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Beale, Steven

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by S.B. Beale

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CHOICE OF CONTROL FUNCTIONS FOR NUMERICAL GRID GENERATION

S.B. Beale

National Research Council
Ottawa, Ontario K1A 0R6

Abstract

This paper considers the use of source terms in grid generation equations. Exponential-type source terms for use in diffusion-source equations are commonly employed in grid generation. A form is presented which guarantees universal convergence and generates orthogonality at the boundary when the boundary points are congregated in the form of a geometric progression. For arbitrary boundary point distribution, a new method of introducing source-terms which ensure orthogonality at all boundaries is described and discussed.

INTRODUCTION

Early grid generation techniques were based on a diffusion formulation (Laplace equation). The term 'control function' is a commonly-used euphemism for functional-modifications to the basic Laplace system, in order to effect control either of the ϕ distribution or the slope of the ϕ distribution at the boundaries,

$$\nabla^2\phi + \ldots \text{other terms} = 0 \qquad (1)$$

In a previous paper [1] a grid generation methodology based on the solution to the general scalar transport equation,

$$\frac{\partial}{\partial t}(\rho\phi) + \vec{\nabla}\cdot(\rho\vec{u}\phi) = \vec{\nabla}\cdot\Gamma\vec{\nabla}\phi + S \qquad (2)$$

was formulated. The scheme used the finite-volume method [2,3,4] to solve for three scalars ϕ (or ϕ, ψ, θ) in a body-fitted co-ordinate grid. The grid was then re-meshed by moving the Cartesian displacement components, x, y, z until nodal values of ϕ , ψ and θ corresponded to desired reference-values of the contravariant displacement components ξ, η, and ζ in the fully-converged state. Conventional CFD codes may thus be applied to the grid generation process. In this paper some rationales are explored to effect grid control using 'control-functions' using the above mentioned re-mesh scheme. Much of the material presented is also relevant to conventional inverse schemes.

Types of boundary conditions

The term 'boundary condition' refers to the solved-for scalars ϕ (not the treatment of the grid at the perimeter). Boundary conditions are normally encoded, as fixed sources, S, or as linearized source-terms,

$$S = C(V - \phi) \qquad (3)$$

or a combination of the two. There may be any number of such sources. Three classes of boundary-value problem are common (i) Prescribed normal gradients or fixed-flux (Neumann problem), (ii) Fixed boundary values (Dirichlet problem). (iii) Mixed (Neumann/Dirichlet) boundary-value problems.

(i) *Neumann problem*. Normal gradients $S = \partial\phi/\partial n$ are prescribed at the boundaries. The case $S = 0$ indicates orthogonality, and is desirable at certain boundaries. The pure Neumann problem where $\partial\phi/\partial n$ is prescribed at all boundaries does not appear to find much use.

(ii) *Dirichlet problem:* ϕ-values are specified at all boundaries. Values must be fixed in a consistent fashion: For instance if ϕ were fixed to, say, 1 and $n\xi$ at all nodes along the west and east boundaries of the grid, respectively, it could be fixed to 2,3,4,…, $n\xi$-3, $n\xi$-2, $n\xi$-1 at each pair of opposing nodes along the north and south boundaries. In 3D, all values around a 'slab' must be fixed to the same consistent value.

(iii) *Mixed Dirichlet/Neumann problem*: The term 'mixed' boundary-condition is used in a specific context here; one which corresponds to the 'ideal' or 'natural' boundary value problem, and which will produce good results under most circumstances. As such, it will generate scalar functions ϕ which, (apart from a scale-factor) correspond to the velocity-potential and stream-function when solving a 2D

Laplace system. Two fixed-value Dirichlet boundary conditions are required in each co-ordinate direction, and additionally two Neumann boundary conditions are also required: The extension to 3D is obvious, there being two Dirichlet and four Neumann conditions for each scalar variable, rotated in a cyclical fashion.

Treatment of the grid at the perimeter

Treatment of the grid is as follows: No grid correction is required if the boundary grid-points are considered fixed. This is typical of all Dirichlet problems and some mixed boundary-value problems (see below). For zero-flux Neumann boundaries, the boundary grid points are allowed to slide along the boundary itself.

