Sharp Estimates of the Embedding Constants for Besov Spaces

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Received: June 6, 2005
Accepted: October 25, 2005

ABSTRACT

Sharp estimates are obtained for the rates of blow up of the norms of embeddings of Besov spaces in Lorentz spaces as the parameters approach critical values.

Key words: Besov spaces, Lorentz spaces, embedding norms, rates of blow-up.

2000 Mathematics Subject Classification: 46E30, 46E35.

Introduction

Our concern is with embeddings of Sobolev type and more particularly with the behaviour of the corresponding embedding constants as the various parameters approach critical values. To provide some background and motivation for this study we begin with classical Sobolev spaces on $\mathbf{R}^n$: all the spaces we shall consider in this paper will consist of real-valued functions. Let $k \in \mathbf{N}$, $1 \le p < \infty$ and denote by $\|\cdot\|_p$ the usual $L_p(\mathbf{R}^n)$ norm. Let w_p^k be the (homogeneous) Sobolev space given by the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|f\|_{w_p^k} := \sum_{|\alpha|=k} \|D^\alpha f\|_p,$$

ISSN: 1139-1138
http://dx.doi.org/10.5209/rev_REMA.2006.v19.n1.16647

in standard notation; let W_p^k be the corresponding inhomogeneous Sobolev space, with norm

$$\|f\|_{W_p^k} := \|f\|_p + \|f\|_{w_p^k}.$$

Given any $r \in (0,\infty)$ and $q \in (0,\infty]$, let $L_{r,q}$ be the usual Lorentz space, defined by the quasinorm

$$\|f\|_{r,q} := \begin{cases} \left(\int_0^\infty \{t^{1/r} f^*(t)\}^q \, \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{0<t<\infty} \{t^{1/r} f^*(t)\}, & q = \infty. \end{cases}$$

Here f^* is the non-increasing rearrangement of f, given by

$$f^*(t) = \inf\{\, \lambda \geq 0 \,:\, \mu_f(\lambda) \leq t \,\}, \qquad t \geq 0,$$

where μ_f is the distribution function of f, defined by

$$\mu_f(\lambda) = |\{\, x \in \mathbf{R}^n \,:\, |f(x)| \geq \lambda \,\}|_n,$$

$|\cdot|_n$ denoting Lebesgue n-measure. Then it is well known that

$$w_p^k \hookrightarrow r_k^{-1} L_{r_k,p}, \quad \text{where } 1/r_k = 1/p - k/n \quad \text{and} \quad 1 < p < n/k. \tag{1}$$

Here by $X \hookrightarrow \lambda Y$, where X and Y are quasi-Banach spaces and $\lambda > 0$, we mean that there is a constant $C > 0$, independent of λ, such that for all $f \in X$, $\lambda \|f\|_Y \leq C\|f\|_X$; by $X \hookrightarrow Y$ we mean that X is continuously embedded in Y. We refer to Talenti [23] for the case $k = 1$; when $k > 1$ the results can be derived by induction (see [11]). It is also a familiar fact that if $q > p$, then

$$(q/r)^{1/q} \|f\|_{r,q} \leq (p/r)^{1/p} \|f\|_{r,p}; \tag{2}$$

see [22, p. 192]. Together with (1) this gives, under the conditions of (1),

$$w_p^k \hookrightarrow r_k^{1/p-1} L_{r_k}. \tag{3}$$

We see that the embedding constants in (1) and (3) blow up as $p \to (n/k)-$, $k < n$. It is natural to ask whether or not these rates of blow up are sharp. Such questions were considered in [9, 10]. To analyse them, we introduce the embedding constants

$$w_1 := \sup_{f \neq 0} \|f\|_{r_k,p} / \|f\|_{w_p^k}$$

and

$$w_2 := \sup_{f \neq 0} \|f\|_{r_k} / \|f\|_{w_p^k}.$$

We use the notation $c \lesssim d$ or $d \gtrsim c$ to mean that c is bounded above by a multiple of d, the multiple being independent of variables in c and d; also $c \approx d$ means that $c \lesssim d$ and $d \lesssim c$. Then, with this notation, it turns out that

$$w_1 \approx r_k \text{ as } p \to (n/k)-, \quad k < n. \tag{4}$$

A simple proof of this follows from the result (see [10, p. 90, Theorem 2.7.2]) that

$$h_1 := \sup_{f \neq 0} \|f\|_q / \|f\|_{H_p^{n/p}} \approx q^{1-1/p} \quad \text{as } q \to \infty \tag{5}$$

(here H_p^s, with $s > 0$ and $1 < p < \infty$, is the usual Bessel-potential space; see [24]) and from the embeddings

$$H_p^{n/p} \hookrightarrow W_p^k \hookrightarrow w_p^k, \quad 1 < p < n/k. \tag{6}$$

Combination of (6), (2) and (5) gives

$$w_1 \gtrsim r_k^{1/p} \sup_{f \neq 0} \|f\|_{r_k} / \|f\|_{H_p^{n/p}} \gtrsim r_k,$$

from which (4) is immediate in view of (1). Analogously,

$$w_2 \approx r_k^{1-1/p} \quad \text{as } p \to (n/k)-, \quad k < n. \tag{7}$$

The same results hold for the inhomogeneous Sobolev spaces W_p^k:

$$W_1 := \sup_{f \neq 0} \|f\|_{r_k,p} / \|f\|_{W_p^k} \approx r_k$$

and

$$W_2 := \sup_{f \neq 0} \|f\|_{r_k} / \|f\|_{W_p^k} \approx r_k^{1-1/p}.$$

This analysis explains what happens when the integration parameter p approaches the critical value n/k from below. However, a question which has attracted much recent attention concerns blow up when the smoothness parameter approaches critical values. To explain this we introduce the (homogeneous) Besov spaces $b_{p,q}^s$. For $0 < s < k$, $1 \leq p < \infty$ and $0 < q \leq \infty$ these may be defined as the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|f\|_{b_{p,q}^s} := \left(\int_0^\infty \{t^{-s} \omega_p^k(t,f)\}^q \, \frac{dt}{t} \right)^{1/q},$$

