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On a Physical Relativization of the Strong Interaction within the Relativistic Quasipotential Approach

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New relativistic Coulomb-like threshold resummation S- and L-factors in quantum chromodynamics are obtained. Consideration is performed within the framework of quasipotential approach in quantum field theory formulated in the relativistic configurational representation in the case of two particles of unequal masses.

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1. Introduction

In two-particle approximation the square of the Bethe–Salpeter (BS) amplitude of two charged particles $\chi_{\rm BS}(x)$ at $x = (\mathbf{r}, \tau) = 0$ and hence at the relative time $\tau = 0$ is an important quantity for $q\bar{q}$ systems. For example, it appears in the expressions for the Drell ratio $R(s)$ [1], for leptonic widths $\Gamma(e^+e^-)$ for 1^- states [2–8]. By considering the total cross section for the production of fermion-antifermion (or quark-antiquark) pairs in e^+e^- annihilation in the kinematic region close to the threshold, we can not cut off the perturbative series in powers of the fine structure constant α (i.e., in the number of loops), even if the expansion parameter α is small [9]. The problem is well-known from QED [10]. This can be seen by considering the contributions of magnetic and electric form factors to the total cross section for the production of fermion-antifermion (or quark-antiquark) pairs in e^+e^- annihilation in the kinematic region close to the threshold $s = 4m^2$ [11]. The reason consists in that the real expansion parameter in the threshold region is α/v where $v = \sqrt{1 - 4m^2/s}$ is a quark velocity, and m is a quark mass. Obviously, it becomes to be singular, when the velocity $v \to 0$. To obtain meaningful result these threshold singularities of the form $(\alpha/v)^n$ have to be summarized. In the nonrelativistic case for the Coulomb interaction

$$V(r) = -\alpha/r \tag{1}$$

this resummation is realized by the known S-factor Gamov–Sommerfeld–Sakharov [12–14]

$$S_{\rm nr} = \frac{X_{\rm nr}}{1 - \exp(-X_{\rm nr})}, \quad X_{\rm nr} = \frac{\pi\alpha}{v_{\rm nr}}. \tag{2}$$

Here $2v_{\rm nr}$ is the relative velocity of two nonrelativistic particles. In the relativistic theory the nonrelativistic approximation needs to be modified. For the first time the relativistic modification of the S-factor (2) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was performed in [15] (see also [16]) and it consisted in the change $v_{\rm nr} \to v$. Just the same form of the S-factor for the interaction of two particles of equal masses was later suggested in [11]. Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in [17]. The relativistic S-factor for two particles of arbitrary masses ($m_1 \neq m_2$) was presented in [18]. The new method to relativistic generalization of the S-factor in the case of two particles of equal masses was developed in [19]. Their method is based on the relativistic quasipotential (RQP) approach [20] in the form suggested in [21]. In the method developed by them, the possibility of transformation

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of quasipotential (QP) equation from the momentum space into relativistic configurational representation in the case of two particles of equal masses [22] has also been used. Moreover, it is important that the potential (1) they used possesses the QCD-like behaviour [23]. This approach gives the following expression for the relativistic S-factor:

$$S(\chi) = \frac{X(\chi)}{1 - \exp\left[-X(\chi)\right]} \tag{3}$$

where

$$X(\chi) = \frac{\pi\alpha}{\sinh\chi},$$

and χ is the rapidity related to the total c. m. energy of interacting particles, $\sqrt{s}$, by $2m\cosh\chi = \sqrt{s}$. The function $X(\chi)$ in Eq. (3) can be expressed in terms of v as $X(\chi) = \pi\alpha\sqrt{1-v^2}/v$. The method proposed by them in [19] has been generalized in [24] successfully to get the following expression for the relativistic L-factor ($\ell \geq 1$) in the case of $m_1 = m_2 = m$:

$$L(\chi) = \prod_{n=1}^{\ell}\left[1 + \left(\frac{\alpha}{2n\sinh\chi}\right)^2\right] S(\chi). \tag{4}$$