Ideally, control functions would always be used in conjunction with sliding boundaries points, corresponding to case (3) above. In this manner control lines/surfaces could readily be concentrated towards one or both Dirichlet boundaries, with orthogonality always preserved along the Neumann boundaries: Unfortunately, practical considerations often constrain the user to employ fixed boundary-point distributions on all sides of the grid. It is in the latter context that the remarks below should be considered.

The re-mesh grid generation scheme allows for grid control to be realised by adjusting the values of the reference $\Delta\xi$, however as in conventional inverse schemes let it be assumed that $\Delta\xi = 1$, below. Under these circumstances it can be safely assumed that $\phi = \xi$ and $\psi = \eta$ in the fully-converged state.

CONTROL FUNCTIONS TO CONTROL THE GRID DISTRIBUTION

In developing a control-function rationale, one may consider solutions to the ordinary differential equation,

$$\frac{d(\rho^* u^* \phi^*)}{dx^*} = \frac{d}{dx^*}\left(\Gamma^* \frac{d\phi^*}{dx^*}\right) - C^*\phi^* + S^* \quad (4)$$

where asterisks denote non-dimensional quantities, e.g.,

$$x^* = \frac{x - x_{\min}}{x_{\max} - x_{\min}} \quad (5)$$

$$\phi^* = \frac{\phi - \phi_{\min}}{\phi_{\max} - \phi_{\min}} \quad (6)$$

Control of ϕ may be effected by prescribing non-trivial values of Γ^*, ρ^*, u^*, C^*, and S^*, which themselves may be functions of ϕ^* or x^*. Entire mathematical handbooks have been written on classes of functions satisfying equations such as Eq. (4). Only simple cases such as diffusion-source equations will be considered below.

Criteria that ϕ should meet are (1) $0 \le \phi^* \le 1$ $\forall\ 0 \le x^* \le 1$ (2) $d\phi^*/dx^* \ge 0$ $\forall\ 0 \le x^* \le 1$. Also (3) the solution for ϕ should itself be grid independent.

Many authors introduce control-functions in the form of a diffusion source formulation [5,6],

Figure 1. Normalised-logarithmic function showing how cells may be concentrated at either end of the grid.

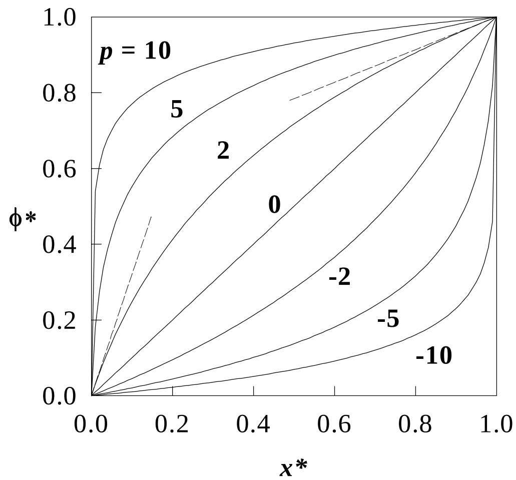
$$\frac{d^2\phi^*}{dx^{*2}} + S^* = 0 \quad (7)$$

It is easy to show that ϕ^* satisfies (1) and (2) only for $-1 \le S^* \le 1$; so the programmer should prevent the user from accidentally selecting values of S^* outside this range. As will be shown below, Eq. (7) is not a particularly good method of controlling grid-lines, at least for constant S^*. However, other functions are often coded as if they were source terms. For example, Thompson et al. [5] propose the use of exponential-type source terms having several forms reducing to one of two forms in 1D,

$$S^* = ae^{b\phi^*} \quad (8)$$

$$S^* = ae^{bx^*} \quad (9)$$

The two-parameters a and b control the distribution. Exponential functions can be designed to effectively control the grid, however it is easy to show that source terms may be generated which violate the criteria (1) and (2). Also use of the two parameters a and b appears to be somewhat ad-hoc.



