A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces

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ABSTRACT

We study the Poincaré inequality in Sobolev spaces with variable exponent. Under a rather mild and sharp condition on the exponent p we show that the inequality holds. This condition is satisfied e. g. if the exponent p is continuous in the closure of a convex domain. We also give an essentially sharp condition for the exponent p as to when there exists an imbedding from the Sobolev space to the space of bounded functions.

Key words: Sobolev spaces, variable exponent, Poincaré inequality, Sobolev imbedding, continuity

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1. Introduction

There has recently been a surge of interest in Sobolev spaces with variable exponent, cf. [4–7, 9–11, 17, 22]. These spaces, introduced in [17], are the natural generalization of Sobolev spaces to the non-homogeneous situation; they have been used e. g. in modeling electrorheological fluids, see the book of M. Růžička, [22]. Lebesgue and Sobolev spaces with variable exponent share many properties with their classical equivalents, but there is also some crucial differences. For instance the Hardy-Littlewood maximal

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operator is bounded on $L^{p(\cdot)}$ if the exponent is 0-Hölder continuous (i. e. satisfies (10)) and $1 < \operatorname{ess\,inf} p \leqslant \operatorname{ess\,sup} p < \infty$, [5]. If the exponent is not 0-Hölder continuous, then the maximal operator need not be bounded on $L^{p(\cdot)}$, [21].

The Poincaré inequality, although of great importance in classical non-linear potential theory (especially in metric spaces) has not been previously studied in the case of variable exponent Sobolev spaces. Our first result, Theorem 2.2, is the following: If $D \subset \mathbb{R}^n$ is smooth domain, say a John domain, and the essential supremum of p is less than the Sobolev conjugate of the essential infimum of p then the Poincaré inequality

$$\|u - u_B\|_{L^{p(\cdot)}(D)} \leqslant C\|\nabla u\|_{L^{p(\cdot)}(D)}$$

holds for every $u \in W^{1,p(\cdot)}(D)$, where $u_B = \fint u(x)dx$. Here the constant C depends on n, p, $\operatorname{diam}(D)$ and the John constant of D. We give an example which shows that the condition for p is sharp even in a ball. It follows from this that if p is continuous in the closure of a convex domain then the Poincaré inequality holds (Corollary 2.7).

In classical theory the constant of the Poincaré inequality is $C \operatorname{diam}(D)$. It is possible to achieve this also for variable exponent Sobolev spaces, as we prove in Corollary 2.10. The price we have to pay is that the exponent p has to be 0-Hölder continuous.

Sobolev imbeddings in variable exponent Sobolev spaces have been studied by many authors in the case when p is less than the dimension, see [6, 9–11]. We give two results in the case when p is greater than the dimension. We prove a result for continuity of the Sobolev functions, namely that every Sobolev function is continuous if the exponent is locally bounded away from the dimension. We show that if a domain satisfies a uniform interior cone condition and $p(x) \geqslant n + f(d(x, \partial G))$ for every x and a certain increasing function f then there exists an imbedding from the variable exponent Sobolev space to L^∞. Our condition is essentially sharp.

Notation

We denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geqslant 2$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote an open ball with center x and radius r by $B(x, r)$.

Let $A \subset \mathbb{R}^n$ and $p\colon A \to [1, \infty)$ be a measurable function (called a *variable exponent* on A). We define $p^+_A = \operatorname{ess\,sup}_{x\in A} p(x)$ and $p^-_A = \operatorname{ess\,inf}_{x\in A} p(x)$. If $A = \mathbb{R}^n$ we write $p^+ = p^+_{\mathbb{R}^n}$ and $p^- = p^-_{\mathbb{R}^n}$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We define the *generalized Lebesgue space* $L^{p(\cdot)}(\Omega)$ to consist of all measurable functions $u\colon \Omega \to \mathbb{R}$ such that

$$\varrho_{p(\cdot)}(\lambda\, u) = \int_\Omega |\lambda\, u(x)|^{p(x)}\, dx < \infty$$

for some $\lambda > 0$. The function $\varrho_{p(\cdot)}\colon L^{p(\cdot)}(\Omega) \to [0, \infty)$ is called the *modular* of the space $L^{p(\cdot)}(\Omega)$. One can define a norm, the so-called *Luxemburg norm*, on this space by the formula $\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}$. Notice that if $p \equiv p_0$ then

$L^{p(\cdot)}(\Omega)$ is the classical Lebesgue space, so there is no danger of confusion with the new notation.

The *generalized Sobolev space* $W^{1,p(\cdot)}(\Omega)$ is the space of measurable functions $u\colon \Omega \to \mathbb{R}$ such that u and the absolute value of the distributional gradient $\nabla u = (\partial_1 u, \dots, \partial_n u)$ are in $L^{p(\cdot)}(\Omega)$. The function $\varrho_{1,p(\cdot)}\colon W^{1,p(\cdot)}(\Omega) \to [0,\infty)$ is defined as $\varrho_{1,p(\cdot)}(u) = \varrho_{p(\cdot)}(u) + \varrho_{p(\cdot)}(|\nabla u|)$. The norm $\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}$ makes $W^{1,p(\cdot)}(\mathbb{R}^n)$ a Banach space.

See [17] for basic properties of variable exponent Lebesgue and Sobolev spaces.

2. The Poincaré inequality

In this section we give a relatively mild condition on the exponent for the Poincaré inequality to hold. We also show that this condition is, in a certain sense, the best possible. For Sobolev functions with zero boundary values the Poincaré inequality was given in [10, Lemma 3.1] and considerably generalized in [14].

Recall the following well known Sobolev-Poincaré inequality. By q^* we denote the Sobolev conjugate of $q < n$, $q^* = nq/(n-q)$.

Lemma 2.1. *Let $D \subset \mathbb{R}^n$ be a bounded John domain. Let $1 \leqslant p < n$ and $p \leqslant q \leqslant p^*$ be fixed exponents. Then*

$$\|u - u_D\|_q \leqslant C(n,p,\lambda)|D|^{1/n+1/q-1/p}\|\nabla u\|_p$$

for all functions $u \in W^{1,p}(D)$, where λ is the John constant.

If $p \geqslant n$ and $q < \infty$ then

$$\|u - u_D\|_q \leqslant C(n,q,\lambda)|D|^{1/n+1/q-1/p}\|\nabla u\|_p$$

for all functions $u \in W^{1,p}(D)$.