(suitably interpreted when $q = \infty$), where $\omega_p^k(t,f) := \sup_{|h| \leq t} \left\| \Delta_h^k f \right\|_p$ is the k^{th}-order L_p-modulus of continuity of f, Δ_h^k being the $k^{th}-$order difference operator with step length h; see after (10) below. It is well known (see [21]) that

$$b_{p,q}^s \hookrightarrow L_{r,q} \quad \text{if } 1/r = 1/p - s/n, \quad 0 < s < n/p,$$

and we wish to find sharp rates of blow up of the embedding constant as $s \to (n/p)-$. Let

$$b_1 := \sup_{f \neq 0} \|f\|_{r,q} / \|f\|_{b^s_{p,q}} \tag{8}$$

and

$$b_2 := \sup_{f \neq 0} \|f\|_r / \|f\|_{b^s_{p,q}}. \tag{9}$$

Since $s \to n/p$ and $0 < s < k$, we must have $n/p \leq k$; it turns out that the results are different for the two possible cases $n/p < k$ and $n/p = k$. When $n/p < k$ it is known that (see [11])

$$b^s_{p,q} \hookrightarrow r^{-c} L_{r,q}, \quad 1/r = 1/p - s/n, \quad 0 < s < n/p < k, \quad c = \max(1, 1/q),$$

and we show that as $s \to n/p < k$, with $1 < p < \infty$ and $0 < q \leq \infty$,

$$b_1 \approx r^c, \quad b_2 \approx r^{(1-1/q)_+}, \quad a_+ = \max(a, 0).$$

If $k = n/p$ the results are different since $\|f\|_{b^s_{p,q}}$ may tend to infinity as $s \to n/p = k$. Putting $s = \sigma k$, $0 < \sigma < 1$, we show that if $1 < p < \infty$, $k = n/p$ and $0 < q \leq \infty$, then, as $\sigma \to 1-$,

$$b^{k\sigma}_{p,q} \hookrightarrow (1-\sigma)^{-a+c} L_{r,q},$$

where

$$1/r = (1-\sigma)/p, \quad a = \min(1/p, 1/q), \quad c = \max(1, 1/q).$$

When $k = 1$, $q = p \geq 1$, these results were proved by Bourgain, Brezis and Mironescu [5], Maz'ya and Shaposhnikova (see [18,19]) and Kolyada and Lerner [15]; the cases $k = n/p \geq 2$, $q \geq 1$ and $k \neq n/p$, $q > 0$ are covered by [11] if $p \geq 1$. In this paper, not only do we establish results for wider exponent ranges, but we also provide different proofs for the aforementioned cases: as in [11] we use real interpolation, but we make substantial use of the (nonlinear) spaces $L_{(r,q)}$ defined to be the set of all functions $f \in L_1 + L_\infty$ such that $f^*(\infty) = 0$ and

$$\|f\|_{L_{(r,q)}} := \begin{cases} \left\{\int_0^\infty t^{q/r} \left(f^{**}(t) - f^*(t)\right)^q \frac{dt}{t}\right\}^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{1/r} (f^{**}(t) - f^*(t)), & q = \infty, \end{cases}$$

is finite (see [2,17,20]). Here $f^{**}(t) = t^{-1} \int_0^t f^*(s) ds$. For the embedding constants b_1, b_2 of (8) and (9) we show that if $1 < p < \infty$, $k = n/p$, $1 \leq q \leq \infty$ and $1/r = (1-\sigma)k/n$, then as $r \to \infty$,

$$b_1 \approx r^{1-1/p}, \quad b_2 \approx r^{1-2/p}.$$

Inequalities of the type

$$f^{**}(t) - f^*(t) \leq 2\{\, f^{**}(t) - f^{**}(2t) \,\} \lesssim t^{-1/p} \omega^k_p(t^{1/n}, f)$$

($f \in C_0^\infty(\mathbf{R}^n)$, $k \in \mathbf{N}$, $1 \le p < n/(k-2)$ if $k \ge 3$, $1 \le p \le \infty$ otherwise) enable the proofs to be streamlined and may have independent interest. Similar inequalities are given by Kolyada in [13, 14] in the cases $n = 1$, $k = 1, 2$; see Remark 1.3.

The paper concludes with a brief discussion of the supercritical case of embeddings of Besov spaces; that is, when the target space is L_∞.

1. Preliminaries

1.1. Real interpolation and Besov spaces

We recall briefly the construction of real interpolation spaces. Let $\vec{A} = (A_0, A_1)$ be a pair of quasi-Banach spaces that are compatible in the sense that both A_0 and A_1 are continuously embedded in some common quasi-Banach space. The K-functional for $\vec{A}$ is defined, for $t > 0$ and $f \in A_0 + A_1$, by

$$K(t, f; \vec{A}) = K(t, f; A_0, A_1) = \inf_{f=f_0+f_1} \{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\}.$$

For $0 < \theta < 1$ and $0 < q \le \infty$, the real interpolation space $\vec{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$ is the set of all $f \in A_0 + A_1$ such that

$$\|f\|_{\vec{A}_{\theta,q}} := \begin{cases} \left(\int_0^\infty \left(t^{-\theta} K(t, f; \vec{A})\right)^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{t>0} t^{-\theta} K(t, f; \vec{A}), & q = \infty, \end{cases}$$

is finite. It is well known (cf. [3, p. 341]) that

$$K(t^k, f; L_p, w_p^k) \approx \omega_p^k(t, f) := \sup_{|h|\le t} \|\Delta_h^k f\|_p, \quad p \in [1, \infty), \tag{10}$$

where Δ_h^k denotes the k^{th} difference operator defined recursively by

$$\Delta_h f(x) = \Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^k = \Delta_h^1 \Delta_h^{k-1} \quad (k \ge 2).$$

The homogeneous Besov spaces $b_{p,q}^s$ on $\mathbf{R}^n$, with $k \in \mathbf{N}$, $0 < s < k$, $1 \le p < \infty$ and $0 < q \le \infty$, are defined as the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm

$$\|f\|_{b_{p,q}^s} := \left(\int_0^\infty (t^{-s}\omega_p^k(t, f))^q \, \frac{dt}{t}\right)^{1/q},$$

interpreted appropriately when $q = \infty$. We shall write b_p^s instead of $b_{p,p}^s$.

From (10) we see that for $0 < s < 1$,

$$b_{p,q}^{sk} = (L_p, w_p^k)_{s,q}.$$

Moreover (see [11])

$$K(t,f;L_p,w_p^1)\approx\left\{t^{-n}\int_{|h|\le t}\|\Delta_h f\|_p^p dh\right\}^{1/p}.$$

Together with Fubini's theorem this shows that

$$\|f\|_{b_p^s}\approx\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|f(x)-f(y|^p}{|x-y|^{n+sp}}\,dxdy\right)^{1/p}, \tag{11}$$

with constants of equivalence independent of s. In fact, in [15] it is shown that

$$(n\varpi_n)^{1/p}2^{-n-2}\|f\|_{b_p^s}\le\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|f(x)-f(y|^p}{|x-y|^{n+sp}}\,dxdy\right)^{1/p}\le((n+p)\varpi_n)^{1/p}\|f\|_{b_p^s},$$

where ϖ denotes the volume of the unit ball in $\mathbf{R}^n$. Hence our norm on b_p^s is equivalent to that used in [5, 6, 18], namely the right-hand side of (11).

1.2. The spaces $L(r,q)$

These spaces will play an important technical rôle in our arguments. For $0<q\le\infty$ and $-\infty<1/r<1$ the (nonlinear) space $L(r,q)$ is defined (following [2, 17, 20]) to be the family of all $f\in L_1+L_\infty$ such that $f^*(\infty)=0$ and

$$\|f\|_{L(r,q)}:=\left\{\int_0^\infty t^{q/r}(f^{**}(t)-f^*(t))^q\frac{dt}{t}\right\}^{1/q}<\infty$$

(with the natural interpretation when $q=\infty$). Here $f^{**}(t)=t^{-1}\int_0^t f^*(s)ds$. Since $\frac{d}{dt}f^{**}(t)=t^{-1}\{f^*(t)-f^{**}(t)\}$, it follows that

$$f^{**}(t)=\int_t^\infty(f^{**}(s)-f^*(s))\,\frac{ds}{s}\quad\text{if } f^*(\infty)=0. \tag{12}$$

Note that an application of l'Hôpital's rule to $f^{**}(t)=t^{-1}\int_0^t f^*(s)\,ds$ shows that the condition $f^*(\infty)=0$ is equivalent to $f^{**}(\infty)=0$. Note also that

$$L(\infty,1)=L_\infty.$$

We shall need the following embedding result.