We should like to remind that the QP wave functions in the momentum space, $\Psi_q(\mathbf{p})$, and in the relativistic configurational representation, $\psi_q(\boldsymbol{\rho})$, are defined by the relation

$$\chi_{\mathrm{BS}}(x=0) = \frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{p}}\Psi_q(\mathbf{p}) = \psi_q(\boldsymbol{\rho})|_{\rho=i}$$

where $d\Omega_{\mathbf{p}} = (md\mathbf{p})/E_p$ is the relativistic three-dimensional volume element in Lobachevsky space realized on the hyperboloid $E_p^2 - \mathbf{p}^2 = m^2$.

Our aim here is to generalize the method proposed in [19] to obtain the relativistic L-factor ($\ell \geq 0$) in the case of two particles of arbitrary masses m_1 and m_2. Consideration is performed on the basis of covariant Hamiltonian formulation of quantum field theory [21, 25] by transition to the three-dimensional relativistic configurational representation for the case of interaction of two relativistic spinless particles having arbitrary masses m_1, m_2 [26, 27].

2. Relativistic threshold resummation S- and L-factors

The basis of our consideration is the integral form of the relativistic Schrödinger equation in the configurational representation with the quasipotential $V(r; E_{q'})$ (we use the system of units $c = \hbar = 1$)

$$\frac{1}{(2\pi)^3}\int d\Omega_{\mathbf{p}'}\left(2E_{q'} - 2E_{p'}\right)\xi(\mathbf{p}',\mathbf{r}) \tag{5}$$
$$\times\int d\mathbf{r}'\xi^*(\mathbf{p}',\mathbf{r}')\psi_{q'}(\mathbf{r}') = \frac{2\mu}{m'}V(r;E_{q'})\psi_{q'}(\mathbf{r})$$

where $\mu = m_1m_2/(m_1+m_2)$ is the usual reduced mass, and $d\Omega_{\mathbf{p}'} = m'd\mathbf{p}'/E_{p'}$ is the relativistic three-dimensional volume element in Lobachevsky space, which is realized on the upper half of the mass hyperboloid $E_{p'}^2 - \mathbf{p}'^2 = m'^2$ of an effective relativistic particle having the mass $m' = \sqrt{m_1 m_2}$ and relative 3-momentum $\mathbf{p}'$, emerging instead of the system of two particles and carrying the total c. m. energy of the interacting particles, $\sqrt{s}$, is proportional to the energy $E_{p'}$ of one effective relativistic particle of mass m' [26, 27]: $\sqrt{s} = \sqrt{{m_1}^2 + \mathbf{p}'^2} + \sqrt{{m_2}^2 + \mathbf{p}'^2} = (m'/\mu)E_{p'}$. The proper Lorentz transformations means a translation in Lobachevsky space. The role of the plane waves corresponding to these translations are played by the functions

$$\xi(\mathbf{p}',\mathbf{r}) = \left(\frac{E_{p'} - \mathbf{p}'\cdot\mathbf{n}}{m'}\right)^{-1-irm'}. \tag{6}$$

By using the expansions

$$\xi(\mathbf{p},\boldsymbol{\rho}) = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}p_{\ell}(\rho,\cosh\chi_p)P_{\ell}\left(\frac{\mathbf{p}\cdot\boldsymbol{\rho}}{p\rho}\right),$$
$$\psi_q(\boldsymbol{\rho}) = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}\frac{\varphi_{\ell}(\rho,\chi_q)}{\rho}P_{\ell}\left(\frac{\mathbf{q}\cdot\boldsymbol{\rho}}{q\rho}\right),$$

and also formula [22]

$$p_\ell(\rho, \cosh\chi) = \frac{(-1)^\ell (\sinh\chi)^\ell}{\rho^{(\ell+1)}}$$
$$\times \left(\frac{d}{d\cosh\chi}\right)^\ell \left(\frac{\sin\rho\chi}{\sinh\chi}\right), (-\rho)^{(\ell)} = i^\ell \frac{\Gamma(\ell + i\rho)}{\Gamma(i\rho)},$$