A function such as the normalised-exponential function,

$$x^* = \frac{1 - e^{p\,\phi^*}}{1 - e^{p}} \qquad (10)$$

may readily be used to congregate a set of points along (0,1) in the form of a geometric progression, for constant $\Delta\phi^*$. Under these circumstances, the inverse, normalised-logarithmic function,

$$\phi^* = \frac{\log(1 + q x^*)}{\log(1 + q)} \qquad (11)$$

$$q = e^{p} - 1 \qquad (12)$$

may be employed to control ϕ. Figure 1 shows this function which satisfies criteria (1) and (2) over $-\infty < p < \infty$. Let it be assumed that it is desired to concentrate cells towards O and consider, say, the case p = 2. With reference to Fig. 1 (a) The ϕ^* function must be such that the tangent at O, $d\phi^*/dx^*$, is reasonably steep, in order to effectively concentrate the grid in this region. (b) The gradient of ϕ^* should be reasonably near to 45º at x^*=1, or at least substantially greater than 0º.

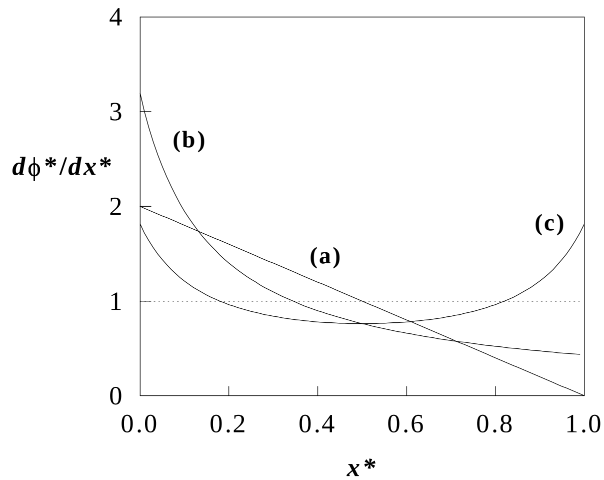


Figure 2. Three kinds of density function (a) Constant *S* (b) according to Eq. (14) (c) according to Eq (16).

The function ϕ^* is a cumulative distribution with associated density function, $d\phi^*/dx^*$. (The statistical analogy should be obvious to the reader.) Figure 2 shows $d\phi^*/dx^*$ vs. x^* for a number of functions. Figure 2(a) shows $d\phi^*/dx^*$ in the form of a straight line, corresponding to constant S, and illustrates why the slope cannot be made steeper than ±1. Figure 2(b) shows the case p=2 in Eq. (11). It can be seen that in order to meet requirements (a) and (b) above, the control-density function possesses the required property of skewness: The sharper the 'spike' at x^*=0 the more concentrated is the grid in this region, conversely at the other end x^*=1, as $d\phi^*/dx^* \to 1$, the grid will tend to become uninfluenced by the effects at x^*=0: A value $d\phi^*/dx^* \to 0$ implies the grid is being excessively pulled away at x^*=1. Thus it is seen that constant S line, Fig. 2(a) is not a particularly good control methodology due to the maximum skewness being restricted by the slope of the $d\phi^*/dx^*$ while the normalised logarithmic function appears to facilitate control of the grid lines in a desirable manner: (However simple and easy-to-code control functions can be constructed using two straight lines of differing slopes.)

The normalised-logarithmic function may be coded in any number of ways, for example as a diffusion-source equation, Eq. (7) with,

$$S^* = \frac{1}{\log(1+q)}\left(\frac{q}{1+qx^*}\right)^2 \qquad (13)$$

or in terms of ϕ^*.

$$S^* = \frac{\left(e^p - 1\right)^2}{p} e^{2p\phi^*} \qquad (14)$$

which is of course just the exponential function of Thompson et. al. [5] with $a = \left(e^p - 1\right)^2 / p$ and $b = 2p$. The forms Eq. (13) and Eq. (14) are recommended, since the criteria (1) and (2) are always ensured.

There is no requirement to encode the normalised logarithmic function as a diffusion-source equation, alternative forms exist e.g., using a diffusion formulation [5] with,

$$\Gamma^* \propto 1 + qx^* \qquad (15)$$

Convection-diffusion, and other formulations may also be constructed. The relative merits of each approach should be based on criteria such as ease of implementation, stability, and rapidity of convergence.