Proof. The case $p < n$ and $q = p^*$ is by B. Bojarski [3, (6.6)]. The case $q < p^*$ follows from this by standard arguments: we choose $s \in [1,n)$ such that $s^* = q$ (or $s = 1$ if $q < 1^*$). By Hölder's inequality and Bojarski's result we obtain

$$\Big(\fint_D |u - u_D|^q dx \Big)^{\frac{1}{q}} \leqslant |D|^{-\frac{1}{s^*}} \Big(\int_D |u - u_D|^{s^*} dx \Big)^{\frac{1}{s^*}} \leqslant C|D|^{-\frac{1}{s^*}} \Big(\int_D |\nabla u|^s dx \Big)^{\frac{1}{s}}$$

$$= C|D|^{\frac{1}{s}-\frac{1}{s^*}} \Big(\fint_D |\nabla u|^s dx \Big)^{\frac{1}{s}} \leqslant C|D| \Big(\fint_D |\nabla u|^p dx \Big)^{\frac{1}{p}},$$

which is clearly equivalent to the inequalities in the theorem. □

Theorem 2.2. *Let $D \subset \mathbb{R}^n$ be a bounded John domain, with constant λ. If $p_D^+ \leqslant (p_D^-)^*$ or $p_D^- \geqslant n$ and $p_D^+ < \infty$ then there exists a constant $C = C(n, p_D^-, p_D^+, \lambda)$ such that*

$$\|u - u_D\|_{p(\cdot)} \leqslant C(1 + |D|)^2 |D|^{\frac{1}{n} + \frac{1}{p_D^+} - \frac{1}{p_D^-}} \|\nabla u\|_{p(\cdot)} \tag{1}$$

for every $u \in W^{1,p(\cdot)}(D)$.

Proof. Assume first that $p_D^+ \leqslant (p_D^-)^*$. Since $p(x) \leqslant p_D^+ \leqslant (p_D^-)^*$ we obtain by [17, Theorem 2.8] and Lemma 2.1 that

$$\begin{aligned}\|u - u_D\|_{p(\cdot)} &\leqslant (1 + |D|)\, \|u - u_D\|_{p_D^+} \\ &\leqslant C(n, p_D^-, \lambda)\, (1 + |D|)\, |D|^{\frac{1}{n} + \frac{1}{p_D^+} - \frac{1}{p_D^-}} \|\nabla u\|_{p_D^-} \\ &\leqslant C(n, p_D^-, \lambda)\, (1 + |D|)^2\, |D|^{\frac{1}{n} + \frac{1}{p_D^+} - \frac{1}{p_D^-}} \|\nabla u\|_{p(\cdot)}.\end{aligned}$$

The case $p_D^- \geqslant n$ is similar, the only difference is that the constant in the second inequality in the above chain of inequalities is $C(n, p_D^+, \lambda)$. □

Remark 2.3. John domains are almost the right class of irregular domains for the classical Sobolev-Poincaré inequality, see [3], [1] and [2, Theorem 4.1].

Previous results on Sobolev imbeddings in the variable exponent setting have been derived in domains whose boundary is locally a graph of a Lipschitz continuous function, see [9–11]. It is therefore of interest to note that every domain, whose boundary is locally the graph of a Lipschitz continuous function, is a John domain, see [19]. In particular every ball is a John domain.

If D is a ball in Theorem 2.2, then the constant in inequality (1) is the classical Sobolev-Poincaré inequality in a ball, see for example [18, Corollary 1.64, p. 38].

The next example shows that if $p_D^- < n$ and $p_D^+ > (p_D^-)^*$ then there need not exist a constant $C > 0$ such that inequality (1) holds for every $u \in W^{1,p(\cdot)}(D)$.

Recall that the *variational capacity* for fixed p, $\operatorname{cap}_p(E, F; D)$, is defined for sets E, F and open D by

$$\operatorname{cap}_p(E, F; D) = \inf_{u \in L(E,F;D)} \int_D |\nabla u|^p dx,$$

where $L(E, F; D)$ is the set of continuous functions u that satisfy $u|_{E \cap D} = 1$, $u|_{F \cap D} = 0$ and $|\nabla u| \in L^{p(\cdot)}(D)$. We use the short-hand notation $\operatorname{cap}(E, F)$ for $\operatorname{cap}(E, F; \mathbb{R}^n)$, similarly for $L(E, F)$. For more information on capacities see [15, Chapter 2] or [20]. The following lemma will be used several times to estimate the gradient of variable exponent functions.

Lemma 2.4 ([15, Example 2.12, p. 35]). *For fixed $p \neq 1, n$, arbitrary $x \in \mathbb{R}^n$ and $R > r > 0$ we have*

$$\operatorname{cap}_p(\mathbb{R}^n \setminus B(x,R), B(x,r)) = \omega_{n-1} \left| \frac{p-n}{p-1} \right|^{p-1} \left| R^{(p-n)/(p-1)} - r^{(p-n)/(p-1)} \right|^{1-p}.$$

Example 2.5. Our aim is construct a sequence of functions in $B = B(0,1) \subset \mathbb{R}^2$ for which the constant in the Poincaré inequality (1) goes to infinity. Let $B_i = B(2^{-i}e_1, \frac{1}{4}2^{-i}) \subset \mathbb{R}^2$ and $B_i' = B(2^{-i}e_1, \frac{1}{8}2^{-i^2}) \subset \mathbb{R}^2$ for every $i = 1, 2, \dots$ and let $1 < p_1 < 2$. Choose a function $u_i \in C_0^\infty(B_i)$ with $u_i|_{B_i'} = 1$ be such that

$$\Big(2 \operatorname{cap}_{p_1}(B_i', \mathbb{R}^2 \setminus B_i)\Big)^{\frac{1}{p_1}} \geqslant \|\nabla u_i\|_{L^{p_1}(B_i)}. \tag{2}$$

