Nonlinear Phenomena in Complex Systems, vol. **17**, no. 4 (2014), pp. 358 - 363

A Solution of the Discrete Wheeler–Dewitt Equation in the Vicinity of Small Scale Factors and Quantum Mechanics in the Space of Negative Constant Curvature

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(Received 05 September, 2014)

The asymptotic of the solution of the discrete Wheeler–DeWitt equation is found in the vicinity of small scale factors. It is shown that the problem is equivalent to the solution of the stationary Schrödinger equation in the (super) space of negative constant curvature. The minimum positive eigenvalue is found from which a continuous spectrum begins.

PACS numbers: 04.60-m, 98.80.Qc, 02.40 Ky
Keywords: Wheeler–DeWitt equation, superspace, spaces of negative constant curvature, quantum mechanics in curved space

The Wheeler–DeWitt equation [1, 2] is a functional equation describing quantum space-time. As the solution of this equation presents great difficulties, it is important to investigate at least asymptotic of the solutions, for example, in the vicinity of small scale factors. Under the scale factor it is understood $a(\boldsymbol{r}) \equiv \gamma^{1/6}(\boldsymbol{r})$ where $\gamma = \det \gamma_{ij}$. A metric tensor $\gamma_{ij}(\boldsymbol{r})$ depends on coordinates $\boldsymbol{r}$, which are defined on a three-dimensional manifold. The Wheeler–DeWitt equation for gravitation and several scalar fields in the vicinity of small scale factors $a \to 0$ looks as

$$\left(\frac{12\gamma^{1/6}}{M_p^2} G_{ijkl} \frac{\delta}{\delta\gamma_{ij}(\boldsymbol{r})} \frac{\delta}{\delta\gamma_{kl}(\boldsymbol{r})} + \frac{\gamma^{-1/3}}{2} \frac{\delta}{\delta\phi(\boldsymbol{r})} \frac{\delta}{\delta\phi(\boldsymbol{r})}\right) \Psi[\gamma, \phi] = 0 \quad (1)$$

where $\phi(\boldsymbol{r}) = \{\phi_1(\boldsymbol{r}), \phi_2(\boldsymbol{r}), \ldots, \phi_N(\boldsymbol{r})\}$ is a set of scalar fields, M_p is the Planck mass and

$$G_{ijkl} = \frac{1}{2}\gamma^{-1/2}\left(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}\right).$$

Eq. (1) is written in the conformal time gauge [3]. As the Wheeler–DeWitt equation contains the functional derivatives acting at the same spatial point, generally, a regularization is required to avoid an occurrence of infinite quantities. Besides, it is necessary to choose an operator ordering procedure. The most natural choice of operator ordering is to form a multivariate Laplacian. Let as write down all variables in the form of a single vector $\boldsymbol{\xi} = \{\gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}, \gamma_{13}, \gamma_{23}, \phi_1, \ldots, \phi_N\}$. Then, Eq. (1) takes the form:

$$G^{AB} \frac{\delta}{\delta\xi^A(\boldsymbol{r})} \frac{\delta}{\delta\xi^B(\boldsymbol{r})} \Psi[\boldsymbol{\xi}(\boldsymbol{r})] = 0 \quad (2)$$

where a matrix G^{AB} is defined by

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$$G = \begin{pmatrix} \tilde{G}_{1111} & \tilde{G}_{1122} & \tilde{G}_{1133} & \tilde{G}_{1112} & \tilde{G}_{1113} & \tilde{G}_{1123} & 0 & 0 \\ \tilde{G}_{2211} & \tilde{G}_{2222} & \tilde{G}_{2233} & \tilde{G}_{2212} & \tilde{G}_{2213} & \tilde{G}_{2223} & 0 & 0 \\ \tilde{G}_{3311} & \tilde{G}_{3322} & \tilde{G}_{3333} & \tilde{G}_{3312} & \tilde{G}_{3313} & \tilde{G}_{3323} & 0 & 0 \\ \tilde{G}_{1211} & \tilde{G}_{1222} & \tilde{G}_{1233} & \tilde{G}_{1212} & \tilde{G}_{1213} & \tilde{G}_{1223} & 0 & 0 \\ \tilde{G}_{1311} & \tilde{G}_{1322} & \tilde{G}_{1333} & \tilde{G}_{1312} & \tilde{G}_{1313} & \tilde{G}_{1323} & 0 & 0 \\ \tilde{G}_{2311} & \tilde{G}_{2322} & \tilde{G}_{2333} & \tilde{G}_{2312} & \tilde{G}_{2313} & \tilde{G}_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \gamma^{-1/3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma^{-1/3} \end{pmatrix}, \qquad (3)$$

$\tilde{G}_{ijkl} = \frac{12\gamma^{1/6}}{M_p^2} G_{ijkl}$, and Eq. (3) is written down for a special case of two scalar fields.