One of the merits of the form given in Eq. (14) is that to attract grid lines to the opposite boundary, the sign of p is simply reversed. Thus to attract grid-lines to both boundaries,

$$S^* = \frac{1}{2}\left(\frac{\left(e^p - 1\right)^2}{p} e^{2p\phi^*} - \frac{\left(e^{-p} - 1\right)^2}{p} e^{-2p\phi^*}\right) (16)$$

Fig. 2(c) shows that this function exhibits the property of kurtosis.

Figure 3 shows a C-grid around an aircraft. The radial ϕ contours (ξ lines) were generated algebraically using trans-finite interpolation, while the circumferential ψ contours (η lines), around the aircraft, were obtained using the method described above. (A feature of the re-mesh scheme is that each scalar variable may be solved-for independently, i.e. in de-coupled fashion.) In Fig. 3(a) no control functions are employed, i.e. the grid corresponds to a Laplace-solution for ψ (only). It can be seen that the ψ-lines cannot readily be concentrated around the aircraft fuselage. By introducing a source term according to Eq. (14) with $p = 2$, Fig. 3(b), it can be seen that the grid may readily be concentrated around the aircraft. Because a fixed (i.e. not sliding) boundary point distribution was specified along AB and AC, it can be seen that there is substantial distortion of the grid in this region. When the fixed boundary point distribution is specified according to Eq. (10) with $p = 2$, Fig. 3(c), not only is the grid concentrated around the aircraft shape, but also the grid is orthogonal at the boundary, as desired.

<u>Calculation of the control functions in the interior of the domain</u>

The example given above is relatively simple, with the same boundary distribution at both ends (AB and AC), so that a single distribution was presumed throughout. More generally, the physical dimensions of the boundaries could vary preventing the establishment of a unique length scale $L = \left|\vec{r}_{\max} - \vec{r}_{\min}\right|$ needed to calculate S from S^*. It is also possible that different boundary distribution be prescribed at opposing boundaries, though this is not recommended.

Under the above circumstances, interior source terms require to be computed from boundary values, using a weighting-function, w. Suppose for example that ψ was fixed to 1 and $n\eta$ at the north and south boundaries. Interior S values would be computed from sources at the the east and west boundaries,

$$\left(\sqrt{g}S\right)_P = (1-w)\left(\sqrt{g}S\right)_{WEST} + w\left(\sqrt{g}S\right)_{EAST} (17)$$

where S_{WEST} and S_{EAST} are obtained using Eq. (14) or other means. The Jacobian $\sqrt{g}$ arises when integrating the finite-volume equations. A reasonable choice for w is just,

$$w = \phi^* = \frac{\phi - \phi_{WEST}}{\phi_{EAST} - \phi_{WEST}} \qquad (18)$$