Lemma 1.1. *Let $1<r<\infty$, $0<q\le\infty$ and put $c=\max(1,1/q)$. Then*

$$L(r,q)\hookrightarrow r^{-c}L_{r,q}.$$

Proof. First suppose that $q \geq 1$ and recall that replacement of f^* by f^{**} in $\|f\|_{r,q}$ gives a quasinorm equivalent to $\|f\|_{r,q}$ (see [3, p. 219, Lemma 4.4.5]). Then from (12) and Minkowski's inequality we have, for all $f \in L(r,q)$,

$$\begin{aligned}\|f\|_{r,q} &\lesssim \int_1^\infty \|f^{**}(s\cdot) - f^*(s\cdot)\|_{r,q} \frac{ds}{s} \\ &\lesssim \int_1^\infty s^{-1/r} \|f\|_{L(r,q)} \frac{ds}{s} = r\|f\|_{L(r,q)}.\end{aligned}$$

If $0 < q < 1$, we integrate by parts and obtain,

$$\|f\|_{r,q}^q \lesssim \int_0^\infty t^{q/r} h^q(t) \frac{dt}{t} = \frac{r}{q}[t^{q/r}h^q(t)]_0^\infty - r\int_0^\infty t^{q/r} h^{q-1}(t) h'(t)\, dt,$$

where

$$h(t) := f^{**}(t), \quad h'(t) = -t^{-1}(f^{**}(t) - f^*(t)).$$

To handle the integrated terms, first note that since

$$t(f^{**}(t) - f^*(t)) = \int_{f^*(t)}^\infty \mu_f(s)\, ds$$

(see [7, equation (6)]), $t(f^{**}(t) - f^*(t))$ is increasing. Thus

$$t(f^{**}(t) - f^*(t))\left(\int_t^\infty s^{q/r-1-q}\, ds\right)^{1/q} \leq \left(\int_t^\infty (f^{**}(s) - f^*(s))^q s^{q/r-1}\, ds\right)^{1/q},$$

and so

$$t^{1/r}(f^{**}(t) - f^*(t)) \lesssim \left(\int_t^\infty (f^{**}(s) - f^*(s))^q s^{q/r-1}\, ds\right)^{1/q}.$$

This shows that

$$f^{**}(t) - f^*(t) = \begin{cases} o(t^{-1/r}) & \text{as } t \to \infty, \\ O(t^{-1/r}) & \text{as } t \to 0. \end{cases}$$

It follows that

$$t^{1/r} f^{**}(t) = \begin{cases} t^{1/r} \int_t^\infty (f^{**}(s) - f^*(s)) \frac{ds}{s} \to 0 & \text{as } t \to \infty, \\ O(1) & \text{as } t \to 0. \end{cases}$$

Hence the integrated terms may be ignored and we have

$$\|f\|_{r,q}^q \lesssim r\int_0^\infty t^{q/r}(f^{**}(t) - f^*(t))^q \frac{dt}{t},$$

as desired. The lemma is proved. □

1.3. Pointwise estimates for the rearrangement

Lemma 1.2. *Let $k \in \mathbf{N}$ and suppose that $1 \le p < \frac{n}{k-2}$ if $k \ge 3$, $1 \le p \le \infty$ if $k = 1, 2$. Then for all $f \in C_0^\infty(\mathbf{R}^n)$,*

$$f^{**}(t) - f^{**}(2t) \lesssim t^{-1/p}\omega_p^k(t^{1/n}, f), \quad 0 < t < \infty. \tag{13}$$

Note that for $n = 1$, similar estimates are proved in [13, Theorem 1], in the case $k = 1$ and [14, pp. 149–150], for $k = 2$; see Remark 1.3 below.

Proof. Let $t > 0$ and let B_h be the ball in $\mathbf{R}^n$ with centre 0, radius h and measure $2t$. Let $u \in \mathbf{R}^n$, $|u| \le h$. Since

$$|f(x)| \le |\Delta_u f(x)| + |f(x+u)|,$$

we have, integrating with respect to u over B_h,

$$2t|f(x)| \le \int_{B_h} |\Delta_u f(x)| du + \int_0^{2t} f^*(s)\, ds.$$

Now integrate with respect to x over a subset E of $\mathbf{R}^n$ with Lebesgue n-measure t and take the supremum over all such sets E. This gives (see [3, p. 53, Proposition 2.3.3])

$$\begin{aligned} 2t[f^{**}(t) - f^{**}(2t)] &\le \int_{B_h} (\Delta_u f)^{**}(t)\, du \\ &= \frac{1}{t}\int_{B_h}\int_0^t (\Delta_u f)^*(s)\, ds du \lesssim \frac{1}{t}\int_{B_h} \|\Delta_u f\|_1\, du \\ &\le 2 \sup_{|u| \le (2t/\varpi_n)^{1/n}} \|\Delta_u f\|_1 = 2\omega_1^1((2t/\varpi_n)^{1/n}, f). \end{aligned} \tag{14}$$

In view of of [4, (5.4.5), p. 332] and the fact that ω_1^1 is an increasing function, we see that (13) follows immediately if $p = k = 1$. If $p > 1$, we apply Hölder's inequality and obtain (13) for $k = 1$. To cover the case $k = 2$ we use the inequality

$$|f(x)| \le \frac{1}{2}|\Delta_u^2 f(x-u)| + \frac{1}{2}[|f(x+u)| + |f(x-u)|]$$

which follows directly from the definition of $\Delta_u^2 f(x-u)$. Integration of this with respect to u over B_h gives

$$2t|f(x)| \le \frac{1}{2}\int_{B_h} |\Delta_u^2 f(x-u)|\, du + \int_0^{2t} f^*(s)\, ds.$$

Hence as before we have

$$2t[f^{**}(t) - f^{**}(2t)] \le \int_{B_h} (\Delta_u^2 f)^{**}(t)\, du. \tag{15}$$

As above we obtain from this the estimate (13) for the case $k = 2$.

