Journal of Siberian Federal University. Mathematics & Physics 2013, 6(4), 437–440

УДК 517.2

Construction of Some Simple Locally Finite Groups

Ülviye B. Güven*
Department of Mathematics,
Middle East Technical University, Ankara, 06 531
Turkey

Otto H. Kegel†
Mathematisches Institut, Eckerstr. 1,
Albert-Ludwigs-Universitat Freiburg,
Eckerstrabe, 1, Freiburg, D-79104,
Germany

Mahmut Kuzucuoğlu‡
Department of Mathematics,
Middle East Technical University, Ankara, 06 531
Turkey

Received 04.08.2013, received in revised form 15.09.2013, accepted 16.10.2013

We construct a proper class of simple locally finite groups. Namely for each infinite cardinal κ, we construct uncountably many pairwise non-isomorphic simple locally finite groups of cardinality κ, as a direct limit of finitary symmetric groups. The construction of the groups of similar kind for countably infinite order has been common knowledge as indicated in [2]. The countable ones are classified using the lattice of Steinitz numbers by Kroshko-Sushchansky in [3]. We give the classification of the uncountable ones by the pair, the cardinality of the group and the characteristic which corresponds to a Steinitz number. We study the structure of the centralizers of arbitrary elements in this new class of groups and correct some of the errors in the section about the centralizers of elements in $S(\xi)$ in [3].

Keywords: simple locally finite group, Steinitz number, construction.

In [2] the construction of countably infinite order, simple locally finite groups as a direct limit of finite symmetric groups is given in the following way: Let Π be the set of sequences consisting of prime numbers and $\xi \in \Pi$. So $\xi = (p_1, p_2, \ldots)$ is a sequence consisting of not necessarily distinct primes p_i.

Let $\alpha \in S_n$. For a natural number $p \in \mathbb{N}$, a permutation $d^p(\alpha) \in S_{pn}$ defined by $(kn+i)^{d^p(\alpha)} = kn+i^\alpha$, $0 \leqslant k \leqslant (p-1)$ and $1 \leqslant i \leqslant n$ is called a **homogenous p-spreading** of the permutation α. So if $\alpha = \begin{pmatrix} 1 \ldots & n \\ i_1 \ldots i_n \end{pmatrix}$, then the homogeneous $p-$spreading of the permutation α is

$$d^p(\alpha) = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \ldots & n & n+1 & \ldots & 2n & \ldots & (p-1)n+1 & \ldots & pn \\ i_1 & \ldots & i_n & n+i_1 & \ldots & n+i_n & \ldots & (p-1)n+i_1 & \ldots & (p-1)n+i_n \end{array} \right)$$

We obtain direct systems by using homogenous p_i-spreadings from the following embeddings where p_i is the i^{th} prime in the sequence ξ.

$$\{1\} \stackrel{d^{p_1}}{\to} S_{p_1=n_1} \stackrel{d^{p_2}}{\to} S_{n_2} \stackrel{d^{p_3}}{\to} S_{n_3} \stackrel{d^{p_4}}{\to} \ldots$$

and

*cbusra@metu.edu.tr
†otto.h.kegel@math.uni-freiburg.de
‡matmah@metu.edu.tr

$$\{1\} \stackrel{d^{p_1}}{\rightarrow} A_{p_1=n_1} \stackrel{d^{p_2}}{\rightarrow} A_{n_2} \stackrel{d^{p_3}}{\rightarrow} A_{n_3} \stackrel{d^{p_4}}{\rightarrow} \dots$$

where $n_i = p_i n_{i-1}$, $i = 1, 2, 3\dots$ and S_{n_i} is the symmetric group on n_i letters, A_{n_i} is the alternating group on n_i letters and $n_0 = 1$. The direct limit groups obtained from the above direct systems are of strictly diagonal type and denoted by $S(\xi)$ and $A(\xi)$, respectively. Observe that $S(\xi) \leqslant Sym(\mathbb{N})$.

In [2, Theorem 6.10] it is proved that the prime $p \in \xi$ repeats infinitely many times in ξ if and only if the gro $S(\xi)$ contains an isomorphic copy of the locally finite, divisible abelian group C_{p^∞}.

