Теорема 2. *Если* $y_1(x)$ *— решение уравнения*

$$y'' + f(x)y' + Ke^y + F(x) = 0,$$

то уравнение

$$y'' + f(x)y' + K\mu e^y + F(x) = 0$$

имеет решение $y = y_1(x) - \ln\mu$.

В работе [3] доказана

Теорема 3. *Дифференциальное уравнение*

$$y_{xx} + f(x,z)y_x + \Phi(y,z) + F(x,z) = 0 \tag{4}$$

допускает группу непрерывных по параметру преобразований тогда и только тогда, когда одновременно выполняются соотношения

$$A^2 f_z + f_x A^1 y + f_x B^1 + f^2 A^1 y + f B^1_x + 2A^3_x - f_x A^1 y - f^2 A^1 y - B^1_{xx} + 3A^1 f + 3A^1 \Phi = 0$$

и

$$(A^1 y + B^1)F_x + A^1(F_z + \Phi_z) + \Phi_y(A^3 y + B^3) + f(A^3_x y + B^3_x) + A^3_{xx} y +$$
$$+ B^3_{xx} - A^3(F + \Phi) + 2(A^1_x y + B^1_x)(F + \Phi) = 0.$$

При этом можно найти функции ξ^1, ξ^2 и η, обеспечивающие существование такого преобразования переменных, относительно которого будет инвариантно дифференциальное уравнение (4).

Литература

1. Ибрагимов Н. Х. *Практический курс дифференциальных уравнений и математического моделирования.* Н. Новгород: Изд-во Нижегородского ун-та им. Н. И. Лобачевского, 2007.

2. Самодуров А. А., Федорако Е. И. *О связи между решениями двух нелинейных дифференциальных уравнений второго порядка* // Докл. БГУИР. 2011. №8(62). С. 5–8.

3. Федорако Е. И. *Непрерывные преобразования дифференциальных уравнений второго порядка с заданной нелинейностью* // Весцi Беларускага дзяржаўнага педагагiчнага унiверсiтэта. 2014. Сер. 3. С. 14–20.

ON THE STABILITY OF GRADIENT-LIKE SYSTEMS

D.N. Cheban[1], **B.S. Kalitine**[2]

[1] State University of Moldova, Moldova
davidcheban@yahoo.com
[2] Belarusian State University, Belarus
Kalitine@yandex.by

This report is dedicated to the study the problem of stability of some classes of gradient-like system of differential equations (both autonomous and non-autonomous cases). We present two main results. The first is a generalization of Absil & Kurdyka theorem [1] about stability of gradient systems with analytic potential for non-gradient systems. Secondly we generalize for some classes of gradient-like non-autonomous systems the well-known Lagrange — Dirichet theorem (see [2]).

Let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{R}^n$ be the real n-dimensional Euclidean space. Consider a system of differential equations

$$\dot{x} = f(x), \quad (x \in U \subset \mathbb{R}^n), \tag{1}$$

where U is an open set containing the origin and $f(0) = 0$. Suppose that the function $f \in \in C(U, \mathbb{R}^n)$ and satisfies the conditions that ensure the uniqueness of solutions in U.

Let $V \in C^1(\mathbb{R}^n, \mathbb{R})$ and we consider the gradient-system

$$\dot{x} = -\nabla V(x), \tag{2}$$

where $\nabla V(x)$ denotes the gradient of V at $x \in \mathbb{R}^n$. We assume that $V(0) = 0$ and $\nabla V(0) = 0$.

Let $\mathfrak{E}$ be the set of all continuous functions $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ possessing the following properties:

1) $\phi(0) = 0$;

2) $\phi(u) > 0$ for all $u > 0$;

3) ϕ is continuously differentiable on $(0, +\infty)$;

4) $\phi'(u) > 0$ for all $u > 0$, where ϕ' is the derivative of ϕ.