Eqs. (1, 2) contain the functional derivatives acting in the same spatial point that demands a regularization. One of ways to remove infinities from Eq. (2) is a discretization, which, for example, can be made by means of a triangulation [4]. For our case, it is sufficient to choose an elementary discretization by introducing a cube spatial grid with the edge length ℓ. One could identify the scale of discretization with the Planck length, however, it is not obligatory. As the space is divided into the cells with the volume $\Delta x \Delta y \Delta z = \ell^3$ and the centers located at points $\boldsymbol{r}_1, \boldsymbol{r}_2 .... \boldsymbol{r}_k$, it is necessary to replace a functional derivative with usual derivative by a rule $\frac{\delta}{\delta \xi^A(r)} \to \frac{1}{\ell^3} \frac{\partial}{\partial \xi_k^A}$ where it is implied that $\boldsymbol{\xi}_k$ is a value of the vector $\boldsymbol{\xi}$ at the point $\boldsymbol{r}_k$, i.e. $\boldsymbol{\xi}_k = \boldsymbol{\xi}(\boldsymbol{r}_k)$. As a result, Eq. (2) will have the same form at all spatial points $\boldsymbol{r}_k$ and its solution will be represented in the form of product of solutions obtained for every spatial point:

$$\Psi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \ldots \boldsymbol{\xi}_k) = \psi(\boldsymbol{\xi}_1) \psi(\boldsymbol{\xi}_2) \ldots \psi(\boldsymbol{\xi}_k).$$

The choice of operator ordering in the form of Laplacian leads to the following equation

$$\frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi^A} \left(\sqrt{G}\, G^{AB} \frac{\partial}{\partial \xi^B} \right) \psi(\boldsymbol{\xi}) = 0 \qquad (4)$$

where $G = \det G_{AB} = 1/\det G^{AB}$. In Eq. (4) and everywhere further, the dependence on a spatial index k is omitted. One has to note that for a case of pure gravitation (i.e. in absence of of scalar fields) the equation coinciding with (4) can be written in the form of

$$\gamma\ \hat{\pi}^{ij} \left(\frac{1}{\gamma} \tilde{G}_{ijkl}\ \hat{\pi}^{kl} \right) \psi(\gamma_{mp}) = 0 \qquad (5)$$

where

$$\hat{\pi}^{ij} = \begin{cases} \frac{\partial}{\partial \gamma_{ij}}, & i = j, \\ \frac{1}{2} \frac{\partial}{\partial \gamma_{ij}}, & i \neq j. \end{cases}$$

Let us find the solution of Eq. (4) in the form of “plane waves ” [5–7]. Here, we introduce the following variables $\tilde{u} = k^{ij}\gamma_{ji}$, $\tilde{v} = k^{ij}\gamma_{jm}k^{mn}\gamma_{nj}$ and $\Phi = p^i\phi_i$ where k^{ij} is some 3×3-dimensional matrix and p^i is a vector of the dimension defined by a number of scalar fields N. Let us represent the state ψ in the form

$$\psi(\gamma_{lm}, \phi_i) = f(\tilde{u}, \tilde{v}, \gamma) \exp(i\Phi). \qquad (6)$$

Substitution of the expression (6) into (4) or, for pure gravitation, into (5) after cumbersome calculations using the Mathematica computer algebra result in the following equation for the function $f(\tilde{u}, \tilde{v}, \gamma)$:

$$\frac{6\gamma^{-1/3}}{M_p^2}\Bigg(-3\gamma^2\frac{\partial^2 f}{\partial\gamma^2}-\left(5+\frac{N}{2}\right)\gamma\frac{\partial f}{\partial\gamma}+2\left(\tilde u^2+\left(\frac{7}{3}-\frac{N}{6}\right)\tilde v\right)\frac{\partial f}{\partial\tilde v}$$
$$+4\left(2\tilde u^2\tilde v-\tilde u^4+8k\tilde u\gamma\right)\frac{\partial^2 f}{\partial\,\tilde v^2}+\left(\frac{7}{3}-\frac{N}{6}\right)\tilde u\frac{\partial f}{\partial\tilde u}+\left(2\tilde v-\tilde u^2\right)\frac{\partial^2 f}{\partial\tilde u^2}$$
$$+4\left(2\tilde u\tilde v-\tilde u^3+6k\gamma\right)\frac{\partial^2 f}{\partial\tilde u\partial\tilde v}-2\tilde u\gamma\frac{\partial^2 f}{\partial\tilde u\partial\gamma}-4\tilde v\gamma\frac{\partial^2 f}{\partial\tilde v\partial\gamma}\Bigg)-\frac{1}{2}p^2\gamma^{-1/3}f=0 \tag{7}$$

where $p^2=\left(p^1\right)^2+\left(p^2\right)^2+\ldots\left(p^N\right)^2$ and $k=\det k^{ij}$.

It is convenient to present a metric tensor in the form of $\gamma_{ij}=a^2\,\tilde\gamma_{ij}$, so that $\det\tilde\gamma_{ij}=1$. The matrix $\tilde\gamma_{ij}$ describes a so-called conformal geometry [8]. Then Eq. (7) rewritten in the terms of new variables $a=\gamma^{1/6}$, $u=k^{ij}\,\tilde\gamma_{ij}=\tilde u\gamma^{-1/3}$, $v=k^{ij}\,\tilde\gamma_{jl}\;k^{lm}\,\tilde\gamma_{mi}=\tilde v\gamma^{-2/3}$ takes the form

$$\tfrac{1}{2M_p^2}\Big(-\tfrac{\partial^2 f}{\partial a^2}-\tfrac{(5+N)}{a}\tfrac{\partial f}{\partial a}+\tfrac{8}{a^2}\Big(5u\,\tfrac{\partial f}{\partial u}+\left(3v-u^2\right)\tfrac{\partial^2 f}{\partial u^2}+\left(3u^2+11v\right)\tfrac{\partial f}{\partial v}$$
$$+2\left(24ku+v^2+6u^2v-3u^4\right)\tfrac{\partial^2 f}{\partial v^2}+2\left(18k+7vu-3u^3\right)\tfrac{\partial^2 f}{\partial u\partial v}\Big)\Big)-\tfrac{1}{2a^2}p^2f=0.$$

A straightforward way is to solve the above equation by the method of variable separation $f\,(a,u,v)=R\,(a)\,g\,(u,v)$. Namely, if the solution $g(u,v)$ of the equation

$$5u\,\tfrac{\partial g}{\partial u}+\left(3v-u^2\right)\tfrac{\partial^2 g}{\partial u^2}+\left(3u^2+11v\right)\tfrac{\partial g}{\partial v}+2\left(24k\,u+v^2+6u^2v-3u^4\right)\tfrac{\partial^2 g}{\partial v^2}$$
$$+2\left(18k+7v\,u-3u^3\right)\tfrac{\partial^2 g}{\partial u\partial v}=-\lambda g, \tag{8}$$

has obtained and the value of the corresponding constant λ has been found, then the equation for the function $R\,(a)$ becomes

$$-\frac{\partial^2 R}{\partial a^2}-\frac{(5+N)}{a}\frac{\partial R}{\partial a}-\frac{8\lambda+M_p^2\,p^2}{a^2}R=0. \tag{9}$$

The solution of (9) can be expressed easy through the Bessel functions. Thus, the main problem is to solve Eq. (8), which describes quantization of the conformal geometry [8] determined by a matrix $\tilde\gamma_{ij}$ with an unit determinant.

It is interesting to note that Eq. (8) can be obtained by another way.

Let us consider the Hamiltonian

$$H = \frac{1}{2}\left(\tilde{\gamma}^{ij}\right)' \tilde{\gamma}'_{ij} = \frac{1}{2}\tilde{\gamma}'_{ik}\,\tilde{\gamma}^{kl}\,\tilde{\gamma}'_{lj}\,\tilde{\gamma}^{ji} \qquad (10)$$

where $\det\tilde{\gamma}_{ij} = 1$, and a prime means a derivative on time. The Hamiltonian (10) corresponds to free motion of a "particle"on a five-dimensional surface $\det\tilde{\gamma}_{ij} = 1$ of constant negative curvature.

