ON INTERSECTION OF TRIPLE OF PREFRATTINI SUBGROUPS
IN FINITE SOLUBLE GROUP

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In this paper we consider finite groups only, so the term “group” always means “finite group”.

D.S. Passman in [1] proved that a p-soluble group G always possesses three Sylow p-subgroups such that their intersection is equal to $O_p(G)$. Later V.I. Zenkov [2] proved the same statement for an arbitrary group. In [3–4] S. Dolfi proved that in every $\pi$-soluble group $G$ there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = O_\pi(G)$ holds.

In connection with these results, in the Kourovka Notebook [5] the author formulated the following Problem 17.55:

Does there exist an absolute constant $k$ such that for any prefrattini subgroup $H$ in any finite soluble group $G$ there exist $k$ conjugates of $H$ whose intersection is $\Phi(G)$, the Frattini subgroup of $G$?

The main goal of this paper is to give an affirmative answer to this question.

Theorem. Let $H$ be a prefrattini subgroup of a soluble group $G$. Then there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = \Phi(G)$ holds.

Corollary 1. Let $H$ be a prefrattini subgroup of a soluble group $G$. If $\Phi(G) = 1$, then there exist elements $x, y \in G$ such that the equality $H \cap H^x \cap H^y = 1$ holds.

Corollary 2. Let $H$ be a prefrattini subgroup of a soluble group $G$. Then the inequalities $|H| \leq \sqrt[3]{|G|^2 \cdot |\Phi(G)|}$ and $|H/\Phi(G)| \leq |G:H|^2$ hold.

Corollary 3. Let $H$ be a prefrattini subgroup of a soluble group $G$. If $\Phi(G) = 1$, then the inequalities $|H| \leq \sqrt[3]{|G|^2}$ and $|H| \leq |G:H|^2$ hold.

The concept of a prefrattini subgroup of a soluble group was introduced by Gaschütz in [6]. Considering a complemented chief factor $L/K$ of a soluble group $G$ as a $G$-module, Gaschütz proved that $G$ has a normal section that is a completely reducible $G$-module whose composition components are $G$-isomorphic to $L/K$, and its composition length $m$ is equal to the number of complemented and $G$-isomorphic to $L/K$ factors of a chief series of $G$. That section is denoted by $Cr_G(L/K)$ and called a crown of $G$ corresponding to $L/K$. A constructive definition of a crown of a soluble group $G$ corresponding to a complemented chief factor $L/K$ is given as follows:

$$Cr_G(L/K) = C_G(L/K)/R,$$

where $R$ is the intersection of the cores of maximal subgroups complementing $L/K$.

A crown of a soluble group $G$ is a crown corresponding to a complemented chief factor of $G$. The set of all crowns of $G$ is denoted by $Cr(G)$. From the Jordan-Hölder theorem it follows that for the construction of $Cr(G)$ it is enough to consider some chief series of $G$ and to choose in it the maximal system $L_1/K_1, \ldots, L_t/K_t$ pairwise non-$G$-isomorphic complemented chief factors. Then we have $Cr(G) = \{Cr_G(L_1/K_1), \ldots, Cr_G(L_t/K_t)\}$.

Definition. Let $G$ be a soluble group, and $Cr(G) = \{Cr_G(L_1/K_1), \ldots, Cr_G(L_t/K_t)\}$. Let $G_i$ be a complement of $Cr_G(L_i/K_i)$ in $G$, where $i \in I = \{1, 2, \ldots, t\}$. Then the subgroup $\bigcap_{i \in I} G_i$ is called a prefrattini subgroup of $G$.

By Definition, every soluble group has at least one prefrattini subgroup.

The following theorem gives basic properties of prefrattini subgroups.

Theorem ([6]). For a soluble group $G$, the following conditions hold:
1) if $H$ is a prefrattini subgroup of $G$ and $N \triangleleft G$, then:
   a) $H^x$ is a prefrattini subgroup of $G$ for any element $x \in G$;
b) $HN/N$ is a prefrattini subgroup of $G/N$;
c) $H$ covers all Frattini chief factors of $G$ and avoids all complemented chief factors of $G$;
d) $Core_G(H) = \Phi(G)$;
e) $|H|$ is the product of the orders of all Frattini chief factors in a chief series of $G$;

2) any two prefrattini subgroups of $G$ are conjugate in $G$.

References