To deal with the situation when $k \geq 3$, we proceed by induction. Suppose that k is even (when it is odd the argument is analogous) and that (13) holds for this k; let $p < n/k$. Then (12) and the inductive hypothesis show that

$$f^{**}(t) \lesssim \int_t^\infty s^{-1/p} \omega_p^k(s^{1/n}, f) \frac{ds}{s} = \int_t^\infty s^{-1/p+k/n} \frac{\omega_p^k(s^{1/n}, f)}{s^{k/n}} \frac{ds}{s}.$$

Since $t \longmapsto t^{-1} K(t, f; L_p, w_p^k)$ is monotonic decreasing (see [3, p. 294]), this gives, with the help of (10),

$$f^{**}(t) \lesssim t^{-1/p} \omega_p^k(t^{1/n}, f), \quad k < n/p.$$

Apply this estimate to $\Delta_u^2 f$: we obtain

$$(\Delta_u^2 f)^{**}(t) \lesssim t^{-1/p} \omega_p^k(t^{1/n}, \Delta_u^2 f), \quad k < n/p.$$

From Lemma 5.4.11 of [3] and this we see that

$$\sup_{|v| \leq t^{1/n}} \sup_{|u| \leq ct^{1/n}} \|\Delta_v^k \Delta_u^2 f\|_p \lesssim \omega_p^{k+2}(t^{1/n}, f)$$

and

$$\left(\Delta_u^2 f\right)^{**}(t) \lesssim t^{-1/p} \omega_p^{k+2}(t^{1/n}, f), \ k < n/p.$$

Together with (15) this gives (13) with k replaced by $k+2$. The proof is complete. ☐

Remark 1.3. When $n = 1$, it is proved respectively in [13, Theorem 1] and [14, pp. 149–150], that

$$f^{**}(t) - f^*(t) \lesssim t^{-1/p} \omega_p^1(t, f) \tag{16}$$

and

$$f^{**}(t) - f^*(t) \lesssim t^{-1/p} \omega_p^2(t, f) \tag{17}$$

These results are equivalent to the cases $k = 1, 2$ respectively of (13), as we shall now show. Since

$$\frac{d}{dt} f^{**}(t) = -t^{-1} \{f^{**}(t) - f^*(t)\}$$

we have that

$$f^{**}(t) - f^{**}(2t) = \int_t^{2t} s^{-1} \{f^{**}(s) - f^*(s)\} \, ds.$$

Hence, from (16) and [3, p. 332, (4.5)],

$$\begin{aligned} f^{**}(t) - f^{**}(2t) &\lesssim \int_t^{2t} s^{-1-\frac{1}{p}} \omega_p^1(s, f) ds \lesssim t^{-1/p} \omega_p^1(2t, f) \\ &\lesssim t^{-1/p} \omega_p^1(t, f), \end{aligned}$$

whence (13) with $k = 1$. In (26) below we prove that

$$f^{**}(t) - f^{*}(t) \le 2\{f^{**}(t) - f^{**}(2t)\}$$

and so (13) with $k = 1$ implies (16). Similarly (17) is equivalent to the case $k = 2$ of (13).

2. Sharp embedding constants for Besov spaces, the subcritical case

Here we consider the well known embedding (see [21])

$$b^s_{p,q} \hookrightarrow L_{r,q}, \quad 1/r = 1/p - s/n, \quad 0 < s < n/p,$$

and aim to find sharp rates of blow up for the embedding constants as $s \to n/p$. As explained in the Introduction, this means that $n/p \le k$, and the cases $n/p < k$ and $n/p = k$ give different results. We introduce the embedding constants

$$b_1 := \sup_{f\neq 0} \|f\|_{r,q}/\|f\|_{b^s_{p,q}} \quad \text{and} \quad b_2 := \sup_{f\neq 0} \|f\|_r/\|f\|_{b^s_{p,q}}.$$

2.1. The case $n/p < k$

Theorem 2.1. *Let $1 < p < \infty$, $0 < q \le \infty$, $0 < s < n/p < k$, $1/r = 1/p - s/n$ and put $c = \max(1, 1/q)$. Then as $s \to n/p$,*

$$b_1 \approx r^c \quad \textit{and} \quad b_2 \approx r^{(1-1/q)_+}. \tag{18}$$

Proof. From [11] we know that $b^s_{p,q} \hookrightarrow r^{-c} L_{r,q}$. This implies that

$$b_1 \lesssim r^c \tag{19}$$

and

$$b_2 \lesssim r^{c-1/q}.$$

Suppose that $0 < q \le 1$. Choose $f \in C^\infty_0(\mathbf{R}^n)$ such that $f = 1$ in a neighbourhood of the origin and $|\operatorname{supp} f|_n > 2$. Then $f^*(s) \gtrsim 1$ for $0 < s < 1$ and hence

$$\|f\|_{r,q} \gtrsim r^{1/q} \quad \text{as } r \to \infty. \tag{20}$$

On the other hand,

$$\omega^k_p(t, f) \lesssim \begin{cases} t^k, & 0 < t < 1, \\ 1, & t > 1. \end{cases}$$

Thus

$$\|f\|_{b^s_{p,q}} \lesssim 1,$$

and from this, (19) and (20) we obtain the required estimate (18) for b_1, when $0 < q \le 1$.

Next let $1 < q \le \infty$. Consider the inhomogeneous Besov space $B^s_{p,q}$, with quasi-norm

$$\|f\|_{B^s_{p,q}} := \|f\|_p + \|f\|_{b^s_{p,q}}.$$

We have

$$B^{n/p}_{p,q} \hookrightarrow B^s_{p,q} \hookrightarrow b^s_{p,q} \quad \text{if} \quad 0 < s < n/p < k,$$

and

$$b_1 \gtrsim \sup_{f \neq 0} \|f\|_{r,q} / \|f\|_{B^{n/p}_{p,q}} \gtrsim r^{1/q} \sup_{f \neq 0} \|f\|_r / \|f\|_{B^{n/p}_{p,q}}. \tag{21}$$

Now we choose a test function f in a way similar to that used in [25]:

$$f(x) := \sum_{j=0}^{l} \psi(2^j x), \quad l \approx r,$$

where $\psi \in C_0^\infty(\mathbf{R}^n)$ is non-negative and $\psi(x) = 1$ when $|x| \le a$, for some $a > 0$. Then $\|f\|_r \gtrsim r$. However, as in [25], we see that $\|f\|_{B^{n/p}_{p,q}} \approx r^{1/q}$. Together with (21) this gives $b_1 \gtrsim r$, from which (18) follows for b_1, when $1 < q \le \infty$.

The proof of (18) for b_2 is analogous. □

2.2. The case $k = n/p$

Here the results are different as $\|f\|_{b^s_{p,q}}$ may tend to ∞ as $s \to n/p = k$. We write $s = \sigma k$, $0 < \sigma < 1$. Our initial aim is to obtain upper estimates for the embedding constants b_1 and b_2; to do this it is convenient to establish some preliminary results.

Lemma 2.2. *Let $f \in C_0^\infty(\mathbf{R}^n)$. Then*

$$f^{**}(t) - f^{**}(2t) \lesssim t^{1/n} |\nabla f|^{**}(t) \tag{22}$$

and

$$f^{**}(t) - f^{**}(2t) \lesssim t^{2/n} |D^2 f|^{**}(t), \tag{23}$$

where $D^2 = \sum_{|\alpha|=2} D^\alpha$.

When $n = 1$, the estimate 22 is equivalent to one given in [12, Lemma 5.1] (see also [1]); see Remark 2.4 below.