Firstly, let us first consider a simple case, when the matrix $\tilde{\gamma}_{ij}$ has dimension 2×2 and the dimension of the surface $\det\tilde{\gamma}_{ij} = 1$ equals two. After some parametrization $\tilde{\gamma}_{ij}\left(\xi^1(t), \xi^2(t)\right)$, one has

$$H = \frac{1}{2}G_{AB}(\tau)\,\xi^{A'}(t)\,\xi^{B'}(\tau)$$

where $G_{AB} = Tr\left[\tilde{\gamma}^{-1}\frac{\partial\tilde{\gamma}}{\partial\xi^A}\tilde{\gamma}^{-1}\frac{\partial\tilde{\gamma}}{\partial\xi^B}\right]$ and a prime means differentiation on time τ.

The generalized momentums is written in the form

$$p_A = \frac{\partial H}{\partial\xi^{A'}} = G_{AB}\xi^{B'}.$$

The Hamiltonian can be expressed through the momentums:

$$H = \frac{1}{2}G^{AB}(\tau)\,p_A(\tau)\,p_B(\tau)\,.$$

Finally, quantization leads to the Schrödinger equation

$$\frac{1}{\sqrt{G}}\frac{\partial}{\partial\xi^B}\left(\sqrt{G}\,G^{AB}\frac{\partial}{\partial\xi^A}\Theta\right) = \lambda\,\Theta.$$

It is convenient to take the coordinates $\{r(\tau), \varphi(\tau)\}$ for the parametrization $\{\xi^1(\tau), \xi^2(\tau)\}$ and to represent a 2×2-matrix with the unit determinant as the product of three matrixes:

$$\tilde{\gamma}_{ij} = \begin{pmatrix} \cos\varphi/2 & -\sin\varphi/2 \\ \sin\varphi/2 & \cos\varphi/2 \end{pmatrix}\begin{pmatrix} \exp r & 0 \\ 0 & \exp(-r) \end{pmatrix}\begin{pmatrix} \cos\varphi/2 & \sin\varphi/2 \\ -\sin\varphi/2 & \cos\varphi/2 \end{pmatrix} = \begin{pmatrix} \xi-\varsigma & \sigma \\ \sigma & \xi+\varsigma \end{pmatrix}$$

where

$$\varsigma = r\cos\varphi,\ \sigma = r\sin\varphi,\ \xi = r.$$

It turns out to be that the surface $\det\tilde{\gamma}_{ij} = 1$ for the 2×2- matrixes in coordinates r, φ is a hyperboloid $\xi^2 - \varsigma^2 - \sigma^2 = 1$. More precisely, the coordinates r, φ parameterize one of its cavities. Let us now search a solution of the Schrödinger equation in the form of $\Theta(r,\varphi) = g\left(k^{ij}\tilde{\gamma}_{ij}\right)$. That leads to the following equation

$$-\frac{1}{2}u^2 g''(u) - u g'(u) = \lambda g(u)$$

where $u = k^{ij}\tilde{\gamma}_{ij}$. A solution of the above equation can be written as

$$g(u) = u^{-1/2+i\sqrt{2\lambda-1/4}}. \qquad (11)$$

Using the matrix k^{ij} in the form of

$$k^{ij} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix},$$

gives

$$u = k^{ij}\,\tilde{\gamma}_{ij} = \xi - n_1\varsigma - n_2\sigma$$

where $n_1 = \cos\theta$, $n_2 = \sin\theta$. Thus, (11) is precisely a plane wave [5–7] on a surface of hyperboloid in a 2 + 1-dimensional Minkowski space.

For the case of the 3 × 3-dimensional matrixes, we will search a solution in the form of $\Theta\left(\xi^1, \xi^2, \xi^3, \xi^4, \xi^5\right) = g\left(u, v\right)$ where $u = k^{ij}\,\tilde{\gamma}_{ij}$, $v = k^{ij}\,\tilde{\gamma}_{jl}\;k^{lm}\,\tilde{\gamma}_{mi}$, and k^{ij} is some matrix. After the cumbersome calculations, we come again to Eq. (8). Since the wave function has five independent coordinates, the function Θ should be defined by the value of the constant λ and some additional four parameters. Thus, it is possible to impose at least two additional conditions on the tensor k^{ij}. It should be noted, that Eq. (8) becomes homogeneous relatively v and u^2 when $k = \det k^{ij} = 0$. This allows finding the solution of (8) at $k = 0$ in the form of

$$g\left(u, v\right) = \left(u^2 - v\right)^{i\alpha/2 - 3/4} s\left(u^2/v\right). \qquad (12)$$