Let $p_2 > 2$ and define $p(x) = p_1 \chi_{B_i \setminus B_i'}(x) + p_2 \chi_{B_i'}(x)$ for $x \in B$ with positive first coordinate. Since $\nabla u_i = 0$ in B_i' we obtain

$$\|\nabla u_i\|_{L^{p(\cdot)}(B_i)} = \|\nabla u_i\|_{L^{p_1}(B_i)}. \tag{3}$$

Let $\tilde{B}_i = B(-2^{-i}e_1, \frac{1}{4}2^{-i})$. We extend u_i to B as an odd function of the first coordinate in $\tilde{B}_i$ and by zero elsewhere. We also extend p to B as an even function of the first coordinate. We denote the extensions by $\tilde{u}_i$ and $\tilde{p}$. By (2) and (3) we obtain

$$2^{1+\frac{1}{p_1}} \Big(\operatorname{cap}_{p_1}(B_i', \mathbb{R}^2 \setminus B_i)\Big)^{\frac{1}{p_1}} \geqslant \|\nabla \tilde{u}_i\|_{L^{\tilde{p}(\cdot)}(B)}.$$

By Lemma 2.4 this yields

$$\|\nabla \tilde{u}_i\|_{L^{\tilde{p}(\cdot)}(B)} \leqslant C(p_1) \left| \tfrac{1}{4} 2^{-i \frac{p_1-2}{p_1-1}} - \tfrac{1}{8} 2^{-i^2 \frac{p_1-2}{p_1-1}} \right|^{\frac{1-p_1}{p_1}}. \tag{4}$$

For large i the right hand side is approximately equal to $C(p_1) 2^{-i^2 \frac{2-p_1}{p_1}}$.

Since $(\tilde{u}_i)_B = 0$, we obtain

$$\|\tilde{u} - (\tilde{u}_i)_B\|_{L^{\tilde{p}(\cdot)}(B)} = \|\tilde{u}\|_{L^{\tilde{p}(\cdot)}(B)} \geqslant |B_i'|^{\frac{1}{p_2}} \approx 2^{-i^2 \frac{2}{p_2}}. \tag{5}$$

By inequalities (4) and (5) we find that

$$\frac{\|\tilde{u} - (\tilde{u}_i)_B\|_{L^{\tilde{p}(\cdot)}(B)}}{\|\nabla \tilde{u}_i\|_{L^{\tilde{p}(\cdot)}(B)}} \geqslant C(p_1) 2^{i^2(\frac{2}{p_1} - 1 - \frac{2}{p_2})} \to \infty$$

as $i \to \infty$ if $\frac{2}{p_1} - 1 - \frac{2}{p_2} > 0$, that is, if $p_2 > \frac{2p_1}{2-p_1} = p_1^*$.

We next show that the condition $p_D^+ \leqslant (p_D^-)^*$ in Theorem 2.2 can be replaced by a set of local conditions.

Theorem 2.6. *Let $D \subset \mathbb{R}^n$ be a bounded John domain. Assume that there exist John domains G_i, $i = 1, \dots, j$, so that $G_i \subset D$ for every i, $D = \cup_{i=1}^{j} G_i$ and either $p^+_{G_i} \leqslant (p^-_{G_i})^*$ or $p^-_{G_i} \geqslant n$ for every i. Then there exists a constant $C > 0$ such that*

$$\|u - u_D\|_{p(\cdot)} \leqslant C\|\nabla u\|_{p(\cdot)} \tag{6}$$

for every $u \in W^{1,p(\cdot)}(D)$. The constant C depends on n, $\operatorname{diam}(D)$, $|G_i|$, p and the John constants of D and G_i, $i = 1, \dots, j$.

Proof. Using the triangle inequality of the norm we obtain

$$\begin{aligned}\|u - u_D\|_{L^{p(\cdot)}(D)} &\leqslant \sum_{i=1}^{j} \|u - u_D\|_{L^{p(\cdot)}(G_i)} \\ &\leqslant \sum_{i=1}^{j} \|u - u_{G_i}\|_{L^{p(\cdot)}(G_i)} + \sum_{i=1}^{j} \|u_D - u_{G_i}\|_{L^{p(\cdot)}(G_i)}.\end{aligned} \tag{7}$$

We estimate the first part of the sum using Theorem 2.2. This yields

$$\begin{aligned}\|u - u_{G_i}\|_{L^{p(\cdot)}(G_i)} &\leqslant C(n, p_{G_i}, |G_i|, \lambda_i)\|\nabla u\|_{L^{p(\cdot)}(G_i)} \\ &\leqslant C(n, p_{G_i}, |G_i|, \lambda_i)\|\nabla u\|_{L^{p(\cdot)}(D)}\end{aligned} \tag{8}$$

for every $i = 1, \dots, j$. Here λ_i is the John constant of G_i. We next estimate the second part of the sum in (7) using the classical Poincaré inequality for the third inequality. We obtain

$$\begin{aligned}\|u_D - u_{G_i}\|_{L^{p(\cdot)}(G_i)} &\leqslant \|1\|_{L^{p(\cdot)}(G_i)} \fint_{G_i} |u(x) - u_D|\,dx \\ &\leqslant \|1\|_{L^{p(\cdot)}(G_i)} |G_i|^{-1} \int_D |u(x) - u_D|\,dx \\ &\leqslant C(n, \operatorname{diam}(D), \lambda)|G_i|^{-1}\|1\|_{L^{p(\cdot)}(G_i)}\|\nabla u\|_{L^1(D)} \\ &\leqslant C(n, \operatorname{diam}(D), \lambda)(1 + |D|)|G_i|^{-1}\|1\|_{L^{p(\cdot)}(G_i)}\|\nabla u\|_{L^{p(\cdot)}(D)}\end{aligned} \tag{9}$$

for every $i = 1, \dots, j$. Here λ is the John constant of D. Now inequality (6) follows by inequalities (7), (8) and (9). □

Corollary 2.7. *Let $D \subset \mathbb{R}^n$ be a bounded convex domain and let $p\colon \overline{D} \to [1, \infty)$ be a continuous exponent. Then there exists a constant $C > 0$ such that*

$$\|u - u_D\|_{p(\cdot)} \leqslant C\|\nabla u\|_{p(\cdot)}$$

for every $u \in W^{1,p(\cdot)}(\mathbb{R}^n)$.