Proof. From (14), if $t > 0$ then

$$f^{**}(t) - f^{**}(2t) \le \frac{1}{2t} \int_{B_h} (\Delta_u f)^{**}(t)\, du,$$

where B_h is the ball in $\mathbf{R}^n$ with centre 0, radius h and measure $2t$. To estimate the right-hand side of this, we note that

$$|(\Delta_u f)(x)| = \left| \int_0^1 \nabla f(x+su) \cdot u \, ds \right| \leq \int_0^1 |\nabla f(x+su)||u| \, ds$$

(see [16, V,4]). Integrate with respect to x over a subset E of $\mathbf{R}^n$ with Lebesgue measure t and take the supremum over all such subsets E. Then by [3, p. 53, Proposition 2.3.3],

$$(\Delta_u f)^{**}(t) \leq \int_0^1 |\nabla f|^{**}(t)|u| \, ds = |\nabla f|^{**}(t)|u|.$$

Hence

$$f^{**}(t) - f^{**}(2t) \leq \frac{1}{2t} \int_{B_h} |\nabla f|^{**}(t)|u| \, du \lesssim t^{1/n} |\nabla f|^{**}(t),$$

as required. The proof of (23) is similar, starting from

$$|(\Delta_u^2 f)(x)| \lesssim \int_0^1 |D^2 f(x+su)||u|^2 \, ds$$

(see [16, V,5]) and using (15). □

We shall also need to generalize the Sobolev spaces we have been considering to those built up over the Lorentz scale. If in the definition of w_p^k we replace the Lebesgue space L_p by the Lorentz space $L_{p,q}$, we obtain a space that we shall denote by $w^k L_{p,q}$. A consequence of Lemma 2.2 is

Proposition 2.3. *Let $1 < p \leq \infty, 0 < q \leq \infty$, $k \in \{1,2\}$ and as in* (1) *define r_k by $1/r_k = 1/p - k/n$. Then for all $f \in C_0^\infty(\mathbf{R}^n)$,*

$$\|f\|_{L(r_k,q)} \lesssim \|f\|_{w^k L_{p,q}}. \tag{24}$$

Proof. From (12), if $t > 0$ then

$$f^{**}(t) - f^{**}(2t) = \int_t^{2t} (f^{**}(\tau) - f^*(\tau)) \frac{d\tau}{\tau}. \tag{25}$$

Since the function g given by

$$g(t) = t(f^{**}(t) - f^*(t))$$

is non-decreasing, (25) implies that

$$f^{**}(t) - f^{**}(2t) = \int_t^{2t} \tau(f^{**}(\tau) - f^*(\tau)) \frac{d\tau}{\tau^2} \geq t(f^{**}(t) - f^*(t)) \cdot \frac{1}{2t};$$

that is,

$$f^{**}(t) - f^*(t) \le 2\{f^{**}(t) - f^{**}(2t)\}. \tag{26}$$

Suppose that $k = 1$. Then using (26) and (22), we see that if $p, q < \infty$, then

$$\begin{aligned}\|f\|^q_{L(r_1,q)} &= \int_0^\infty t^{q/r_1}(f^{**}(t) - f^*(t))^q \,\frac{dt}{t} \lesssim \int_0^\infty t^{q/r_1}(f^{**}(t) - f^{**}(2t))^q \,\frac{dt}{t} \\ &\lesssim \int_0^\infty \big(t^{1/p}|\nabla f|^{**}(t)\big)^q \,\frac{dt}{t} \lesssim \|f\|_{w^1 L_{p,q}}.\end{aligned}$$

If either p or q is infinite the proof is adapted in a natural manner.

The proof when $k = 2$ is similar, using this time (23) instead of (22). □

Remark 2.4. In [12, Lemma 5.1], it is proved that

$$f^*(t) - f^*(2t) \lesssim t^{1/n}|\nabla f|^{**}(t). \tag{27}$$

This is equivalent to (22). For, it follows from

$$f^{**}(2t) = \frac{1}{2t}\bigg\{\int_0^t f^*(s)\,ds + \int_t^{2t} f^*(s)\,ds\bigg\}$$

that

$$f^*(t) \le 2f^{**}(2t) - f^*(2t)$$

and hence, on using (26), if (22) holds, we have

$$\begin{aligned}f^*(t) - f^*(2t) &\le 2\{f^{**}(2t) - f^*(2t)\} \\ &\le 4\{f^{**}(2t) - f^*(4t)\} \\ &\lesssim t^{1/n}|\nabla f|^{**}(2t) \\ &\le t^{1/n}|\nabla f|^{**}(t).\end{aligned}$$

Conversely, (27) implies that

$$\begin{aligned}f^{**}(t) - f^{**}(2t) &= t^{-1}\int_0^t \{f^*(s) - f^*(2s)\}\,ds \\ &\lesssim t^{-1}\int_0^t s^{1/n}|\nabla f|^{**}(s)\,ds \\ &\le t^{1/n}|\nabla f|^{**}(t)\end{aligned}$$

since $s|\nabla f|^{**}(s)$ is non-decreasing.

Proposition 2.5. *Let $1 < p < \infty$ and let r_1 be defined by $1/r_1 = 1/p - 1/n$. Then for all $f \in C_0^\infty(\mathbf{R}^n)$,*

$$I := t\bigg\{\int_{t^n}^\infty \tau^{p/r_1}\,(f^{**}(\tau) - f^*(\tau))^p \,\frac{d\tau}{\tau}\bigg\}^{1/p} \lesssim \omega_p^1(t, f).$$

Proof. Let

$$g_t(x) := t^{-n} \int_0^t \cdots \int_0^t f(x+y)\, dy_1 \cdots dy_n.$$

Then

$$\begin{aligned} I^p &\lesssim t^p \int_{t^n}^{\infty} \tau^{p/r_1} (f^{**}(\tau) - f^{**}(2\tau))^p \, \frac{d\tau}{\tau} \\ &\lesssim t^p \int_{t^n}^{\infty} \tau^{p/r_1} \big((f-g_t)^{**}(\tau) + g_t^{**}(\tau) - f^{**}(2\tau)\big)^p \, \frac{d\tau}{\tau}. \end{aligned} \tag{28}$$

We claim that

$$g_t^{**}(\tau) \le f^{**}(\tau).$$

To justify this, note that

$$g_t^{**}(\tau) \le t^{-n} \int_0^t \cdots \int_0^t f^{**}(\tau) dy_1 \cdots dy_n = f^{**}(\tau).$$

Then (28) gives

$$I^p \lesssim \|f - g_t\|_p^p + t^p \int_{t^n}^{\infty} \tau^{p/r_1} (g_t^{**}(\tau) - g_t^{**}(2\tau))^p \, \frac{d\tau}{\tau}. \tag{29}$$

Moreover, from (25) we have

$$f^{**}(t) - f^{**}(2t) \lesssim t^{-1} \int_0^{2t} \{f^{**}(\tau) - f^*(\tau)\}\, d\tau. \tag{30}$$

Together with Minkowski's inequality, (29) and (30) give

$$I \lesssim \|f - g_t\|_p + t\|g_t\|_{L(r_1,p)},$$

and using (24) this shows that

$$I \lesssim \|f - g_t\|_p + t\|g_t\|_{w_p^1}.$$

Since $\|f - g_t\|_p \le \omega_p^1(t,f)$ and $\|g_t\|_{w_p^1} \le \omega_p^1(t,f)$, the proof is complete. □

A basic result concerning the embeddings of the spaces $w^k L_{p,q}$ is given in the following Lemma (see also [20]).