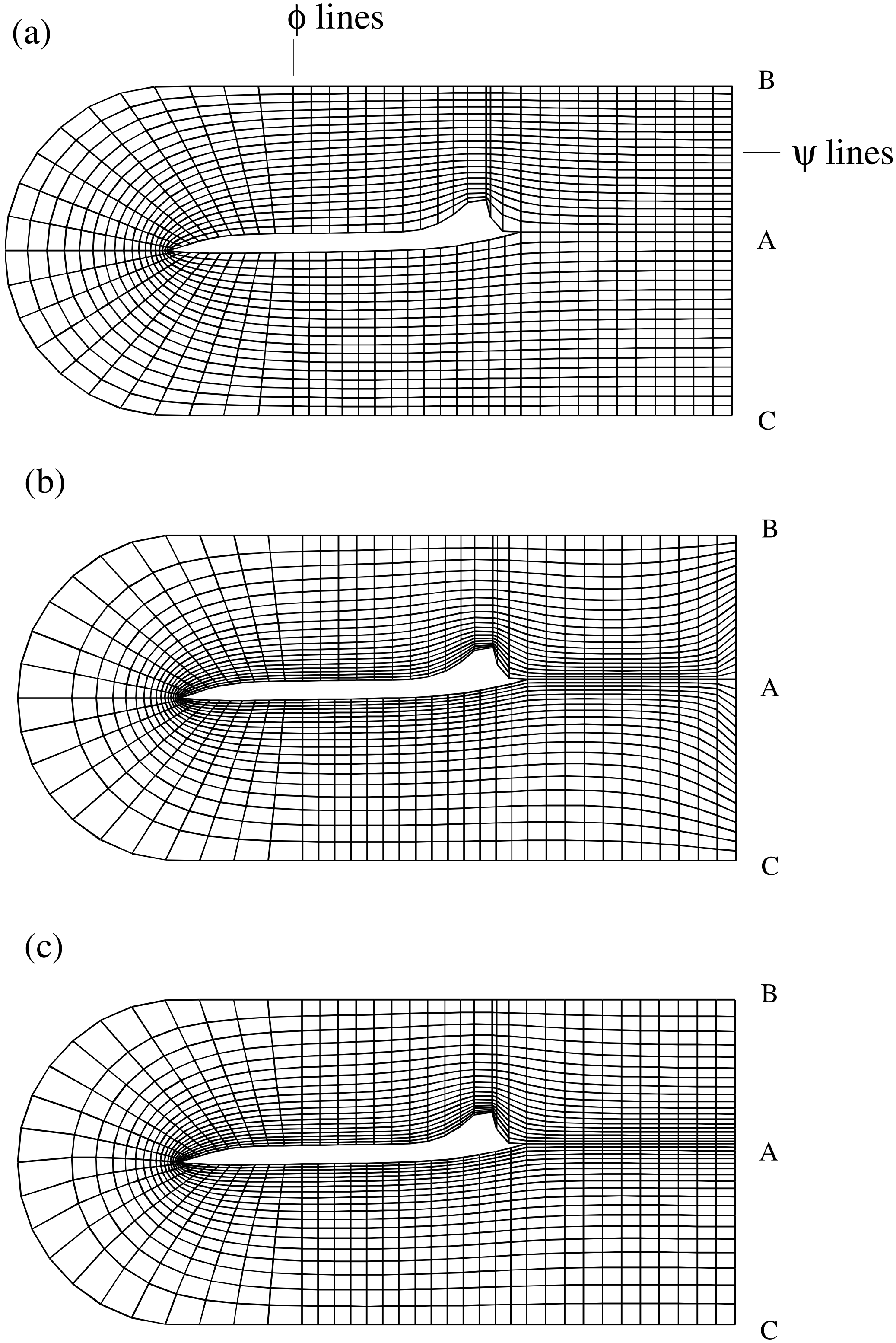


Figure 3. Control function used to control grid distribution. (a) Laplace grid (b) Diffusion-source formulation according to Eq. (14) with *p*=2 (c) Diffusion-source formulation with boundary point distribution along AB and AC prescribed according to Eq. (10) with *p*=2.