Proof. Since p is continuous we find for every $x \in \overline{D}$ a constant $r(x) > 0$ such that either

$$p^+_{B(x,r(x))\cap D} \leqslant (p^-_{B(x,r(x))\cap D})^* \quad \text{or} \quad p^-_{B(x,r(x))\cap D} \geqslant n.$$

Since $\overline{D}$ is compact it is possible to find finite covering of D with balls $B(x, r(x))$. It is easy to see that each $B(x, r(x)) \cap D$ is a John domain and hence the corollary follows by Theorem 2.6. □

Sometimes it is useful to have better control over the constant in the Poincaré inequality as the domain D changes than we have in (1). In the fixed exponent case the constant of the Poincaré inequality is $C \operatorname{diam}(D)$. We show that this kind of constant is also possible for variable exponent Sobolev spaces. The price we have to pay for this is that the exponent p has to satisfy a much stronger condition in Theorem 2.8 than in Theorem 2.2; in Theorem 2.2 the exponent p could be discontinuous even in every point, but in Theorem 2.8 the exponent is 0-Hölder continuous.

Theorem 2.8. *Let $D \subset \mathbb{R}^n$ be a bounded uniform domain. Let $p\colon D \to \mathbb{R}$ be such that $1 < p^-_D \leqslant p^+_D < \infty$. Assume that there exists a constant $C > 0$ such that*

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|} \tag{10}$$

for every $x, y \in D$ with $|x - y| \leqslant \frac{1}{2}$. Then the inequality

$$\|u - u_D\|_{p(\cdot)} \leqslant C \operatorname{diam}(D) \Big(1 + \max\Big\{|D|^{1/p^+_D - 1/p^-_D}, |D|^{1/p^-_D - 1/p^+_D}\Big\}\Big) \|\nabla u\|_{p(\cdot)}, \tag{11}$$

holds for every $u \in W^{1,p(\cdot)}(D)$. Here the constant C depends on the dimension n, the uniform constant of D and p.

Proof. Since $W^{1,p(\cdot)}_0(D) \hookrightarrow W^{1,1}(D)$ we obtain as in the proof of [12, Theorem 11] for every $u \in W^{1,p(\cdot)}(D)$ that

$$|u(x) - u(y)| \leqslant C|x - y|(\mathbb{M}\nabla u(x) + \mathbb{M}\nabla u(y)) \tag{12}$$

for almost every $x, y \in D$. Here $\mathbb{M}$ is the Hardy-Littlewood maximal operator:

$$\mathbb{M}\nabla u(x) = \sup_{r>0} \fint_{B(x,r)} |\nabla u(y)|\,dy,$$

with the understanding that $\nabla u = 0$ outside D. The constant C depends on the dimension n and the uniform constants of D.

Integrating inequality (12) over y we obtain

$$\begin{aligned}\Big|u(x) - \fint_D u(y)\,dy\Big| &\leqslant \fint_D |u(x) - u(y)|\,dy \\ &\leqslant C \operatorname{diam}(D)\Big(\mathbb{M}\nabla u(x) + \fint_D \mathbb{M}\nabla u(y)\,dy\Big).\end{aligned}$$

By Hölder's inequality [17, Theorem 2.1] this yields

$$|u(x) - u_D| \leqslant C \operatorname{diam}(D)\Big(\mathbb{M}\nabla u(x) + \frac{C(p)\|1\|_{L^{p'(\cdot)}(D)}}{|D|}\|\mathbb{M}\nabla u\|_{p(\cdot)}\Big).$$

Since the previous inequality holds point-wise, it is clear that we have an inequality also for the Lebesgue norms of both sides:

$$\begin{aligned}\|u - u_D\|_{p(\cdot)} &\leqslant C \operatorname{diam}(D)\left(\|\mathbb{M}\nabla u\|_{p(\cdot)} + \frac{C}{|D|}\|1\|_{p'(\cdot)}\|1\|_{p(\cdot)}\|\mathbb{M}\nabla u\|_{p(\cdot)}\right)\\ &\leqslant C \operatorname{diam}(D)\left(1 + |D|^{-1}\max\{|D|^{1+1/p_D^+-1/p_D^-}, |D|^{1+1/p_D^--1/p_D^+}\}\right)\|\mathbb{M}\nabla u\|_{p(\cdot)}\end{aligned}$$

By [5, Theorem 3.5] (see also [7, Remark 2.2]) the Hardy-Littlewood maximal operator is bounded, and so we obtain

$$\|u - u_D\|_{p(\cdot)} \leqslant C \operatorname{diam}(D)\left(1 + \max\left\{|D|^{1/p_D^+-1/p_D^-}, |D|^{1/p_D^--1/p_D^+}\right\}\right)\|\nabla u\|_{p(\cdot)},$$

where the constant C depends on the dimension n, the uniform constant of D and p. □

Remark 2.9. We refer to [19] for basic properties of uniform domains: Every uniform domain is a John domain. Every domain, whose boundary is locally a graph of a Lipschitz continuous function, is a uniform domain. In particular if D is a ball then the constant in (11) depends on the dimension n and p.

Corollary 2.10. *Let p be as in the previous theorem. If B is a ball with $|B| \leqslant 1$ then*

$$\|u - u_B\|_{p(\cdot)} \leqslant C \operatorname{diam}(B)\|\nabla u\|_{p(\cdot)},$$

where the constant C does not depend on B.

Proof. Since $|B| \leqslant 1$ we have

$$\max\left\{|B|^{1/p_B^+-1/p_B^-}, |B|^{1/p_B^--1/p_B^+}\right\} = |B|^{1/p_B^+-1/p_B^-}.$$

Since p is 0-Hölder continuous, (10), we obtain by [5, Lemma 3.2] that there exists a constant $C > 0$, depending only on the dimension n and the constant in (10), such that $|B|^{1/p_B^+-1/p_B^-} \leqslant C$ for every ball B. Hence $|B| \leqslant 1$ implies that the constant in (11) is less than $C \operatorname{diam}(B)$. □

3. Continuity

The functions in the classical Sobolev space $W^{1,p}$ are continuous if $p > n$. In this section we consider when functions in variable exponent Sobolev space are continuous.

Theorem 3.1. *Suppose that $p > n$ is locally bounded away from n in D. Then $W^{1,p(\cdot)}(D) \subset C(D)$.*

Proof. Let $x \in D$ and consider the ball $B = B(x, \delta(x)/2)$. Define $q = \operatorname{ess\,inf}_{y\in B} p(y)$. Then, by [17, Theorem 2.8],

$$W^{1,p(\cdot)}(B) \hookrightarrow W^{1,q}(B) \subset C(B).$$

Therefore every function in $W^{1,p(\cdot)}(D)$ is continuous at x, and since x was arbitrary, the claim follows. □

The following corollary is immediate.