Substitution of (12) into (8) results in the equation for the function $s\left(z\right)$

$$3\left(z-1\right)\left[2\left(z-2\right)\left(z-1\right)z s''\left(z\right) + \left(z\left(4z-7\right) + 2\right) s'\left(z\right)\right] - \left(\lambda - \frac{\alpha^2}{2} - \frac{9}{8}\right) s\left(z\right) = 0, \qquad (13)$$

with the general solution expressed through the hypergeometric function

$$s(z) = \frac{c_1}{\sqrt{2-\frac{2}{z}}}\,{}_2F_1\left(\tfrac{1}{12}\left(9 - 2\sqrt{3}\,i\sqrt{2\lambda-3-\alpha^2}\right), \tfrac{1}{12}\left(2\sqrt{3}\,i\sqrt{2\lambda-3-\alpha^2}+9\right); \tfrac{3}{2}; \tfrac{z}{2(z-1)}\right) + c_2\,{}_2F_1\left(\tfrac{1}{12}\left(3 - 2\sqrt{3}\,i\sqrt{2\lambda-3-\alpha^2}\right), \tfrac{1}{12}\left(2\sqrt{3}\,i\sqrt{2\lambda-3-\alpha^2}+3\right); \tfrac{1}{2}; \tfrac{z}{2(z-1)}\right). \qquad (14)$$

Thus, the wave function Θ depends on parameters α, λ and some additional three parameters contained in the tensor k^{ij}, which obeys the condition $k = 0$ and some two additional conditions.

Here, we do not discuss a question about normalization of the wave function in detail. The detailed consideration of quantization on a hyperboloid is presented in Ref. [9]. Here, we note that for the wave function to be normalized, it is necessary, that the normalization of wave function requires positivity of the expression $2\lambda - 3 - \alpha^2$ under the square root in Eq. (14). The minimal possible value of the constant $\lambda = 3/2$ is reached at $\alpha = 0$.

In the given work we have solved the discrete Wheeler–DeWitt equation in the vicinity of small scale factors by the method of the variables separation. It is shown, that the constant λ of the variables separation cannot be infinitely small. The minimal admissible value of λ arises because the quantization is carried out on a surface of constant negative curvature.

It means that, if the scale ℓ of discretization is of order of the inverse Planck mass, there exists an energy density $\rho = \frac{8}{M_p^2\ell^6 a^6}\lambda \approx \frac{12M_p^4}{a^6}$ in early Universe due to fluctuations of the conformal geometry. Let us emphasize that there is a time, when the matter fields as well as the gravitational waves do not oscillate so that the well-known vacuum energy corresponding to zero oscillations of the field oscillators is not formed yet.

Acknowledgement

The authors are grateful to Victor Red'kov for the discussion.

References

[1] J.A. Wheeler. Superspace and Nature of Quantum Geometrodynamics. In: *Battelle Rencontres.* Eds. C. DeWitt, J.A. Wheeler. (Benjamin, New York, 1968).

[2] B.S. DeWitt. Phys. Rev. **160**, 1113 (1967).

[3] S. L. Cherkas, V. L. Kalashnikov. Gen. Rel. Grav. **44**, 3081 (2012).

[4] H. W. Humber. Gen. Rel. Grav. **41**, 817 (2009).

[5] I.M. Gelfand, M.I. Graev, N.J. Vilenkin. *Integral geometry and the related questions of the theory of representations.* (State Edition of Physics and Mathematics literature, Moscow, 1962). (In Russian).

[6] I.S. Shapiro. Dokl. Akad. Nauk SSSR, 106, 647 (1956).

[7] E. M. Ovsiyuk, N. G. Tokarevskaya, V. M. Red'kov. J. Nonlin. Phenomena in Complex Syst. **12**, 1 (2009).

[8] J. W. Jr. York. Phys. Rev. Lett. **26**, 1656 (1971).

[9] V.S. Otchik, V.M. Red'kov. Preprint of Institute of Physics Akad. Nauk BSSR, no. 298. (Institute of Physics, Minsk, 1983).