Eq. (5) can be transformed to the form

$$\frac{2}{\pi}\int\limits_0^\infty d\chi' \frac{(\sinh\chi')^{2\ell+2}(-1)^{\ell+1}}{\rho^{(\ell+1)}} \left(2\cosh\chi - 2\cosh\chi'\right) \left(\frac{d}{d\cosh\chi'}\right)^\ell \left(\frac{\sin\rho\chi'}{\sinh\chi'}\right)$$
$$\times \left(\frac{d}{d\cosh\chi'}\right)^\ell \frac{1}{\sinh\chi'} \int\limits_0^\infty d\rho' \frac{\rho'\sin\rho'\chi'}{(-\rho')^{(\ell+1)}} \varphi_\ell(\rho', \chi) = \frac{V(\rho; E_q)\varphi_\ell(\rho, \chi)}{\rho}$$

where we introduced the notation:

$$\mathbf{p}' = m'\mathbf{p}, \mathbf{p} = \sinh(\chi')\mathbf{n}_p, |\mathbf{n}_p| = 1, \boldsymbol{\rho}' = m'\mathbf{r}',$$
$$\rho' = |\boldsymbol{\rho}'|, r' = |\mathbf{r}'|, d\mathbf{r}' = m'^{-3} d\boldsymbol{\rho}',$$
$$d\Omega_{\mathbf{p}'} = m'^3 d\Omega_{\mathbf{p}}, d\Omega_{\mathbf{p}} = \frac{d\mathbf{p}}{E_p}, E_{p'} = m'E_p,$$
$$E_p = \sqrt{1 + \mathbf{p}^2} = \cosh\chi', V(r; E_{q'}) = m'V(\rho; E_q),$$
$$E_{q'} = m'E_q, E_q = \sqrt{1 + \mathbf{q}^2} = \cosh\chi,$$
$$\xi(\mathbf{p}, \boldsymbol{\rho}) = (E_p - \mathbf{p}\cdot\mathbf{n})^{-1-i\rho}, \psi_{q'}(\mathbf{r}) = \psi_q(\boldsymbol{\rho}).$$

We will seek for a solution of RQP equation (2) with the potential (1) in the form

$$\varphi_\ell(\rho, \chi) = \frac{(-\rho)^{(\ell+1)}}{\rho} \int\limits_{\alpha_-}^{\alpha_+} d\zeta e^{i\rho\zeta} R_\ell(\zeta, \chi) \qquad (7)$$

where ζ-integration is performed in the complex plane over a contour with end points $\alpha_\pm = -R \pm i\varepsilon, R \to +\infty, \varepsilon \to +0$ (see Fig. 1). The resulting solution Eq. (2) at arbitrary $\ell \geq 0$ can be represented in terms of the hypergeometrical function as

$$\varphi_\ell(\rho, \chi) = N_\ell(\chi)(-\rho)^{(\ell+1)} e^{i\rho\chi + iA\chi + i\pi(\ell+1)} \qquad (8)$$
$$\times F\left(\ell + 1 - iA, \ell + 1 - i\rho; 2\ell + 2; 1 - e^{-2\chi}\right)$$

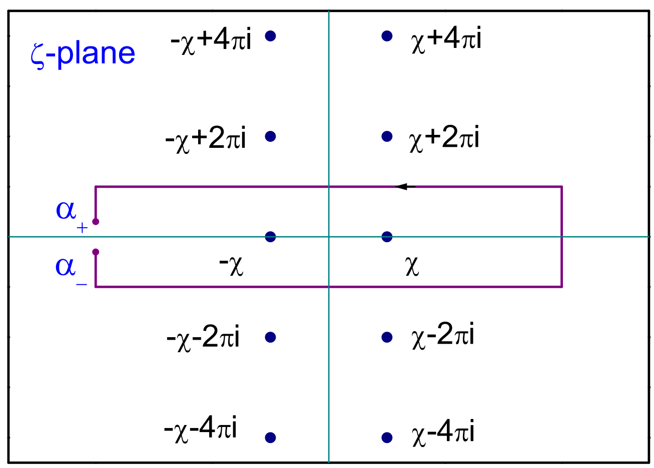

FIG. 1. Contour of integration in Eq. (7) and singularities of the function $R_\ell(\zeta, \chi)$in the complex ζ-plane.