As the set of Steinitz numbers is uncountable, we have uncountably many pairwise non-isomorphic simple locally finite groups of the type $S(\xi)$.

Recall that the formal product $n = 2^{r_2}3^{r_3}5^{r_5}\dots$ of prime powers with $0 \leqslant r_p \leqslant \infty$ for all primes p is called a **Steinitz number** (supernatural number). If $\alpha = 2^{r_2}3^{r_3}5^{r_5}\dots$ and $\beta = 2^{s_2}3^{s_3}5^{s_5}\dots$ are two Steinitz numbers, then $\alpha|\beta$ if and only if $r_p \leqslant s_p$ for all primes p. The set of Steinitz numbers forms a partially ordered set with respect to the above division. Moreover they form a lattice if we define the meet and the join as $\alpha \wedge \beta = 2^{min\{r_2,s_2\}}3^{min\{r_3,s_3\}}5^{min\{r_5,s_5\}}\dots$ and $\alpha \vee \beta = 2^{max\{r_2,s_2\}}3^{max\{r_3,s_3\}}5^{max\{r_5,s_5\}}\dots$.

For each sequence ξ we define $Char(\xi) = p_1^{r_{p_1}} p_2^{r_{p_2}}\dots$ where r_{p_i} is the number of times that the prime p_i repeat in ξ. If it repeats infinitely often, then we write p_i^∞. Therefore for each $\xi \in \Pi$, there corresponds a Steinitz number $Char(\xi)$. For the group $S(\xi)$ obtained from the sequence ξ we define $Char(S(\xi)) = Char(\xi)$.

In [3] the class of groups $S(\xi)$ is classified by using the lattice isomorphism between the lattice of Steinitz numbers ordered with respect to division of two Steinitz numbers and the lattice of groups $S(\xi)$ ordered with respect to being a subgroup.

By a type of a permutation $\alpha \in S_n$ we mean a vector $t(\alpha) = (k_1, k_2, \dots, k_n)$ where k_i is the number of cycles of length i in the cycle decomposition of α. By a principal beginning α_0 of an element $\alpha \in S(\xi)$ we mean the element $\alpha_0 \in S_n$ where n is the smallest positive integer such that α_0 cannot be obtained by a homogeneous spreading in the sequence ξ.

In [3] the structure of the centralizers of elements in $S(\xi)$ is discussed and the following [3, Lemma 4.4] is stated. Let $\alpha \in S(\xi)$ with the principal beginning α_0. If $k_1, \dots, k_s$ are non-zero components of $t(\alpha_0)$, then

$$C_{S(\xi)}(\alpha) \cong \underset{j=1}{\overset{s}{Dr}} \, (C_{k_j} \bar{\wr} \, S(\xi))$$

here $\bar{\wr}$ denotes the restricted wreath product of the cyclic group C_{k_j} with $S(\xi)$.

But it is well known that the center of the restricted wreath product is the trivial group when the second group $S(\xi)$ has infinite order see [1, Exercise 6.2.3 p.48] and [4, Corollary 4.4]. Therefore such a restricted wreath product cannot be a centralizer of a non-trivial element, as the element α must be in the center.

Moreover the characteristic of the group $S(\xi)$ which appears on the right hand side as a component in the centralizer may change depending on the type of the element and the $Char(\xi)$. The cyclic group C_{k_j} should be C_j in [3, Lemma 4.4].

The corrected form of the centralizer of an element is the following.

Theorem 1. *Let ξ be an infinite sequence, $g \in S(\xi)$ and type of principal beginning $g_0 \in S_{n_k}$ be $t(g_0) = (r_1, r_2, \dots, r_{n_k})$. Then*

$$C_{S(\xi)}(g) \cong \underset{i=1}{\overset{n_k}{Dr}} \, C_i(C_i \bar{\wr} \, S(\xi_i))$$

where $Char(\xi_i) = \dfrac{Char(\xi)}{n_k} r_i$ *for* $i = 1, \dots n_k$. *If* $r_i = 0$, *then we assume that corresponding factor is* $\{1\}$.

