ON FREE LEFT n-DINILPOTENT DOPPELALGEBRAS

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Recall that a doppelalgebra [1] is a nonempty set with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms $(x \dashv y) \vdash z = x \dashv (y \vdash z)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)$. As usual, $\mathbb{N}$ denotes the set of all positive integers.

Lemma. *In a doppelalgebra $(D, \dashv, \vdash)$ for any $n > 1, n \in \mathbb{N}$, and any $x_i \in D$, $1 \leq i \leq n+1$, and $*_j \in \{\dashv, \vdash\}$, $1 \leq j \leq n$, any parenthesizing of*

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1}$$

gives the same element from D.

A doppelalgebra $(D, \dashv, \vdash)$ will be called left dinilpotent, if for some $n \in \mathbb{N}$ and any $x_1, ..., x_n$, $x \in D$ the following identities hold:

$$(x_1 *_1 ... *_{n-1} x_n) \dashv x = x_1 *_1 ... *_{n-1} x_n = (x_1 *_1 ... *_{n-1} x_n) \vdash x,$$

where $*_1, ..., *_{n-1} \in \{\dashv, \vdash\}$. The least such n we shall call the left dinilpotency index of $(D, \dashv, \vdash)$. For $k \in \mathbb{N}$ a left dinilpotent doppelalgebra of left dinilpotency index $\leq k$ is said to be left k-dinilpotent. The notion of a left dinilpotent doppelalgebra is an analog of the notion of a left nilpotent semigroup [2]. It is clear that operations of any left 1-dinilpotent doppelalgebra coincide and it is a left zero semigroup. The class of all left n-dinilpotent doppelalgebras forms a subvariety of the variety of doppelalgebras. A doppelalgebra which is free in the variety of left n-dinilpotent doppelalgebras will be called a free left n-dinilpotent doppelalgebra.

Let X be an arbitrary nonempty set and let ω be an arbitrary word in the alphabet X. The length of ω will be denoted by l_ω. Let further $F[X]$ be the free semigroup on X, T be the free monoid on the two-element set $\{a, b\}$ and $\theta \in T$ be an empty word. Fix $n \in \mathbb{N}$. If $l_w \geq n$ for $w \in F[X]$, by $\overset{n}{\overrightarrow{w}}$ denote the initial subword with the length n of w. By definition, the length l_θ of θ is equal to 0 and $\overset{0}{\overrightarrow{u}} = \theta$ for all $u \in T \backslash \{\theta\}$. Define operations $\dashv$ and $\vdash$ on

$$L_n = \{(w, u) \in F[X] \times T \ \mid l_w - l_u = 1, \, l_w \leq n\}$$

by

$$(w_1, u_1) \dashv (w_2, u_2) = \begin{cases} (w_1 w_2, u_1 a u_2), & l_{w_1} + l_{w_2} \leq n, \\ (\overset{n}{\overrightarrow{w_1 w_2}}, \overset{n-1}{\overrightarrow{u_1 a u_2}}), & l_{w_1} + l_{w_2} > n, \end{cases}$$

$$(w_1, u_1) \vdash (w_2, u_2) = \begin{cases} (w_1 w_2, u_1 b u_2), & l_{w_1} + l_{w_2} \leq n, \\ (\overset{n}{\overrightarrow{w_1 w_2}}, \overset{n-1}{\overrightarrow{u_1 b u_2}}), & l_{w_1} + l_{w_2} > n \end{cases}$$

for all $(w_1, u_1), (w_2, u_2) \in L_n$. The obtained algebra will be denoted by $FDDA_n^l(X)$.

Theorem. *$FDDA_n^l(X)$ is the free left n-dinilpotent doppelalgebra.*

We also consider separately free left n-dinilpotent doppelalgebras of rank 1 and characterize the least left n-dinilpotent congruence on a free doppelalgebra. In order to construct free right n-dinilpotent doppelalgebras and characterize the least right n-dinilpotent congruence on a free doppelalgebra we use the duality principle.

References

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