where $A = \alpha\mu/(m'\sinh\chi)$ and constant of normalization $N_\ell(\chi)$ is found from the condition

$$\lim_{\alpha\to 0} \varphi_\ell(\rho, \chi) \qquad (9)$$
$$= \rho p_\ell(\rho, \cosh\chi) \xrightarrow[\rho\to\infty]{} \frac{\sin(\rho\chi - \pi\ell/2)}{\sinh\chi}.$$

The relativistic L-factor is connected with the RQP partial wave function $\varphi_\ell(\rho,\chi)$ as follows:

$$L_{\rm uneq}(\chi) \qquad (10)$$
$$= \lim_{\rho\to i}\left|\frac{\Gamma(2\ell+2)}{(2\sinh\chi)^\ell\Gamma^2(\ell+1)}\left(\Delta^*\right)^\ell\left[\frac{\varphi_\ell(\rho\,,\chi)}{\rho}\right]\right|^2,$$
$$\Delta^* = \frac{1}{i}\left[\exp\left(i\frac{\partial}{\partial\rho}\right)-1\right].$$

By using Eqs. (8–10), we finally find the following expression for the relativistic L-factor:

$$L_{\rm uneq}(\chi) \qquad (11)$$
$$= \prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha\mu}{m' n\sinh\chi}\right)^2\right]S_{\rm uneq}(\chi),$$

$$S_{\rm uneq}(\chi) = \lim_{\rho\to i}\left|\frac{\varphi_0(\rho,\chi)}{\rho}\right|^2 \qquad (12)$$
$$= \frac{X_{\rm uneq}(\chi)}{1-\exp\left[-X_{\rm uneq}(\chi)\right]}, X_{\rm uneq}(\chi) = \frac{2\pi\alpha\mu}{m'\sinh\chi}.$$

The function $\sinh\chi$ in Eqs. (11) and (12) can be expressed in terms of the relative velocity of an effective relativistic particle with mass m', $u'_{\rm rel}$, determined by the relation

$$u'_{\rm rel} = \frac{2u}{\sqrt{1-u^2}}, \qquad (13)$$
$$u = \sqrt{1-\frac{4m'^2}{s-(m_1-m_2)^2}},$$

in the form $\sinh\chi = \mu u'_{\rm rel}/m'$.

Thus, in terms of relative velocity of an effective relativistic particle (13) the factors (11) and (12) are given by expressions

$$L_{\rm uneq}(u'_{\rm rel}) = \prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha}{n u'_{\rm rel}}\right)^2\right]S_{\rm uneq}(u'_{\rm rel}), \qquad (14)$$

$$S_{\rm uneq}(u'_{\rm rel}) = \frac{X_{\rm uneq}(u'_{\rm rel})}{1-\exp\left[-X_{\rm uneq}(u'_{\rm rel})\right]}, \qquad (15)$$
$$X_{\rm uneq}(u'_{\rm rel}) = \frac{2\pi\alpha}{u'_{\rm rel}}.$$

The S-factor in Eq. (15) only formally has the same form, as the nonrelativistic S-factor (2). However, the S-factor in Eq. (15) has an obviously relativistic nature since as the argument r in the Coulomb potential (1) and the relative velocity of (13), both are relativistic invariants.

The relativistic factors (11) and (12) [or (14), (15)] have the following important properties:

• In the nonrelativistic limit, $u \ll 1$, they reproduce the well-known nonrelativistic result.

