T}_0(\alpha) = \mathcal{S}_1(\alpha) \subseteq \mathcal{S}(\alpha), \quad (0 \leq \alpha < 1), \quad (1.10)$$

$$\mathcal{T}_1(\alpha) = \mathcal{K}_1(\alpha) \subseteq \mathcal{K}(\alpha), \quad (0 \leq \alpha < 1), \quad (1.11)$$

$$\mathcal{T}_0(\alpha) = \mathcal{S}(\alpha) \subseteq \mathcal{S}(0) = \mathcal{S}, \quad (0 \leq \alpha < 1), \quad (1.12)$$

$$\mathcal{T}_1(\alpha) = \mathcal{K}(\alpha) \supseteq \mathcal{K}(0) = \mathcal{K}, \quad (0 \leq \alpha < 1). \quad (1.13)$$

The important subclasses in the Geometric function theory such as the multivalently starlike functions  $\mathcal{S}_p(\alpha)$  of order  $\alpha$  ( $0 \leq \alpha < p; p \in \mathcal{N}$ ) in  $\mathcal{U}$ , the multivalently convex functions  $\mathcal{K}_0(\alpha)$  of order  $\alpha$  ( $0 \leq \alpha < p; p \in \mathcal{N}$ ) in  $\mathcal{U}$ , the multivalently starlike functions  $\mathcal{S}_p$  in  $\mathcal{U}$ , the multivalently convex functions  $\mathcal{K}_p$  in  $\mathcal{U}$ , the starlike functions in  $\mathcal{S}(\alpha)$  of order  $\alpha$  ( $0 \leq \alpha < 1$ ) in  $\mathcal{U}$ , the convex functions  $\mathcal{K}(\alpha)$  of order  $\alpha$  ( $0 \leq \alpha < 1$ ) in  $\mathcal{U}$ , the starlike functions  $\mathcal{S}$  in  $\mathcal{U}$ , and the convex functions  $\mathcal{K}$  in  $\mathcal{U}$ , are seen to be easily identifiable with the aforementioned classes (*cf.*, *e.g.*, [1], [2], and [11]).

The purpose of considering inequality (1.4) is to obtain general results which combine certain types of inequalities concerning functions belonging to the classes  $\mathcal{S}_p(\alpha)$ ,  $\mathcal{K}_p(\alpha)$ ,  $\mathcal{S}_p$ ,  $\mathcal{K}_p$ ,  $\mathcal{S}(\alpha)$ ,  $\mathcal{K}(\alpha)$ ,  $\mathcal{S}$ , and  $\mathcal{K}$ . Some interesting corollaries are also deduced from our main results. Other interesting results involving certain inequalities and/or multivalent functions were also studied, for example, by Owa *et al.* ([8], [9], [10]), and Irmak *et al.* ([3], [4], [5]).

### 2. Main Result

Before stating and proving our main result, we require the following assertion (popularly known as Jacks’s Lemma):

**Lemma** (cf., [6], [7]). *Let the function  $x(z)$  be non-constant and regular in the unit disc  $\mathcal{U}$ , such that  $w(0) = 0$ . If  $|w(z)|$  attains its maximum value on the circle  $|z| = r < 1$  at a point  $z_0$ , then*

$$z_0 w'(z_0) = c w(z_0), \quad (c \geq 1). \tag{2.1}$$

**Theorem.** *Let a function  $f(z)$  belong to the class  $\mathcal{T}(p)$ . Define a function  $F(z)$  by*

$$F(z) = (1 - \lambda)f(z) + \lambda z f'(z), \quad (0 \leq \lambda \leq 1), \tag{2.2}$$

and if  $F(z)$  satisfies anyone of the following inequalities:

$$\left| \frac{1 + \frac{zF''(z)}{F'(z)} - p}{\frac{zF'(z)}{F(z)} - p} - 1 \right| < \frac{1}{2p - \alpha}, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.3}$$

$$\left| 1 + z \left( \frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)} \right) \right| < \frac{p - \alpha}{2p - \alpha}, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.4}$$