Lemma 2.6. *Let* $k \in \mathbf{N}$, $0 < q \le \infty$ *and suppose that* $1 < p < n/(k-2)$ *if* $k \ge 3$, *and* $1 < p \le \infty$ *otherwise. Let* r_k *be given by* $1/r_k = 1/p - k/n$, *as in* (1). *Then*

$$w^k L_{p,q} \hookrightarrow L(r_k, q).$$

Proof. The cases $k = 1, 2$ follow directly from Proposition 2.3. For the remaining cases, we use induction and suppose that the Lemma is true for some k, with $r_k > 0$. By the inductive hypothesis applied to the second-order derivatives of $f \in w^{k+2}L_{p,q}$, we have

$$\|D^2 f\|_{L_{r_k,q}} \lesssim \|f\|_{w^{k+2}L_{p,q}},$$

and hence, by (26) and (23),

$$\|f\|_{L(r_{k+2},q)} \lesssim \|f\|_{w^{k+2}L_{p,q}}.$$

The proof is complete. □

After this preparation we can give the promised embedding result.

Theorem 2.7. *Let $1 < p < \infty$, $0 < q \le \infty$ and $k = n/p$. Then as $\sigma \to 1-$,*

$$b^{k\sigma}_{p,q} \hookrightarrow (1-\sigma)^{-a+c} L_{r,q}, \tag{31}$$

where

$$1/r = (1-\sigma)/p$$

and

$$a = \min(1/p, 1/q), \quad c = \max(1, 1/q).$$

Proof. First suppose that $k = 1$. From Proposition 2.5, with $f \in b^{\sigma}_{p,q}$,

$$\|f\|^q_{b^{\sigma}_{p,q}} \gtrsim \int_0^\infty t^{(1-\sigma)q} \left\{ \int_{t^n}^\infty u^{p/r_1} (f^{**}(u) - f^*(u))^p \,\frac{du}{u} \right\}^{q/p} \frac{dt}{t} := I.$$

To estimate I from below, first suppose that $q \le p$ and apply Minkowski's inequality:

$$\begin{aligned} I &\gtrsim \int_0^\infty t^{(1-\sigma)q/n} \left\{ \int_1^\infty (ut)^{p/r_1} \,(f^{**}(ut) - f^*(ut))^p \,\frac{du}{u} \right\}^{q/p} \frac{dt}{t} \\ &\gtrsim \left\{ \int_1^\infty \left(\int_0^\infty t^{(1-\sigma)q/n} (ut)^{q/r_1} (f^{**}(ut) - f^*(ut))^q \,\frac{dt}{t} \right)^{p/q} \frac{du}{u} \right\}^{q/p} \\ &\gtrsim \left(\int_1^\infty u^{-(1-\sigma)p/n} \,\frac{du}{u} \right)^{q/p} \|f\|^q_{L(r_1,q)}. \end{aligned}$$

Thus

$$\|f\|_{b^{\sigma}_{p,q}} \gtrsim (1-\sigma)^{-1/p} \|f\|_{L(r_1,q)}, \quad q \le p. \tag{32}$$

If $q > p$, we integrate by parts. Let $h(t) := \int_t^\infty u^{p/r_1} (f^{**}(u) - f^*(u))^p \,\frac{du}{u}$. Then

$$I \gtrsim \int_0^\infty t^{(1-\sigma)q/n} h^{q/p}(t) \,\frac{dt}{t} \gtrsim -(1-\sigma)^{-1} \int_0^\infty t^{(1-\sigma)q/n} h^{(q/p)-1}(t) h'(t) dt.$$

Since $t \longmapsto g(t) := t\{f^{**}(t) - f^*(t)\}$ is non-decreasing, we have

$$h(t) = \int_t^\infty u^{-p/n} \Big(\frac{g(u)}{u}\Big)^p \, du \geq g^p(t) \int_t^\infty u^{-p/n-p} \, du,$$

and so

$$h(t) \gtrsim t^{1-p/n}\{f^{**}(t) - f^*(t)\}^p.$$

Hence

$$I \gtrsim (1-\sigma)^{-1} \int_0^\infty u^{q/r_1}(f^{**}(u) - f^*(u))^q \, \frac{du}{u}$$

and so

$$\|f\|_{b^\sigma_{p,q}} \gtrsim (1-\sigma)^{-1/q} \|f\|_{L(r_1,q)}, \quad q > p. \tag{33}$$

The Theorem for the case $k = 1$ now follows immediately from (32), (33), and Lemma 1.1.

Now suppose that $k \geq 2$. We claim that

$$b^{k\sigma}_{p,q} \hookrightarrow (w^{k-1}_p, w^k_p)_{\sigma_1,q}, \quad 1 - \sigma_1 = (1-\sigma)k. \tag{34}$$

In fact, using the embedding

$$(L_p, w^k_p)_{\theta,1} \hookrightarrow w^{k-1}_p, \quad \theta = 1 - 1/k,$$

we obtain (34) by reiteration (see [11]):

$$b^{k\sigma}_{p,q} \hookrightarrow ((L_p, w^k_p)_{\theta,1}, w^k_p)_{\sigma_1,q} \hookrightarrow (w^{k-1}_p, w^k_p)_{\sigma_1,q}, \quad 1 - \sigma_1 = (1-\sigma)k.$$

Moreover, since we know the theorem to be true when $k = 1$, we have

$$b^{\sigma_1}_{p,q} \hookrightarrow (1-\sigma_1)^{-a} L_{r_{\sigma_1},q}, \quad 1/r_{\sigma_1} = 1/p - \sigma_1/n. \tag{35}$$

Now (34) and (35) lead to

$$b^{k\sigma}_{p,q} \hookrightarrow (1-\sigma)^{-a} w^{k-1} L_{r_{\sigma_1},q},$$

which together with Lemma 2.6 and Lemma 1.1 give (31). The proof is complete. □

As observed earlier, when $k = 1$, $q = p \geq 1$, these results were proved in [5], [18] and [15]; the cases $k = n/p \geq 2$, $q \geq 1$ and $k \neq n/p$, $q > 0$ are covered by [11] if $p \geq 1$. The proof here is different. Note also that when $p = 1$ and $k = n$ the following better result is proved in [11]:

Theorem 2.8. *Let $k = n$ and put $1/r_\sigma = 1 - \sigma$, $0 < \sigma < 1$. Then*

$$b^{n\sigma}_{1,q} \hookrightarrow L_{r_\sigma,q}.$$

Now we can give the promised sharp estimates for the rates of blow up of the embedding constants b_1 and b_2, defined in (8) and (9), for the case $k = n/p$.