• In the relativistic limit, $u \to 1$, the factors (11) and (12) [or (14), (15)] go to unity.

• In the case of equal masses they coincide with S-factor (3) and L-factor (4).

• In the ultrarelativistic limit, as it was argued in [28, 29], the bound state spectrum vanishes since $m' \to 0$. This feature reflects an essential difference between potential models and quantum field theory where an additional dimensional parameter appears. One can conclude that within a potential model, the S- and L-factors which correspond to the continuous spectrum should go to unity in the limit $m' \to 0$. Thus, in contrast to the nonrelativistic case, the relativistic factors (11) and (12) [or (14), (15)] reproduce both the known nonrelativistic and the expected ultrarelativistic limits.

To illustrate the differences between the nonrelativistic S-factor (2) and the new relativistic S-factor in Eq. (15) in more detail, in Fig. 2 we plot the behavior of these factors as functions of u at different values of the parameter α (the numbers at the curves). From this figure one can see that in the region of nonrelativistic values of u, $u \le 0.2$ where the influence of the S-factor is high, the difference between (2) and (15) is practically absent. However, when α increases, the expression (2) gives a less suitable result in the region of large values u, in particular, as $u \to 1$.

Thus, the above performed analysis demonstrates that the factor in Eq. (15) coincides in form with the factor (2). However, the relative velocity of an effective relativistic particle (13) emerging instead of the system of two particles, now plays the role of the

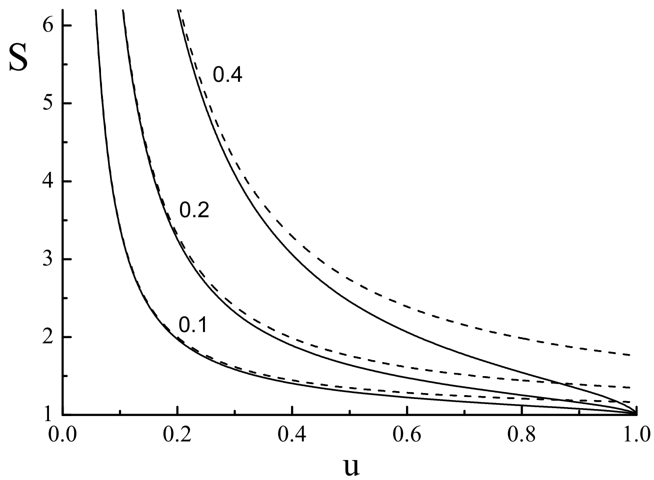

FIG. 2. Behavior of the S-factor at different values of the parameter α (the numbers at the curves). The solid lines correspond to the new relativistic S-factor (15) and the dashed lines to the nonrelativistic S-factor (2).

parameter of velocity, but not the relativistic relative velocity of interacting particles, $\mathbf{v}$. These new relativistic threshold S- and L-factors could have a significant impact in interpreting strong-interaction physics.

3. Conclusion

The new relativistic threshold resummation factors (14) and (15) for the interaction of two relativistic particles of unequal masses were obtained. To reach this aim the relativistic quasipotential equation in relativistic configuration representation [26] with the Coulomb potential for the interaction of two relativistic particles of unequal masses was used.

The new relativistic factors obtained here reproduce both the known nonrelativistic and expected ultrarelativistic limits and correspond to the QCD-like Coulomb potential. The new S-factor coincides in form with the nonrelativistic (2); however, the role of the parameter of velocity is played not by the relative velocity of interacting particles, $\mathbf{v}$, but by the relative velocity (13) of an effective relativistic particle emerging instead of the system of two particles.

It was shown that there is a difference (see Fig. 2) between the expression (15) obtained here and the nonrelativistic (2). As the new relativistic factors (14) and (15) were obtained within the framework of completely covariant method, one can expect that these factors takes into account more adequately relativistic nature of interaction.

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