$$\left| \frac{F(z)}{zF'(z)} \left( 1 + \frac{zF''(z)}{F'(z)} \right) - 1 \right| < \frac{p - \alpha}{(2p - \alpha)^2}, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.5}$$

$$\left| \frac{zF'(z)}{F(z)} \left[ 1 + z \left( \frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)} \right) \right] \right| < p - \alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.6}$$

$$\Re \left\{ \frac{zF'(z)}{F(z)} \left( \frac{1 + \frac{zF''(z)}{F'(z)} - p}{\frac{zF'(z)}{F(z)} - p} - 1 \right) \right\} < 1, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.7}$$

then  $f(z) \in \mathcal{T}_\lambda(p; \alpha)$ .

**Proof.** Let  $f(z) \in \mathcal{T}(p)$ . then from (1.1) and (2.1), we find that

$$\begin{aligned} \frac{zF'(z)}{F(z)} &= \frac{z f'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} \\ &= \frac{p[1 + \lambda(p - 1)] + \sum_{k=p}^\infty k[1 + \lambda(k - 1)]a_k z^{k-p}}{1 + \lambda(p - 1) + \sum_{k=p}^\infty [1 + \lambda(k - 1)]a_k z^{k-p}}, \end{aligned} \tag{2.8}$$

$(z \in \mathcal{U}; 0 \leq \lambda \leq 1; p \in \mathcal{N}).$

Now, define a function  $w(z)$  by

$$\frac{zF'(z)}{F(z)} - p = (p - \alpha)w(z), \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.9}$$

then the function  $w(z)$  is analytic in  $\mathcal{U}$ , and  $w(0) = 0$ . It follows from the above definition (2.9) that

$$1 + \frac{zF''(z)}{F'(z)} - p = (p - \alpha)w(z) \left( 1 + \frac{zw'}{w(z)} \frac{1}{p + (p - \alpha)w(z)} \right), \tag{2.10}$$

$(z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N})$ .

Hence, from (2.8) and (2.9), we have

$$F_1(z) = \frac{1 + \frac{zF''(z)}{zF'(z)} - p}{\frac{zF'(z)}{F(z)} - p} - 1 = \frac{zw'(z)}{w(z)} \frac{1}{p + (p - \alpha)w(z)}, \tag{2.11}$$

$$F_2(z) = 1 + z \left( \frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)} \right) = \frac{(p - \alpha)zw'(z)}{p + (p - \alpha)w(z)}, \tag{2.12}$$

$$F_3(z) = \frac{F(z)}{zF'(z)} \left( 1 + \frac{zF''(z)}{F'(z)} \right) - 1 = \frac{(p - \alpha)zw'(z)}{[p + (p - \alpha)w(z)]^2}, \tag{2.13}$$

$$F_4(z) = \frac{zF'(z)}{F(z)} \left[ 1 + z \left( \frac{F''(z)}{F'(z)} - \frac{F'(z)}{F(z)} \right) \right] = (p - \alpha)zw'(z), \tag{2.14}$$

$$F_5(z) = \frac{zF'(z)}{F(z)} \left( \frac{1 + \frac{zF''(z)}{F'(z)} - p}{\frac{zF'(z)}{F(z)} - p} - 1 \right) = \frac{zw'(z)}{w(z)}, \tag{2.15}$$

$(z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N})$ .

We claim that  $|w(z)| < 1$  in  $\mathcal{U}$ . For otherwise (by Jack's Lemma) there exists a point  $z_0 \in \mathcal{U}$  such that

$$z_0w'(z_0) = cw(z_0), \tag{2.16}$$

where

$$|w(z_0)| = 1, (c \geq 1).$$

Therefore, the equations (2.11)-(2.15) in conjunction with (2.16) yield:

$$|F_1(z_0)| = \left| \frac{z_0w'(z_0)}{w(z_0)} \frac{1}{p + (p - \alpha)w(z_0)} \right| = \frac{c|w(z_0)|}{|p + (p - \alpha)w(z_0)|} \geq \frac{1}{2p - \alpha}, \tag{2.17}$$