Corollary 2.9. *Let* $1 < p < \infty$, $1 \le q \le p$, $k = n/p$, $0 < \sigma < 1$ *and* $1/r = (1-\sigma)k/n$. *Then as* $r \to \infty$,

$$b_1 \approx r^{1-1/p}$$

and

$$b_2 \approx r^{1-2/p}.$$

Proof. The estimates from above follow from Theorem 2.7 and (2). For the rest, we use the test function f, where $f(x) = \phi(|x|^{k-\lambda})$, $0 < \lambda < k = n/p$, $\lambda \to k$, where ϕ is a smooth function supported in $\{ t \in \mathbf{R} : |t| \le 2 \}$ and $\phi(t) = t$ for $|t| < 1$. Then we have (20); and the derivatives of f of order k behave like $(k-\lambda)|x|^{-\lambda}$ for $0 < |x| < 2^{1/(k-\lambda)}$. In particular,

$$\|f\|_{w_p^k} \lesssim (k-\lambda)^{1-1/p}.$$

Hence

$$\omega_p^k(t,f) \lesssim \begin{cases} t^k(k-\lambda)^{1-1/p} & \text{if} \quad 0 < t < 1 \\ 1 & \text{if} \quad t > 1. \end{cases}$$

Thus

$$\|f\|_{b^\lambda_{p,q}} \lesssim (k-\lambda)^{1-1/p-1/q}. \tag{36}$$

Taking $\lambda = \sigma k$ we see that (36) and (20) give

$$b_1 \gtrsim r^{1-1/p}, \ b_2 \gtrsim r^{1-2/p} \quad \text{as } r \to \infty, \quad \text{if } k = n/p. \tag{37}$$

This finishes the proof. □

Note that (37) holds for $0 < q \le \infty$ and $1 < p < \infty$. This is needed in the proof of Corollary 2.13 below.

When $p = 1$ we have a complete result, valid for all $q > 0$, due to Theorem 2.8:

Corollary 2.10. *Let* $k = n$, $p = 1$, $0 < q \le \infty$ *and* $1/r = 1 - \sigma$, $0 < \sigma < 1$. *Then as* $r \to \infty$,

$$b_1 \approx 1 \quad \textit{and} \quad b_2 \approx r^{-1}.$$

It follows that when $1 \le q \le p$, the results are sharp. We now try to improve Theorem 2.7 for the case $q > p > 1$, interpolating the sharp result given by that Theorem for $q = p$. To this end we need the following uniform reiteration theorem.

Theorem 2.11. *Let* (A_0, A_1) *be a quasi-Banach pair. Let*

$$0 < \theta_0 < \theta_1 < 1, \quad \lambda := \theta_1 - \theta_0 \approx 1 - \theta_0 \approx 1 - \theta_1 \tag{38}$$

and suppose that $0 < \sigma < 1$, $q > p > 1$. *Then*

$$\left(\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p}\right)_{\sigma,q} = \lambda^{1/q-1/p}\vec{A}_{\theta,q}, \tag{39}$$

where $\theta = (1-\sigma)\theta_0 + \sigma\theta_1$, *uniformly with respect to* $\lambda \to 0$.

Proof. Step 1. Here we show that the usual reiteration formula of Holmstedt (see [4, p. 52]) for the K-functional is uniform with respect to $\lambda \to 0$ under the conditions (38):

$$K^p(t, f; \vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p}) \approx \int_0^{t^{1/\lambda}} u^{-\theta_0 p} K^p(u, f)\,\frac{du}{u} + t^p \int_{t^{1/\lambda}}^{\infty} u^{-\theta_1 p} K^p(u, f)\,\frac{du}{u}, \tag{40}$$

where $K(u, f) = K(u, f; A_0, A_1)$. Indeed, if $f = f_0 + f_1$, then

$$\begin{aligned} I_0^p := &\int_0^{t^{1/\lambda}} u^{-\theta_0 p} K^p(u, f)\,\frac{du}{u} \\ \lesssim &\int_0^{t^{1/\lambda}} u^{-\theta_0 p} K^p(u, f_0)\,\frac{du}{u} + \int_0^{t^{1/\lambda}} u^{\lambda p} u^{-\theta_1 p} K^p(u, f_1)\,\frac{du}{u}, \end{aligned}$$

and so

$$I_0 \lesssim \|f_0\|_{\vec{A}_{\theta_0,p}} + t\|f_1\|_{\vec{A}_{\theta_1,p}}.$$

Analogously,

$$I_1^p := t^p \int_{t^{1/\lambda}}^{\infty} u^{-\theta_1 p} K^p(u, f)\,\frac{du}{u}$$

satisfies

$$I_1 \lesssim \|f_0\|_{\vec{A}_{\theta_0,p}} + t\|f_1\|_{\vec{A}_{\theta_1,p}}.$$

The upper estimate implicit in (40) now follows.

For the reverse inequality, choose a representation $f = f_0 + f_1$ such that

$$K(t^{1/\lambda}, f) \approx \|f_0\|_{A_0} + t^{1/\lambda}\|f_1\|_{A_1}.$$

Then

$$K(s, f_0) \le \|f_0\|_{A_0} \lesssim K(t^{1/\lambda}, f) \quad \text{and} \quad K(s, f_1) \lesssim s t^{-1/\lambda} K(t^{1/\lambda}, f).$$

It follows that

$$\begin{aligned} \|f_0\|^p_{\vec{A}_{\theta_0,p}} \lesssim &\int_0^{t^{1/\lambda}} u^{-\theta_0 p} K^p(u, f)\,\frac{du}{u} + \int_0^{t^{1/\lambda}} u^{-\theta_0 p} K^p(u, f_1)\,\frac{du}{u} \\ &+ \int_{t^{1/\lambda}}^{\infty} u^{-\theta_0 p} K^p(t^{1/\lambda}, f)\,\frac{du}{u}. \end{aligned}$$

Since

$$\int_0^{t^{1/\lambda}} u^{-\theta_0 p} K^p(u, f_1)\,\frac{du}{u} \lesssim \int_0^{t^{1/\lambda}} u^{(1-\theta_0)p} t^{-p/\lambda} K^p(t^{1/\lambda}, f)\,\frac{du}{u}$$
$$\lesssim (1-\theta_0)^{-1} t^{-\theta_0 p/\lambda} K^p(t^{1/\lambda}, f),$$

we see that

$$\|f_0\|^p_{\vec{A}_{\theta_0,p}} \lesssim I_0^p + \{\theta_0(1-\theta_0)\}^{-1} t^{-\theta_0 p/\lambda} K^p(t^{1/\lambda}, f). \tag{41}$$

Analogously,

$$t^p \|f_1\|^p_{\vec{A}_{\theta_1,p}} \lesssim I_1^p + \{\theta_1(1-\theta_1)\}^{-1} t^{-\theta_0 p/\lambda} K^p(t^{1/\lambda}, f). \tag{42}$$

Noticing that

$$I_0^p \gtrsim (1-\theta_0)^{-1} t^{-\theta_0 p/\lambda} K^p(t^{1/\lambda}, f),$$

we see that under the conditions (38), the estimates (41) and (42) become

$$\|f_0\|^p_{\vec{A}_{\theta_0,p}} \lesssim I_0^p \quad \text{and} \quad t^p \|f_1\|^p_{\vec{A}_{\theta_1,p}} \lesssim I_1^p + I_0^p.$$

This completes the proof of (40).