Corollary 3.2. *Suppose that p is continuous in D. Then $W^{1,p(\cdot)}(D) \subset C(D)$ if $p(x) > n$ for every $x \in D$.*

We next use a classical example to show that the assumption that p is locally bounded away from n in D is not superfluous when p is not continuous.

Example 3.3. Let $B = B(0, 1/16)$, $\varepsilon > 0$ and suppose that

$$p(x) \leqslant \overline{p}(|x|) = n + (n-1-\varepsilon)\frac{\log_2 \log_2(1/|x|)}{\log_2(1/|x|)}$$

for $x \in B \setminus \{0\}$ and $p(0) > n$. We show that then $W^{1,p(\cdot)}(B) \not\subset C(B)$.

Define $u(x) = \cos(\log_2 |\log_2|x||)$ for $x \in B \setminus \{0\}$ and $u(0) = 0$. Clearly u is not continuous at the origin. So we have to show that $u \in W^{1,p(\cdot)}(B)$. It is clear that u has partial derivatives, except at the origin.

Since u is bounded it follows that $u \in L^{p(\cdot)}(B)$. We next estimate the gradient:

$$|\nabla u(x)| = \left|\sin(\log_2|\log_2|x||) \cdot \frac{1}{|x|\log_2|x|}\right| \leqslant \left|\frac{1}{|x|\log_2|x|}\right|.$$

We therefore find that

$$\begin{aligned}\int_B |\nabla u(x)|^{p(x)} dx &\leqslant \int_B \frac{dx}{(|x||\log_2|x||)^{p(x)}} \\ &= \omega_{n-1}\int_0^{1/16} \frac{r^{n-1}dr}{(r|\log_2 r|)^{\overline{p}(r)}} \\ &= \omega_{n-1}\sum_{i=5}^{\infty}\int_{2^{-i-1}}^{2^i} \frac{r^{n-1}dr}{(r|\log_2 r|)^{\overline{p}(r)}}.\end{aligned}$$

Since $1/(r|\log_2 r|) > 1$ we may increase the exponent $\overline{p}$ for an upper bound. In the annulus $B(0, 2^{-i}) \setminus B(0, 2^{-i-1})$ we have $i \leqslant \log_2(1/|x|) \leqslant i+1$. Since $y \to \log_2(y)/y$ is decreasing we find that

$$\overline{p}(x) \leqslant n + (n-1-\varepsilon)\frac{\log_2 i}{i}$$

in the same annulus. We can therefore continue our previous estimate by

$$\begin{aligned}\int_B |\nabla u(x)|^{p(x)} dx &\leqslant \sum_{i=5}^{\infty} \int_{2^{-i-1}}^{2^{-i}} \frac{r^{n-1} dr}{(r|\log_2 r|)^{n+(n-1-\varepsilon)\log_2(i)/i}} \\ &\leqslant C \sum_{i=5}^{\infty} \int_{2^{-i-1}}^{2^{-i}} \frac{2^{-i(n-1)} dr}{(i2^{-i})^{n+(n-1-\varepsilon)\log_2(i)/i}} \\ &= C \sum_{i=5}^{\infty} 2^{(n-1-\varepsilon)\log_2(i)} i^{-n-(n-1-\varepsilon)\log_2(i)/i} \\ &= C \sum_{i=5}^{\infty} i^{-1-\varepsilon} i^{-(n-1-\varepsilon)\log_2(i)/i} \leqslant C \sum_{i=5}^{\infty} i^{-1-\varepsilon} < \infty.\end{aligned}$$

4. Sobolev imbedding theorems

We start by introducing a relative variational $p(\cdot)$-pseudocapacity, and proving some basic properties for it. This capacity is quite similar to the Sobolev $p(\cdot)$-capacity studied by P. Harjulehto, P. Hästö, M. Koskenoja and S. Varonen in [13].

Let $F, E \subset \mathbb{R}^n$ be closed disjoint sets and D be a domain in $\mathbb{R}^n$. The *variational $p(\cdot)$-pseudocapacity* is defined as

$$\psi_{p(\cdot)}(F, E; D) = \inf_{u \in L(F,E;D)} \|\nabla u\|_{L^{p(\cdot)}(D)},$$

where $L(F, E; D)$ is as before (see Section 2). For $L(F, E; D) = \emptyset$ we define $\psi_{p(\cdot)}(F, E; D) = \infty$. We write $L(E, x; D)$ for $L(F, \{x\}; D)$ etc.

Remark 4.1. Including $C(D)$ in the definition of the capacity is somewhat strange in this context, since we do not, in general, know whether continuous functions are dense in $W^{1,p(\cdot)}(D)$, but see [8]. However, since we are interested in the case when $p > n$, the assumption makes sense, by Theorem 3.1.

The reason for calling the function $\psi_{p(\cdot)}(F, E; D)$ a pseudocapacity is that it is defined as a capacity but using the norm instead of the modular. This corresponds to introducing an exponent $1/p$ to the capacity in the fixed exponent case. Because of this we cannot expect the pseudocapacity to have all the usual properties of a capacity. It nevertheless has many of them:

Theorem 4.2. *Let $F, E \subset \mathbb{R}^n$ be closed sets and D be a domain in $\mathbb{R}^n$. Then the set function $(F, E) \mapsto \psi_{p(\cdot)}(F, E; D)$ has the following properties:*

(i) $\psi_{p(\cdot)}(\emptyset, E; D) = 0$.

(ii) $\psi_{p(\cdot)}(F, E; D) = \psi_{p(\cdot)}(E, F; D)$.

(iii) *Outer regularity, i. e.* $\psi_{p(\cdot)}(F, E_1; D) \leqslant \psi_{p(\cdot)}(F, E_2; D)$.

(iv) *If E is a subset of $\mathbb{R}^n$, then*

$$\psi_{p(\cdot)}(F, E; D) = \inf_{\substack{E \subset U \\ U \text{ open}}} \psi_{p(\cdot)}(F, U; D).$$

(v) *If $K_1 \supset K_2 \supset \dots$ are compact, then*

$$\lim_{i\to\infty} \psi_{p(\cdot)}(F, K_i; D) = \psi_{p(\cdot)}\Big(F, \bigcap_{i=1}^{\infty} K_i; D\Big).$$

(vi) *Suppose that $p > n$ is locally bounded away from n. If $E_i \subset \mathbb{R}^n$ for every $i = 1, 2, \dots$, then*

$$\psi_{p(\cdot)}\Big(F, \bigcup_{i=1}^{\infty} E_i; D\Big) \leq \sum_{i=1}^{\infty} \psi_{p(\cdot)}\left(F, E_i; D\right).$$

Proof. Assertion (i) is clear since we may use a constant function. Assertion (ii) is clear since if $u \in L(F, E; D)$ then $1 - u \in L(E, F; D)$. Assertion (iii) follows since $L(F, E_2; D) \subset L(F, E_1; D)$.