Remark 1. Not that the function $\phi(u) := cu^\alpha/\alpha$ $(c, \alpha > 0)$ belongs to $\mathfrak{E}$. In particular, if $\alpha = 1$, then $\phi(u) = cu$ belongs to $\mathfrak{E}$. The function $\phi(u) = \ln(1 + u)$ also belongs to $\mathfrak{E}$ and so on.

Theorem 1. *Suppose that for* (1) *there exists a neighborhood* U *of* $x = 0$ *and continuously differentiable function* $V : U \to \mathbb{R}_+$ *such that*

1) $V(0) = 0$;

2) *there exists a function* $\phi \in \mathfrak{E}$ *such that*

$$\left.\frac{d\phi(V(x))}{dt}\right|_{(1)} \leqslant -||f(x)||, \quad \forall x \in U.$$

Then the trivial solution of (1) *is stable.*

Remark 2. If there exist a neighborhood U of $x = 0$ and continuously differentiable function $V : U \to \mathbb{R}_+$, $c > 0$ and $\rho \in (0, 1)$ such that $V(0) = 0$, $V(x) \geqslant 0$ for all $x \in U$ and $dV(x)/dt|_{(1)} \leqslant -c||f(x)||V(x)^\rho$, then condition 2) of Theorem 1 holds. In fact. It easy to see that if we take $\phi(u) := c^{-1}(1 - \rho)^{-1}u^{1-\rho}$, then $\phi \in \mathfrak{E}$ and $d\phi(V(x))/dt|_{(1)} \leqslant -||f(x)||$ for all $x \in U$.

Example 1. Consider a gradient system of differential equations (2). Suppose that the potential V of gradient system (2) has a local minimum at the origin. We take V as a Lyapunov function, then $V(x) \geqslant 0$ and $V(0) = 0$ and all the conditions of Theorem 1 are fulfilled. This means in particular that if a function $V(x)$ is analytic in a neighborhood of the origin and has a local minimum at this point, then the trivial solution of the gradient system is stable. In the work [1] it was proved that for real analytic system this statement is reversible.

An m-dimensional torus is denoted by $\mathcal{T}^m := \mathbb{R}^m/2\pi\mathbb{Z}^m$. Let $(\mathcal{T}^m, \mathbb{T}, \sigma)$ be an irrational winding of $\mathcal{T}^m$ with the frequency $\nu := (\nu_1, \nu_2, \dots, \nu_m)$, i. e., $\sigma(t, \theta) := (\nu_1 t + \theta_1, \nu_2 t + \theta_2, \dots \dots, \nu_m t + \theta_m)$ for all $t \in \mathbb{R}$ and $\theta := (\theta_1, \theta_2, \dots, \theta_m) \in \mathcal{T}^m$, where $\nu_1, \nu_2, \dots, \nu_m$ are some irrationally independent real numbers.

A point $x \in X$ (respectively, a function $\varphi \in C(\mathbb{R}; \mathbb{R}^n)$) is called quasi-periodic with the frequency $\nu := (\nu_1, \nu_2, \dots, \nu_m) \in \mathcal{T}^m$, if there exists a continuous function $\Phi : \mathcal{T}^m \to X$ (respectively, $\Phi : \mathcal{T}^m \mapsto \mathbb{R}^m$) and a point $\theta_0 \in \mathcal{T}^m$ such that $\pi(t, x) := \Phi(\sigma(t, \theta_0))$ (respectively, $\varphi(t) = \Phi(\sigma(t, \theta_0))$) for all $t \in \mathbb{R}$, where $(\mathcal{T}^m, \mathbb{T}, \sigma)$ is an irrational winding of the torus $\mathcal{T}^m$.

Let $F \in C^1(\mathbb{R}^n \times \mathcal{T}^m, \mathbb{R})$ and $\nabla F(0, \theta) = 0$ for all $\theta \in \mathcal{T}$. Below we will study the problem of stability of trivial solution for system

$$\begin{cases} x'' + \nabla_x F(\theta, x) = 0 & (x \in \mathbb{R}^n), \\ \theta' = \Phi(\theta) & (\theta \in \mathcal{T}^m), \end{cases} \tag{3}$$

where $\Phi \in C^(\mathcal{T}^m, \mathbb{R}^m)$.