For the construction of uncountable simple locally finite groups, let κ be an arbitrary infinite cardinal number. Our methods are similar to [3]. Let $FSym(\kappa)$ denote the finitary symmetric group and $Alt(\kappa)$ denote the alternating group on the set κ. As before, let Π be the set of sequences of prime numbers and $\xi \in \Pi$. Then ξ is a sequence of not necessarily distinct primes. Let $\alpha \in FSym(\kappa)$, ($Alt(\kappa)$). For a natural number $p \in \mathbb{N}$, a permutation $d^p(\alpha) \in FSym(\kappa p)$ defined by $(\kappa s+i)^{d^p(\alpha)} = \kappa s+i^{\alpha}$, $i \in \kappa$ and $0 \leqslant s \leqslant p-1$ is called **homogeneous p-spreading** of the permutation α. We divide the ordinal κp into p equal parts and on each part we repeat the permutation diagonally as in the finite case. So if $\alpha = \begin{pmatrix} 1 \dots n \\ i_1 \dots i_n \end{pmatrix} \in FSym(\kappa)$, then the homogeneous $p-$spreading of the permutation α is

$$d^p(\alpha) = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \dots & n & \kappa+1 & \dots & \kappa+n & \dots & \kappa(p-1)+1 & \dots & \kappa(p-1)+n \\ i_1 & \dots & i_n & \kappa+i_1 & \dots & \kappa+i_n & \dots & \kappa(p-1)+i_1 & \dots & \kappa(p-1)+i_n \end{array} \right)$$

with the obvious meaning that the elements in $\kappa p \setminus supp(d^p(\alpha))$ are fixed.

We continue to take the embeddings using homogeneous p-spreadings with respect to the given sequence of primes in ξ. From the given sequence of embeddings, we have direct systems and hence direct limit groups $FSym(\kappa)(\xi)$ and ($Alt(\kappa)(\xi)$) respectively. Observe that $FSym(\kappa)(\xi)$ and $Alt(\kappa)(\xi)$ are subgroups of $Sym(\kappa\omega)$ where ω is the first infinite ordinal.

Then we have the following Theorem.

Theorem 2. *Let $\xi \in \Pi$ be an infinite sequence.*

(i) $FSym(\kappa)(\xi) = Alt(\kappa)(\xi)$ if the prime 2 repeats infinitely often in ξ.

(ii) If the prime 2 repeats only finitely many times, then
$|FSym(\kappa)(\xi) : Alt(\kappa)(\xi)| = 2$

(iii) $Alt(\kappa)(\xi)$ is a simple locally finite group of cardinality κ.

(iv) The permutations $\alpha, \beta \in FSym(\kappa)(\xi)$ are conjugate in $FSym(\kappa)(\xi)$ if and only if the restrictions α' and β' of α and β are conjugate in $FSym(\kappa n_i)$ for some $i \in \mathbb{N}$.

The centralizers of elements in $FSym(\kappa)$ are known. Indeed if $\alpha \in FSym(\kappa)$ with $|supp(\alpha)| = n$, then α is contained in a subgroup of $FSym(\kappa)$ which is isomorphic to S_n. Assume that the type of α in this finite symmetric group is $t(\alpha) = (r_1, r_2, \dots, r_n)$. Then

$$C_{FSym(\kappa)}(\alpha) \cong (\mathop{Dr}_{i=1}^{n} (C_i \wr S_{r_i})) \times FSym(\kappa \setminus supp(\alpha))$$

If $r_i = 0$, then we assume that the corresponding factor in the direct product is $\{1\}$.

Now we state the structure of the centralizer of an arbitrary element in $FSym(\kappa)(\xi)$.