Next we prove (iv). It is clear that

$$\psi_{p(\cdot)}(F, E; D) \leqslant \inf_{\substack{E \subset U \\ U \text{ open}}} \psi_{p(\cdot)}(F, U; D).$$

Let $\varepsilon > 0$. Assume that $u \in L(F, E; D)$ is such that

$$\|\nabla u\|_{p(\cdot)} \leqslant \psi_{p(\cdot)}(F, E; D) + \varepsilon.$$

Since u is continuous, $\{u > 1 - \varepsilon\}$ is an open set containing E. Hence we obtain

$$\begin{aligned} \inf_{\substack{E \subset U \\ U \text{ open}}} \psi_{p(\cdot)}(F, U; D) &\leqslant \psi_{p(\cdot)}(F, \{u > 1 - \varepsilon\}; D) \\ &\leqslant \Big\| \nabla \min\{\frac{u}{1 - \varepsilon}, 1\} \Big\|_{p(\cdot)} \leq (1 - \varepsilon)^{-1} \|\nabla u\|_{p(\cdot)} \\ &\leqslant (1 - \varepsilon)^{-1} \psi_{p(\cdot)}(F, E; D) + \frac{\varepsilon}{1 - \varepsilon}. \end{aligned}$$

Letting $\varepsilon \to 0$ yields assertion (iv).

We then prove (v). It is clear that

$$\psi_{p(\cdot)}(F, \cap_{i=1}^{\infty} K_i; D) \leqslant \lim_{i\to\infty} \psi_{p(\cdot)}(F, K_i; D)$$

Let $\varepsilon > 0$. Assume that $u \in L(F, \cap_{i=1}^{\infty} K_i; D)$ is such that

$$\|\nabla u\|_{p(\cdot)} \leqslant \psi_{p(\cdot)}(F, \cap_{i=1}^{\infty} K_i; D) + \varepsilon.$$

When i is large the set K_i lies in the closed set $\{u \geqslant 1 - \varepsilon\}$; therefore

$$\begin{aligned}\lim_{i\to\infty} \psi_{p(\cdot)}(F, K_i; D) &\leqslant \psi_{p(\cdot)}(F, \{u \geqslant 1-\varepsilon\}; D)\\ &\leqslant \Big\|\nabla \min\{\frac{u}{1-\varepsilon}, 1\}\Big\|_{p(\cdot)} \leq (1-\varepsilon)^{-1}\|\nabla u\|_{p(\cdot)}\\ &\leqslant (1-\varepsilon)^{-1}\psi_{p(\cdot)}(F, \cap_{i=1}^{\infty} K_i; D) + \frac{\varepsilon}{1-\varepsilon}.\end{aligned}$$

Letting $\varepsilon \to 0$ yields assertion (v).

To prove (vi) let $\varepsilon > 0$ and choose functions $u_i \in L(F, E_i; D)$ such that

$$\|\nabla u_i\|_{p(\cdot)} \leqslant \psi_{p(\cdot)}(F, E_i; D) + \varepsilon/2^i,$$

for $i = 1, \ldots$. Let $v_i = u_1 + \ldots + u_i$. Then (v_i) is a Cauchy sequence, and so it converges to a function $v \in W^{1,p(\cdot)}(D)$. Define $\tilde{v}(x) = \min\{v(x), 1\}$, so that $|\tilde{v}| \in L^{p(\cdot)}(D)$ by [13, Theorem 2.2]. It is clear that $\tilde{v}|_{F\cap D} = 0$ and $\tilde{v}|_{E\cap D} = 1$, where $E = \cup E_i$. Since $p > n$ is locally bounded away from n, it follows from Theorem 3.1 that every function in $W^{1,p(\cdot)}(D)$ is continuous, and so we have $\tilde{v} \in L(F, \bigcup E_i; D)$, from which the claim easily follows, since

$$\|\nabla \tilde{v}\|_{p(\cdot)} \leqslant \sum_{i=1}^{\infty} \|\nabla u_i\|_{p(\cdot)} \leqslant \sum_{i=1}^{\infty} \psi_{p(\cdot)}(F, E_i; D) + \varepsilon. \qquad \square$$

Using the pseudocapacity we can start our study of Sobolev-type imbeddings. The following result is the direct generalization of [20, 5.1.1, Theorem 1].

Theorem 4.3. *If $p^+ < \infty$, then the following two conditions are equivalent:*

(i) $W^{1,p(\cdot)}(D) \cap C(D) \hookrightarrow L^{\infty}(D)$.

(ii) *There exist $r, k > 0$ such that $\psi_{p(\cdot)}(\overline{D} \setminus B(x, r), x; D) \geqslant k$ for every $x \in D$.*

Proof. Suppose that (2) holds, with constants $r, k > 0$. Let $u \in W^{1,p(\cdot)}(D) \cap C(D)$ and let $y \in D$ be a point with $u(y) \neq 0$. Fix a function $\eta \in C_0^{\infty}(B(0,1))$ with $0 \leqslant \eta \leqslant 1$ and $\eta(0) = 1$. Define $v(x) = \eta\big((x-y)/r\big)u(x)/u(y)$. It is clear that $v \in W^{1,p(\cdot)}(D)$

and since $v(y) = 1$ and $v(x) = 0$ for $x \notin B(y, r)$ we see that $v \in L(\overline{D} \setminus B(y, r), y; D)$. It follows that

$$k \leqslant \psi_{p(\cdot)}(\overline{D} \setminus B(y, r), y; D) \leqslant \|\nabla v\|_{p(\cdot)}.$$

Then we calculate that

$$\begin{aligned} k|u(y)| &\leqslant \|\nabla\big(u(x)\eta((x - y)/r)\big)\|_{p(x)} \\ &\leqslant \sup_{x\in D} \eta(x)\|\nabla u\|_{p(\cdot)} + \frac{1}{r} \sup_{x\in D} \nabla\eta(x)\|u\|_{p(\cdot)} \\ &\leqslant \max\Big\{ \sup_{x\in D} \eta(x), \frac{1}{r} \sup_{x\in D} \nabla\eta(x)\Big\}\|u\|_{1,p(\cdot)}, \end{aligned}$$

so that $|u(y)|$ is bounded by a constant independent of y.