Everywhere below we will suppose that the functions F and Φ are regular, i.e., for every $(x_0, x_0', \theta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$ system (3) admits a unique solution $(\varphi(t, x_0, x_0', \theta), \varphi'(t, x_0, x_0', \theta), \sigma(t, \theta))$

defined on $\mathbb{R}_+$. This means that system (3) generates a semi-group dynamical system $(X, \mathbb{R}_+, \pi)$ on the space $X := \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$, where $(\mathcal{T}^m, \mathbb{R}, \sigma)$ is a dynamical system associated by equation

$$\theta' = \Phi(\theta), \tag{4}$$

$(\varphi(t, x_0, x_0', \theta), \varphi'(t, x_0, x_0', \theta))$ is a unique solution of equation

$$x'' + \nabla F(\sigma(t, \theta), x) = 0 \quad (\theta \in \mathcal{T}^m) \tag{5}$$

passing through the point (x_0, x_0') at the initial moment $t = 0$,

$$\pi(t, (x_0, x_0', \theta)) := (\varphi(t, x_0, x_0', \theta), \varphi'(t, x_0, x_0', \theta))$$

for all $(t, x_0, x_0', \theta) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$.

Remark 3. 1. By arguments above autonomous system (3) and non-autonomous equation (5) (in fact the family of non-autonomous equations depending on parameter $\theta \in \mathcal{T}^m$) are equivalent.

2. If equation (5) admits a trivial solution, then the set $\{(0, 0)\} \times \mathcal{T}^m \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{T}^m$ is an invariant subset (invariant torus) of system (3).

Definition. Recall that the trivial solution of equation (5) (or equivalently, the invariant torus of system (3)) is said to be uniformly (with respect to $\theta \in \mathcal{T}^m$) Lyapunov stable, if for arbitrary $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $||x_0||^2 + ||x_0'||^2 < \delta^2$ implies $||\varphi(t, x_0, x_0', \theta)||^2 + ||\varphi'(t, x_0, x_0', \theta)||^2 < \varepsilon^2$ for all $t \in \mathbb{R}_+$ and $\theta \in \mathcal{T}^m$.

Denote by $\mathcal{K}$ the set of all continuous functions $a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ possessing the following properties: $a(0) = 0$; a is monotonically strictly increasing.

Theorem 2. *Suppose that following conditions hold:*

1) $F(0, \theta) = 0$ *and* $\nabla_x F(0, \theta) = 0$ *for all* $\theta \in \mathcal{T}^m$;

2) *there exists a function* $a \in \mathcal{K}$ *such that* $F(x, \theta) \geqslant a(||x||)$ *for all* $x \in \mathbb{R}^n$ *and* $\theta \in \mathcal{T}^m$;

3) $\langle \nabla_\theta F(x, \theta), \Phi(\theta) \rangle \leqslant 0$ *for all* $(x, \theta) \in \mathbb{R}^n \times \mathcal{T}^m$.

Then the trivial solution of equation (5) *is uniformly Lyapunov stable.*

References

1. Absil P.-A., Kurdyka K. *On the stable equilibrium points of gradient systems* // Systems & Control Letters. 2006. V. 55. P. 573–577.

2. Rouch N., Habets P., Laloy M. *Stability Theory by Liapunov s Direct Metod.* Berlin: Springer, 1977.

ON POSITIVE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR SINGULAR IN PHASE VARIABLES NONLINEAR DIFFERENTIAL SYSTEMS

I. Kiguradze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
kig@rmi.ge

In the finite interval $[a, b]$ we consider the problem

$$\frac{du_i}{dt} = f_i(t, u_1, \ldots, u_n) \quad (i = 1, \ldots, n), \tag{1}$$

$$u_i(a) = \varphi_i(u_i(b)) \quad (i = 1, \ldots, n), \tag{2}$$