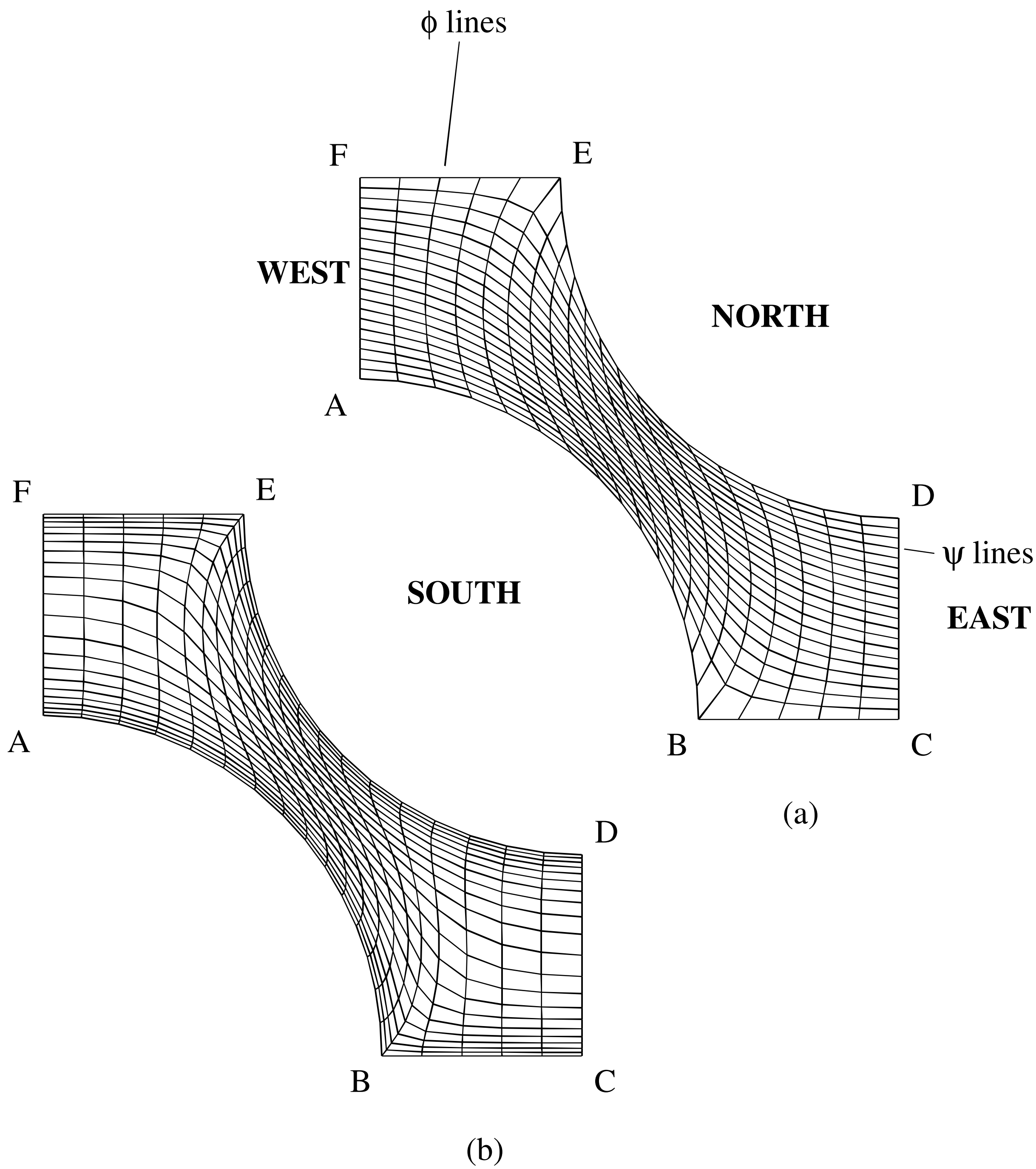


Figure 4. Control functions used to procure orthogonality. (a) Dirichlet problem with no source terms (b) Mixed (Dirchlet/Neumann) boundary problem with source terms prescribed according to Eq. (19).

where $\phi^* = \xi^*$ in the fully-converged state. It is worth noting that the 2D distribution is now a function of changes in ψ. Thus if the ψ distribution changed, the ϕ distribution would also change (a little), violating the criterion of grid independence (3) above. The effect is probably minor, since it is the weighting function, w, not the end-values of S, which are affected. Nonetheless it is better if a single distribution be prescribed for the whole domain.

When sliding boundaries are prescribed, it is possible to control the ϕ and ψ distributions, and always ensure orthogonality at the boundaries. Often it is necessary to presume a fixed boundary point distribution. For simple geometries, it is still possible to control the form of both the distribution and to generate orthogonality, if the control-functions and the prescribed boundary point distribution are chosen in a consistent fashion. For more complex boundary distributions, some sort of 'automatic' procedure to generate orthogonality is frequently sought.

CONTROL FUNCTIONS TO PROCURE ORTHOGONALITY

For problems involving fixed grid-points around arbitrary curved surfaces, control-functions are usually prescribed that will in some fashion procure orthogonality at the boundaries. Previous authors [7,8] have treated the problem as a Dirichlet problem, and impose explicit geometric constraints in the inverse forms. Note however that the Dirichlet problem and the mixed boundary value problem are the same, the problem being one of finding the control terms that satisfies both the requirement of orthogonality (zero-flux), and that of fixed values ϕ□=□1,2,3,... $n\xi$ along the boundary.

The author treats the case as a mixed boundary value problem, i.e., Neumann at the appropriate boundaries: With, ϕ fixed to, say, 1 and $n\xi$ at the west and east boundaries, it is desired not only that there be zero-flux of ϕ along the south and north boundaries, but also that the nodal ϕ values converge on values 1,2,3,…$n\xi$. Assuming a diffusion-source formulation, Eq. (7), the goal is to find S such that the nodal values of ϕ along the north and south (Neumann) boundaries converge on these reference values.