Suppose conversely that (1) holds and let C be a constant such that $\|u\|_\infty \leqslant C\|u\|_{1,p(\cdot)}$ for all $u \in W^{1,p(\cdot)}(D)$. For functions in $v \in L(\overline{D} \setminus B(x, r), x; D)$ this gives

$$1 = \|v\|_\infty \leqslant C\|v\|_{1,p(\cdot)} \leqslant C(\|\chi_{B(x,r)}\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}).$$

Since $p^+ < \infty$ we can choose r small enough that $\|\chi_{B(x,r)}\|_{p(\cdot)} \leqslant 1/(2C)$. For such r we have $\|\nabla u\|_{p(\cdot)} \geqslant 1/(2C)$. It follows that

$$\psi_{p(\cdot)}(\overline{D} \setminus B(x, r), x; D) = \inf_{u\in L(\overline{D}\setminus B(x,r),x;D)} \|\nabla u\|_{p(\cdot)} \geqslant 1/(2C)$$

for the same r. □

Remark 4.4. Since we do not know whether $C^\infty(D)$ is dense in $W^{1,p(\cdot)}(D)$ we have only proved the theorem for continuous functions in $W^{1,p(\cdot)}(D)$. If p is such that $C(D)$ is dense in $W^{1,p(\cdot)}(D)$, for instance if p is locally bounded above n, then we may replace condition (1) by $W^{1,p(\cdot)}(D) \hookrightarrow L^\infty(D)$.

Define $D = B(1/16) \setminus \{0\}$ and let p be as in Example 3.3. Then the standard example $u(x) = \log|\log(x)|$ shows that $W^{1,p(\cdot)}(D) \not\hookrightarrow L^\infty$, the calculations being as in the theorem. We next show that the exponent p from the theorem is almost as good as possible. We need the following lemma.

Lemma 4.5. *Let $\{a_i\}$ be a partition of unity and $k > m - 1$. Then*

$$\sum_{i=0}^{\infty} a_i^m i^k \geqslant \Big(\sum_{i=0}^{\infty} i^{-k/(m-1)}\Big)^{1-m}.$$

Proof. Fix an integer i and consider the function

$$a \mapsto (a_i + a)^m i^k + (a_{i+1} - a)^m (i + 1)^k,$$

for $-a_i < a < a_{i+1}$. We find that this function has a minimum at $a = 0$ if and only if

$$\left(\frac{a_i}{a_{i+1}}\right)^{m-1} = \left(\frac{i+1}{i}\right)^k. \tag{13}$$

Let $\{a_i\}$ be a minimal sequence, so that (13) holds for every $i \geqslant 0$. This partition is given by $a_i = i^{-k/(m-1)}a_0$ for $i > 0$ and $a_0 = (\sum i^{-k/(m-1)})^{-1}$ and so we easily calculate the lower bound as given in the lemma. □

We next give a simple sufficient condition for the imbedding $W^{1,p(\cdot)}(D) \hookrightarrow L^\infty(D)$ to hold in a regular domain:

Theorem 4.6. *Suppose that D satisfies a uniform interior cone condition. If $p^+ < \infty$ and*

$$p(x) \geqslant n + (n-1+\varepsilon)\frac{\log_2 \log_2(c/\delta(x))}{\log_2(c/\delta(x))}$$

for some fixed $0 < \varepsilon < n-1$ and constant $c > 0$ then $W^{1,p(\cdot)}(D) \hookrightarrow L^\infty(D)$. Here $\delta(x)$ denotes the distance of x from the boundary of D

Proof. Note first that the claim trivially holds in compact subsets of D which satisfy the cone condition, since p is bounded away from n in such sets. Therefore it suffices to prove the claim for $\delta(x)$ less than some constant.

By the uniform interior cone condition there exist real values $0 < \alpha < \pi/2$ and $r > 0$ and a unit vector field v_x such that for every $x \in D$ the cone

$$C_x = \{y \in B(x,r) \colon \langle x-y, v_x\rangle > |x-y|\cos\alpha\}$$

lies completely in D, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product.

Fix $z \in D$. Consider the cone

$$C = \{y \in B(z,r/2) \colon \langle z-y, v_z\rangle > |z-y|\cos(\alpha/3)\}$$

and, for $i = 2, 3, \ldots$, the annuli

$$A_i = \big(B(z, 2^{-i+1}r) \setminus B(z, 2^{-i}r)\big) \cap C.$$

To simplify notation let us assume that $z = 0$, $r = 1$ and $v_z = e_1$; the proof in the general case is essentially identical. Since $A_i \subset C \subset D$ we have $d(A_i, \partial D) \geqslant d(A_i, \partial C)$. We can estimate the latter distance as shown in Figure 1. This gives $d(A_i, \partial D) \geqslant 2^{-i}\sin(\alpha/3)$ so that

$$p(x) \geqslant n + (n-1+\varepsilon)\frac{\log_2(i+c)}{i+c}$$

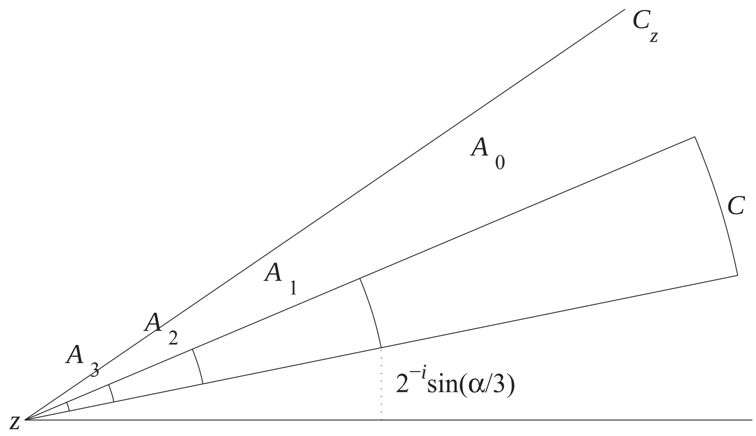

Figure 1: The cone C and the distance to the boundary

for $x \in A_i$ and some c depending on α. Let us define $q_i = n + (n-1+\varepsilon)\frac{\log_2(i+c)}{i+c}$ and a new variable exponent by

$$q(x) = \begin{cases} q_i & \text{if } x \in A_i \text{ for some } i \\ p(x) & \text{otherwise} \end{cases}$$