Step 2. Here we prove (39), assuming that $q > p$. By (40) we have

$$A := \int_0^\infty t^{-\sigma q} I_0^q \,\frac{dt}{t} = \lambda \int_0^\infty t^{-\lambda\sigma q} g^{q/p}(t) \frac{dt}{t},$$

where $g(t) := \int_0^t u^{-\theta_0 p} K^p(u, f)\,\frac{du}{u}$. Integrating by parts and using the facts that

$$g'(t) = t^{-\theta_0 p - 1} K^p(t, f), \quad g(t) \geq (1-\theta_0)^{-1} t^{-\theta_0 p} K^p(t, f),$$

we obtain

$$\|f\| := \|f\|_{(\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})_{\sigma,q}} \gtrsim (1-\theta_0)^{1/q-1/p} \|f\|_{\vec{A}_{\theta,q}}, \quad \theta = (1-\sigma)\theta_0 + \sigma\theta_1. \tag{43}$$

For the reverse estimate, we write

$$A = \lambda \int_0^\infty t^{-\lambda\sigma q} \left(\int_0^1 (tu)^{-\theta_0 p} K^p(tu, f)\,\frac{du}{u} \right)^{q/p} \frac{dt}{t}.$$

By Minkowski's inequality (see, for example, p. 530 of [8]),

$$\begin{aligned}
A &\leq \lambda \left(\int_0^1 \left(\int_0^\infty (tu)^{-\theta_0 q} K^q(tu, f) t^{-\lambda\sigma q}\,\frac{dt}{t} \right)^{p/q} \frac{du}{u} \right)^{q/p} \\
&= \lambda \left(\int_0^1 \left(\int_0^\infty v^{-\theta q} K^q(v, f)\,\frac{dv}{v} \right)^{p/q} u^{\lambda\sigma p - 1} du \right)^{q/p} \\
&= \lambda \|f\|^q_{\vec{A}_{\theta,q}} \left(\int_0^1 u^{\lambda\sigma p - 1} du \right)^{q/p} \\
&= (\sigma p)^{-q/p} \|f\|^q_{\vec{A}_{\theta,q}} \lambda^{1-q/p}
\end{aligned} \tag{44}$$

Let

$$B = \lambda \int_0^\infty t^{(1-\sigma)\lambda q} \left(\int_0^1 (tu)^{-\theta_1 p} K^p(tu, f) \, \frac{du}{u} \right)^{q/p} \frac{dt}{t}.$$

By Minkowski's inequality again,

$$B \lesssim \lambda^{1-q/p} \|f\|^q_{\vec{A}_{\theta,q}}.$$

Together with (44) this gives

$$\|f\| \lesssim \lambda^{1/q-1/p} \|f\|_{\vec{A}_{\theta,q}}. \tag{45}$$

Finally, (39) follows from (43) and (45). □

Theorem 2.12. *Let $1 < p < q \leq \infty$, $0 < \sigma < 1$, $1/r = (1-\sigma)/p$ and $k = n/p$. Then as $\sigma \to 1$,*

$$b^{k\sigma}_{p,q} \hookrightarrow (1-\sigma)^{1-1/p} L_{r,q}.$$

Proof. Let $\theta_0 = 2\sigma - 1$, $\theta_1 = (1+\sigma)/2$. Then Theorem 2.7 gives

$$(L_p, w^k_p)_{\theta_j,p} \hookrightarrow (1-\sigma)^{1-1/p} (L_p, L_\infty)_{\theta_j,p}, \quad j = 0, 1.$$

Applying Theorem 2.11 with $\sigma = 2/3$ we have

$$(L_p, w^k_p)_{\sigma,q} \hookrightarrow (1-\sigma)^{1-1/p} (L_p, L_\infty)_{\sigma,q}.$$

Since (see [11]) $(L_p, L_\infty)_{\sigma,q} = L_{r,q}$, uniformly with respect to $\sigma \approx 1$, the desired embedding follows. □

As a consequence of this we have

Corollary 2.13. *Let $1 < p < \infty$, $p < q$, $1 \leq q \leq \infty$, $k = n/p$, $0 < \sigma < 1$ and $1/r = (1-\sigma)k/n$. Then as $r \to \infty$,*

$$b_1 \approx r^{1-1/p} \quad \textit{and} \quad b_2 \approx r^{1-2/p}.$$

The case $0 < q < 1$ remains to be settled.

3. Sharp embedding constants for Besov spaces, the supercritical case

It is well known (see [24]) that if $s > n/p$, then

$$B^s_{p,q} \hookrightarrow L_\infty,$$

where $B^s_{p,q}$ is the inhomogeneous Besov space, defined by means of Fourier decompositions. The problem of finding sharp rates of blow up for the corresponding embedding constants as $s \to (n/p)+$ was considered quite recently by Triebel [25]. In fact, he dealt not only with Besov but also with Lizorkin-Triebel spaces, both types of spaces being considered on bounded Lipschitz domains. Here we consider the same problem for the slightly larger Besov spaces $\tilde{b}^s_{p,q}$ $(1 \le p < \infty)$, defined by means of the norm

$$\|f\|_{\tilde{b}^s_{p,q}} := \left(\int_0^1 \{t^{-s}\omega_p^{k+1}(t,f)\}^q \,\frac{dt}{t}\right)^{1/q} + \|f\|_{L_1+L_\infty}, \quad s < k+1.$$

Note that using monotonicity, we can replace the above integral by a sum and then conclude that the scale $\tilde{b}^s_{p,q}$ is increasing with respect to q. It is also decreasing with respect to s. It turns out that the results concerning the embedding constants are the same as in [25]. Let

$$b_3 := \sup_{f\neq 0} \|f\|_\infty / \|f\|_{\tilde{b}^s_{p,q}}.$$

Theorem 3.1. *Let $1 \le p < \infty$, $0 < q \le \infty$, $n/p < k+1$ and $\sigma > 0$. Then as $\sigma \to 0$,*

$$\tilde{b}^{n/p+\sigma}_{p,q} \hookrightarrow \sigma^{(1-1/q)_+} L_\infty.$$

Proof. Using (12), (26) and Lemma 1.2, we have

$$f^{**}(t) \lesssim \int_t^1 u^{-1/p}\omega_p^{k+1}(u^{1/n}, f)\,\frac{du}{u} + f^{**}(1), \quad k < n/p + 1.$$

From this and monotonicity the result follows for the case $q \le 1$. If $q > 1$ we also have to apply the Hölder inequality. □

As a consequence we have

Corollary 3.2. *Under the conditions of the last theorem,*

$$b_3 \approx \sigma^{-(1-1/q)_+}.$$

Part of this follows from [25], since $B^s_{p,q} \hookrightarrow \tilde{b}^s_{p,q}$, uniformly with respect to $s \to (n/p)+$ (see [24, p. 110]).

Acknowledgements. G. E. K. is grateful for a research grant from the School of Mathematics, Cardiff University, which enabled him to visit the Universities of Cardiff and Sussex in January and February 2005, when much of this work was done.

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