$$|F_2(z_0)| = \left| \frac{(p - \alpha)z_0w'(z_0)}{p + (p - \alpha)w(z_0)} \right| = \frac{c(p - \alpha)|w(z_0)|}{|p + (p - \alpha)w(z_0)|} \geq \frac{p - \alpha}{2p - \alpha}, \tag{2.18}$$

$$|F_3(z_0)| = \left| \frac{(p - \alpha)z_0w'(z_0)}{p + (p - \alpha)w(z_0)} \right| = \frac{c(p - \alpha)|w(z_0)|}{|p + (p - \alpha)w(z_0)|^2} \geq \frac{p - \alpha}{(2p - \alpha)^2}, \tag{2.19}$$

$$|F_4(z_0)| = c(p - \alpha)|w(z_0)| \geq p - \alpha, \tag{2.20}$$

$$\Re\{F_5(z_0)\} = \Re\left\{ \frac{zw'(z_0)}{w(z_0)} \right\} = c \geq 1, \tag{2.21}$$

$$(0 \leq \alpha < p; p \in \mathcal{N}),$$

which contradict our assumptions (2.3)-(2.7), respectively. Therefore  $|w(z)| < 1$  holds true for all  $z \in \mathcal{U}$ . From the definition (2.9) yields

$$\left| \frac{zF'(z)}{F(z)} - p \right| = (p - \alpha) |w(z)| < p - \alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.22}$$

which implies that

$$\Re \left\{ \frac{zF'(z)}{F(z)} \right\} = \Re \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} \right\} > \alpha, \tag{2.23}$$

$(z \in \mathcal{U}; 0 \leq \lambda \leq 1; 0 \leq \alpha < p; p \in \mathcal{N}),$

and hence  $f(z) \in \mathcal{T}_\lambda(p; \alpha)$ .

We mention now some interesting corollaries for the classes  $\mathcal{T}_\lambda(\alpha)$ ,  $\mathcal{S}_p(\alpha)$ ,  $\mathcal{K}_p(\alpha)$ ,  $\mathcal{S}_p$ ,  $\mathcal{K}_p$ ,  $\mathcal{S}(\alpha)$ ,  $\mathcal{K}(\alpha)$ ,  $\mathcal{S}$ , and  $\mathcal{K}$  which are easily deducible from the main result.

**Corollary 1.** *Let a function  $f(z)$  defined by (1.1) belong to the class  $\mathcal{T}(p)$ . If  $f(z)$  satisfies anyone of the following inequalities:*

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p} - 1 \right| < \frac{1}{2p - \alpha}, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}) \tag{2.24}$$

$$\left| 1 + z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right| < \frac{p - \alpha}{2p - \alpha}, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.25}$$

$$\left| \frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| < \frac{p - \alpha}{(2p - \alpha)^2}, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}) \tag{2.26}$$

$$\left| \frac{zf'(z)}{f(z)} \left[ 1 + z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right] \right| < p - \alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.27}$$

$$\Re \left\{ \frac{zf'(z)}{f(z)} \left( \frac{1 + \frac{zf''(z)}{f'(z)} - p}{\frac{zf'(z)}{f(z)} - p} - 1 \right) \right\} < 1, \quad (z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}), \tag{2.28}$$

then  $f(z) \in \mathcal{S}_p(\alpha)$ .

**Corollary 2.** *Let a function  $f(z)$  defined by (1.1) belong to the class  $\mathcal{T}(p)$ . If  $f(z)$  satisfies anyone of the following inequalities:*

$$\left| \frac{1 + \frac{z[2f''(z) + zf'''(z)]}{f'(z) + zf''(z)} - p}{1 + \frac{zf''(z)}{f'(z)} - p} - 1 \right| < \frac{1}{2p - \alpha}, \tag{2.29}$$

$$(z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}),$$

$$\left| z \left( \frac{2f''(z) + zf'''(z)}{f'(z) + zf''(z)} - \frac{f''(z)}{f'(z)} \right) \right| < \frac{p - \alpha}{2p - \alpha}, \tag{2.30}$$