By Theorem 4.3 we know that it suffices to find a lower bound for $\|\nabla u\|_{1,p(\cdot)}$ with $u \in L(\overline{D} \setminus B(0,r), 0; D)$ since, by Theorem 3.1, $W^{1,p(\cdot)}(D) \subset C(D)$. Since $\|u\|_{1,p(\cdot)} \geqslant c\|u\|_{1,q(\cdot)}$, we see that it suffices to estimate $\psi_{q(\cdot)}(\overline{D} \setminus B(0,R), 0; B(0,R) \cap D)$ for small R in order to prove the theorem. Moreover, by monotony, we need only consider $\psi_{q(\cdot)}(\overline{D} \setminus B(0,R), 0; B(0,R) \cap C)$. For every function $u \in W^{1,q(\cdot)}(C)$ we have

$$\|u\|_{1,q(\cdot)} \geqslant \min\{1, \varrho_{1,q(\cdot)}(u)\},$$

by [17, Theorem 2.8]. Therefore we see that it suffices to show that $\varrho_{1,q(\cdot)}(u) > c$ for every $u \in L(\overline{D} \setminus B(0,R), 0; B(0,R) \cap C)$ in order to get $\psi_{q(\cdot)}(\overline{D} \setminus B(0,R), 0; B(0,R) \cap C) \geqslant \min\{1, c\} > 0$, which will complete the proof.

It is clear that $|\nabla u| \geqslant |\partial u/\partial r|$, the radial component of the gradient, so that

$$\int_{A_i} |\nabla u|^{q_i}\, dx \geq \int_{A_i} \left|\frac{\partial u}{\partial r}\right|^{q_i} dx.$$

It is then easy to see that the function minimizing the sum over the integrals should depend only on the distance from the origin, not on the direction. For such a function let us denote the value at any point of distance 2^{-i} from the origin by v_i.

Consider then a function v which equals v_{i-1} on $S(0,2^{-i+1})$ and v_i on $S(0,2^{-i})$. Using Lemma 2.4 we find that

$$\begin{aligned}\int_{A_i} |\nabla v|^{q_i}\,dx &\geqslant (v_{i-1}-v_i)^{q_i}\operatorname{cap}_{q_i}(\mathbb{R}^n\setminus B(0,2^{-i+1}),B(0,2^{-i}))\\ &= (v_{i-1}-v_i)^{q_i}\omega_{n-1}\left(\frac{q_i-n}{q_i-1}\right)^{q_i-1}\left(2^{(q_i-n)/(q_i-1)}-1\right)^{1-q_i}2^{i(q_i-n)}\\ &\geqslant c(v_{i-1}-v_i)^{q_i}2^{i(q_i-n)},\end{aligned}$$

where the constant c does not depend on q_i. It follows that

$$\varrho_{1,q(\cdot)}(v)\geqslant\sum_{i=2}^{\infty}\int_{A_i}|\nabla u|^{q_i}\,dx\geqslant c\sum_{i=2}^{\infty}(v_{i-1}-v_i)^{q_i}2^{i(q_i-n)}.$$

Since the lower bound depends only on the v_i, we see that

$$\inf_{u\in L}\varrho_{1,q(\cdot)}(u)\geqslant c\inf_{\{v_i\}}\sum_{i=2}^{\infty}(v_{i-1}-v_i)^{q_i}2^{i(q_i-n)},$$

where the second infimum is over sequences $\{v_i\}$ with $v_i\leqslant v_{i-1}$, $v_0=1$ and $\lim_{i\to\infty}v_i=0$. Let us set $a_i=v_{i-1}-v_i$ so that $a_i\geqslant 0$ and $\sum a_i=1$. Then we need to estimate

$$\inf_{\{a_i\}}\sum_{i=2}^{\infty}a_i^{q_i}2^{i(q_i-n)},$$

with the infimum over partitions of unity $\{a_i\}$. Let N be such that

$$\frac{\varepsilon}{3}\geqslant q_i-n=(n-1+\varepsilon)\frac{\log_2(i+c)}{i+c}\geqslant(n-1+\varepsilon/2)\frac{\log_2(i)}{i}$$

for $i\geq N$. Note that such an N can be chosen independent of z. Since $a_i\leqslant 1$ we have $a_i^{q_i}\geqslant a_i^{n+\varepsilon/3}$ for such terms. Then we find that

$$\inf_{\{a_i\}}\sum_{i=2}^{\infty}a_i^{q_i}2^{i(q_i-n)}\geqslant\inf_{\{a_i\}}\sum_{i=2}^{N-1}a_i^{q_i}2^{i(q_i-n)}+\sum_{i=N}^{\infty}a_i^{n+\varepsilon/3}i^{n-1+\varepsilon/2}.$$

The first sum on the left-hand-side is finite, hence

$$\sum_{i=2}^{N-1}a_i^{q_i}2^{i(q_i-n)}\geqslant\sum_{i=2}^{N-1}a_i^{q}\geqslant N^{1-q}\bigg(\sum_{i=2}^{N-1}a_i\bigg)^{q},$$

where $q=\max_{2\leqslant i\leqslant N-1}q_i$. It follows from Lemma 4.5 that

$$\sum_{i=N}^{\infty}a_i^{n+\varepsilon/3}i^{n-1+\varepsilon/2}\geqslant c\bigg(\sum_{i=N}^{\infty}a_i\bigg)^{n+\varepsilon/3}.$$

Combining these estimates we see that

$$\inf_{\{a_i\}} \sum_{i=2}^{\infty} a_i^{q_i} 2^{i(q_i-n)} \geqslant N^{1-q} \Big(\sum_{i=2}^{N-1} a_i \Big)^q + c \Big(\sum_{i=N}^{\infty} a_i \Big)^{n+\varepsilon/3}$$

is uniformly bounded from below by a positive constant, since the sum of the a_i's is 1. We have thus shown that the condition of Theorem 4.3 holds, which concludes the proof. □

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