Theorem 3. *Let ξ be an infinite sequence. If $\alpha \in FSym(\kappa)(\xi)$ with principal beginning $\alpha_0 \in FSym(\kappa n_i)$, $t(\alpha_0) = (r_1, \dots, r_n)$, and $|supp(\alpha_0)| = n$. Then*

$$C_{FSym(\kappa)(\xi)}(\alpha) \cong (\mathop{Dr}_{i=1}^{n} C_i(C_i \wr S(\xi_i))) \times FSym(\kappa)(\xi')$$

where $Char(\xi_i) = \dfrac{Char(\xi)}{n_i} r_i$ and $Char(\xi') = \dfrac{Char(\xi)}{n_i}$. If $r_i = 0$, then we assume that the corresponding factor in the direct product is $\{1\}$.

Lemma 1. *The group C_{p^∞} is contained in $FSym(\kappa)(\xi)$ if and only if the prime p repeats infinitely often in ξ.*

Proof. In the construction of the groups $FSym(\kappa)(\xi)$ when we take the direct limit of finitary symmetric groups, we also take the direct limit of finite symmetric groups as we did in the finite case $S_{n_i} \to S_{n_{i+1}}$. So for any infinite ξ, isomorphic copy of the groups $S(\xi)$ are obtained as subgroups of $FSym(\kappa)(\xi)$. Therefore if the prime p repeats infinitely often in ξ, then by [2, Theorem 6.10] C_{p^∞} is a subgroup of $FSym(\kappa)(\xi)$.

On the other hand if the prime p repeats only finitely many times then one can imitate the proof of [2, Theorem 6.10] and show that C_{p^∞} is not a subgroup of $FSym(\kappa)(\xi)$. □

Hence by the above Lemma we may conclude that as the cardinality of the set of the Steinitz numbers is uncountable, the class of groups $\{Alt(\kappa)(\xi) \mid \xi \in \Pi\}$ forms uncountably many pairwise non-isomorphic simple, locally finite, non-linear groups of cardinality κ for any given infinite cardinal κ. Hence they form a proper class.

The groups $FSym(\kappa)(\xi)$ and $Alt(\kappa)(\xi)$ are different from $S(\xi)$ and $A(\xi)$ by the cardinality properties whenever κ is an uncountable cardinal. Observe also that $FSym(\kappa)(\xi)$ contains an isomorphic copy of $S(\xi)$ as a subgroup. Since $FSym(\omega)(\xi)$ and $S(\xi)$ act on different sets clearly $FSym(\omega)(\xi)$ is not a subgroup of $S(\xi)$.

We may define $Char(FSym(\kappa)(\xi)) = Char(\xi)$ as before.

The following theorem gives the characterization of the groups $FSym(\kappa)(\xi)$ in terms of the lattice of Steinitz numbers and the cardinality κ. Therefore for any given infinite cardinal κ, there exist uncountably many pairwise non-isomorphic locally finite simple non-linear groups.

Theorem 4. *Let κ be a fixed infinite cardinal. There is a lattice isomorphism between the lattice of groups $\Sigma = \{FSym(\kappa)(\xi) \mid \xi \in \Pi\}$ ordered with respect to being a subgroup and the lattice $\mathcal{S}$ of Steinitz numbers ordered with respect to division in Steinitz numbers.*

Proof. One may prove this along the lines of [3, Theorem 2]. □

Acknowledgements: The third author would like to express his sincere thanks to the colleagues and the staff of FRIAS for warm hospitality during his visit to Freiburg.

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Построение некоторых простых локально-конечных групп

Юлвие Б. Гювен
Отто Н. Кегель
Махмуд Кузучуоглу

В статье строится собственный класс простых локально конечных групп, а именно, для каждого бесконечного кардинального числа κ мы строим несчетное множество попарно неизоморфных простых локално конечных групп мощности κ, как индуктивный предел финитарных симметрических групп. Как указано в [2], кострукция групп такого типа бесконечной счетной мощности хорошо известна. В счетном случае они классифицированы Крошко-Сущанским [3] с помощью решетки чисел Стейница. Мы классифицируем их в несчетном случае по мощности группы и характеристике, соответствующей числу Стейница. Мы исследуем структуру централизаторов произвольных элементов в группах из этого нового класса и исправляем некоторые ошибки в параграфе о централизаторах элементов из $S(\xi)$ в [3].

Ключевые слова: простые локально-конечные группы, числа Стейница, построение.