The solution is simple. Let S^0 be a guessed value for the source term and S' be a source-term correction factor. At the end of each 'sweep' in the solution procedure, the source term is re-computed as,

$$\sqrt{g}S = \sqrt{g}\left(S^0 + S'\right) \qquad (19)$$

The current value of S being taken as S^0 for the next cycle etc. The correction factor, S', is obtained using,

$$\sqrt{g}S' = C\left(\xi - \phi_P\right) \qquad (20)$$

where ϕ_P is the current nodal value and ξ = 1,2,3,..$n\xi$ are the desired reference values. This ensures convergence on the reference value, while at the same time ensuring that a zero-flux boundary condition is imposed. The reader will note that although ϕ is solved at the north and south boundaries, no grid correction is applied here, i.e. the boundary point distribution is fixed to the initial locations. The source terms, S, are evaluated along both the south and north boundaries, interior values being interpolated as described above. The coefficient C is fixed according to,

$$C = \frac{\partial S}{\partial \phi} \approx a_E + a_W \qquad (21)$$

where a_E and a_W are just the linking coefficients in the finite-volume equations for ϕ,

$$a_E = \left(\sqrt{g}g^{11}\right)_e \qquad (22)$$

i.e. the diagonal (orthogonal) diffusion terms. Figure 4 shows a grid in a tube bank with small pitch-to-diameter ratio. Figure 4(a) shows the Laplace solution (no source terms). Values of ϕ were fixed to 1,2,3,…$n\xi$ along the south (ABC) and north (DEF) boundaries, using a large coefficient in Eq. (3). Because the initial grid locations at the north and south boundaries are incompatible with the 'natural' solution to the mixed boundary value problem, there is a high degree of non-orthogonality both in the interior of the grid and at the boundaries.

Figure 4(b) shows that by introducing source terms according to Eq. (20), it is possible to procure orthogonality at the boundaries, in spite of the high degree of shear in the body of the grid itself. As discussed above, nodal values of ϕ along the south and north boundaries (ABC and DEF) were not fixed using a large coefficient, but were solved-for using the finite-volume procedure. The boundary points themselves were however fixed. A similar procedure allowed the ψ distribution to be concentrated along the east (AF) and east (CD) boundaries, and at the same time automatically ensure orthogonality.

The simple iterative procedure described above was found to converge without difficulty. While easy to implement in a conventional finite-volume approach, (combined with a grid correction procedure), it is not immediately apparent how it could be readily utilised in an inverse-method grid-generation program. Most readers should be familiar with the diffusion-source formulation presented above. It is worth noting that, the same end was achieved using a convection-diffusion formulation; *u*-values being obtained as, $u = u^0 + u'$ where $u = C(\phi_P - \xi)$. At the time of writing, there does not appear to be any inherent advantage of one approach over another. The matter is the subject for research.

CONCLUSION

A number of methods of introducing control functions into finite-volume equations governing grid generation were described. Attention was focused on the prescription of source-terms in diffusion-source equations, with prescribed (fixed) boundary point locations. It was shown that by prescribing the source terms according to the inverse function used to generate the boundary point distribution good quality grids could be generated. The example of a normalised-logarithmic distribution was shown to meet the criteria for ensuring convergence. It was shown that this could be implemented in the form of a source term containing an inverse square function of x, or equivalently with an exponential function of ϕ. Many other types of control functions are possible and should be considered.

For the case of arbitrary point distribution along curved lines, an automatic procedure based on an iterative solution to the mixed Dirichlet/Neumann problem with fixed boundary points (i.e. no grid correction) was described. The procedure was found to converge without difficulty.

While there has been much attention in the literature about the need for orthogonality along mesh boundaries, the reader should appreciate that no control function can correct the presence of significant non-orthogonality in the interior of the domain, when inappropriate choices are made for fixed boundary-point locations. Such non-orthogonalities may impede convergence for even the best flow solver. Under these circumstances, the user should consider the use of sliding boundaries, or manually re-adjust the fixed boundary points where possible, in order to remedy the problem.

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