$(z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}),$

$$\left| \frac{f'(z)}{f'(z) + zf''(z)} \left( 1 + \frac{z[2f''(z) + zf'''(z)]}{f'(z) + zf''(z)} \right) - 1 \right| < \frac{p - \alpha}{(2p - \alpha)^2}, \tag{2.31}$$

$(z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}),$

$$\left| z \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left( \frac{2f''(z) + zf'''(z)}{f'(z) + zf''(z)} - \frac{f''(z)}{f'(z)} \right) \right| < p - \alpha, \tag{2.32}$$

$(z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}),$

$$\Re \left\{ \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left( \frac{1 + \frac{z[2f''(z) + zf'''(z)]}{f'(z) + zf''(z)} - p}{1 + \frac{zf''(z)}{f'(z)} - p} - 1 \right) \right\} < 1, \tag{2.33}$$

$(z \in \mathcal{U}; 0 \leq \alpha < p; p \in \mathcal{N}),$

then  $f(z) \in \mathcal{K}_p(\alpha)$ .

**Corollary 3.** Let a function  $f(z)$  defined by belong to the class  $\mathcal{T}$ . If  $f(z)$  satisfies anyone of the following inequalities:

$$\left| \frac{\frac{zf''(z)}{zf'(z)}}{\frac{zf'(z)}{f(z)} - 1} - 1 \right| < \frac{1}{2 - \alpha}, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \tag{2.34}$$

$$\left| 1 + z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right| < \frac{1 - \alpha}{2 - \alpha}, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \tag{2.35}$$

$$\left| \frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| < \frac{1 - \alpha}{(2 - \alpha)^2}, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \tag{2.36}$$

$$\left| \frac{zf'(z)}{f(z)} \left[ 1 + z \left( \frac{f''(z)}{f'(z)} - \frac{f'(z)}{f(z)} \right) \right] \right| < 1 - \alpha, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \tag{2.37}$$

$$\Re \left\{ \frac{zf'(z)}{f(z)} \left( \frac{\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)} - 1} - 1 \right) \right\} < 1, \quad (z \in \mathcal{U}; 0 \leq \alpha < 1), \tag{2.38}$$

then  $f(z) \in \mathcal{S}(\alpha)$ .

**Corollary 4.** Let a function  $f(z)$  defined belong to the class  $\mathcal{T}$ . If  $f(z)$  satisfies anyone of the following inequalities:

$$\left| \frac{f'(z)}{zf''(z)} \left( \frac{2f''(z) + zf'''(z)}{f'(z) + zf''(z)} \right) - 1 \right| < \frac{1}{2 - \alpha}, \tag{2.39}$$

$(z \in \mathcal{U}; 0 \leq \alpha < 1),$

$$\left| z \left( \frac{2f''(z) + zf'''(z)}{f'(z) + zf''(z)} - \frac{f''(z)}{f'(z)} \right) \right| < \frac{1-\alpha}{2-\alpha}, \quad (2.40)$$

$$(z \in \mathcal{U}; 0 \leq \alpha < 1),$$

$$\left| \frac{f'(z)}{f'(z) + zf''(z)} \left( 1 + \frac{z[2f''(z) + zf'''(z)]}{f'(z) + zf''(z)} \right) - 1 \right| < \frac{1-\alpha}{(2-\alpha)^2}, \quad (2.41)$$

$$(z \in \mathcal{U}; 0 \leq \alpha < 1),$$

$$\left| z \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left( \frac{2f''(z) + zf'''(z)}{f'(z) + zf''(z)} - \frac{f''(z)}{f'(z)} \right) \right| < 1 - \alpha, \quad (2.42)$$

$$(z \in \mathcal{U}; 0 \leq \alpha < 1),$$

$$\Re \left\{ \left( 1 + \frac{zf''(z)}{f'(z)} \right) \left( \frac{f'(z)[2f''(z) + zf'''(z)]}{f''(z)[f'(z) + zf''(z)]} - 1 \right) \right\} < 1, \quad (2.43)$$

$$(z \in \mathcal{U}; 0 \leq \alpha < 1),$$

then  $f(z) \in \mathcal{K}(\alpha)$ .

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