SUBGROUP-CLOSED LATTICE AND K-LATTICE FORMATIONS

S.F. Kamornikov[1], Xiaolan Yi[2]

[1]Gomel Branch of International University "MITSO", 246019, Gomel, Belarus
sfkamornikov@mail.ru
[2]Zhejiang Sci-Tech University, 310018, Hangzhou, China
yixiaolan2005@126.com

Throughout this paper, all groups are finite.

One of the most striking results in the theory of subnormal subgroups is the celebrated "join" theorem, proved by H. Wielandt in 1939: the subgroup generated by two subnormal subgroups of a finite group is itself subnormal. As a result, the set $sn(G)$ of all subnormal subgroups of a group G is a sublattice of the subgroup lattice.

The Wielandt's theorem was developed in the formation theory using concepts of $\mathfrak{F}$-subnormality and K-$\mathfrak{F}$-subnormality.

The first concept was proposed by R. Carter and T. Hawkes. Let $\mathfrak{F}$ be a non-empty formation. A subgroup H of a group G is said to be $\mathfrak{F}$-*subnormal* in G if either $H = G$ or there exists a maximal chain of subgroups

$$H = H_0 \subset H_1 \subset \cdots \subset H_n = G$$

such that $H_i^{\mathfrak{F}} \subseteq H_{i-1}$ for all $i = 1, \ldots, n$. The set of all $\mathfrak{F}$-subnormal subgroups of a group G is denoted by $sn_{\mathfrak{F}}(G)$.

It is rather clear that the $\mathfrak{N}$-subnormal subgroups of a group G for the formation $\mathfrak{N}$ of all nilpotent groups are subnormal, and they coincide in the soluble universe. However the equality $sn_{\mathfrak{N}}(G) = sn(G)$, does not hold in general.

To avoid the above situation, O.H. Kegel introduced a little bit different notion of $\mathfrak{F}$-subnormality. It unites the notions of subnormal and $\mathfrak{F}$-subnormal subgroup.

A subgroup H of a group G is called $\mathfrak{F}$-*subnormal in sense of Kegel* (or simply K-$\mathfrak{F}$-*subnormal*) in G if there exists a chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that H_{i-1} is either normal in H_i or $H_i^{\mathfrak{F}} \subseteq H_{i-1}$ for all $i = 1, \ldots, n$. We shall write $H \in sn_{K-\mathfrak{F}}(G)$ and denote $sn_{K-\mathfrak{F}}(G)$ the set of all K-$\mathfrak{F}$-subnormal subgroups of a group G.

Obviously, $sn_{K-\mathfrak{N}}(G) = sn(G)$ for every group G.

Let $\mathfrak{F}$ be a formation. One might wonder whether the set of $\mathfrak{F}$-subnormal subgroups of a group forms a sublattice of the subgroup lattice. As simple exemples show, the answer is in general negative.

Therefore the following question naturally arises:

Which are the formations $\mathfrak{F}$ for which the set $sn_{\mathfrak{F}}(G)$ is a sublattice of the subgroup lattice of G for every group G?

This question was first proposed by L.A. Shemetkov in his monograph [1, Problem 12] in 1978 and it appeared in the Kourovka Notebook [2, Problem 9.75] in 1984.

In 1992, A. Ballester-Bolinches, K. Doerk, and M.D. Perez-Ramos [3] gave the answer to that question in the soluble universe for subgroup-closed saturated formations. In 1993, A.F. Vasil'ev, S.F. Kamornikov, and V.N. Semenchuk [4] published the solution of the Shemetkov's problem in the general finite universe for subgroup-closed saturated formations. The following important result was obtained in 2002. A.F. Vasil'ev and the first author in [5] characterized the subgroup-closed lattice formations which are soluble.

In 1978, O.H. Kegel [6] showed that if $\mathfrak{F}$ is a subgroup-closed formation such that $\mathfrak{F}\mathfrak{F} = \mathfrak{F}$, then the set of all K-$\mathfrak{F}$-subnormal subgroups of a group G is a sublattice of the subgroup lattice of G for every group G. He also asks in [6] for other formations enjoying the lattice property for K-$\mathfrak{F}$-subnormal subgroups:

Which are the formations $\mathfrak{F}$ for which the set $sn_{K-\mathfrak{F}}(G)$ is a sublattice of the subgroup lattice of G for every group G?

In 1993, A.F. Vasil'ev, the first author, and V.N. Semenchuk [4] gave the answer to that question in the general finite universe for subgroup-closed saturated formations.

This paper can be considered as a further great step of the programme aiming to the classification of all lattice and K-lattice formations. We say that $\mathfrak{F}$ is a *lattice* (respectively, K-*lattice*) formation if the set of all $\mathfrak{F}$-subnormal (respectively, K-$\mathfrak{F}$-subnormal) subgroups is a sublattice of the lattice of all subgroups in every group.

Here we solve the Shemetkov's problem and the Kegel's question for all subgroup-closed formations.

Theorem. *Let $\mathfrak{F}$ be a subgroup-closed formation. The following statements are pairwise equivalent:*

1. *The set of all K-$\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice of every group.*

2. *The set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the subgroup lattice of every group.*

3. *$\mathfrak{F} = \mathfrak{M} \times \mathfrak{K} \times \mathfrak{L}$ for some subgroup-closed formations $\mathfrak{M}$, $\mathfrak{K}$ and $\mathfrak{L}$ satisfying the following conditions:*

(a) $\pi(\mathfrak{M}) \cap \pi(\mathfrak{K}) = \emptyset$, $\pi(\mathfrak{K}) \cap \pi(\mathfrak{L}) = \emptyset$ *and* $\pi(\mathfrak{M}) \cap \pi(\mathfrak{L}) = \emptyset$.

(b) $\mathfrak{M} = \mathfrak{S}_{\pi(\mathfrak{M})}\mathfrak{M}$ *is a saturated formation, and it is an $\mathfrak{M}^2$-normal Fitting class.*

(c) *Every non-cyclic $\mathfrak{M}$-critical group G with $\Phi(G) = 1$ is a primitive group of type 2 such that $G/Soc(G)$ is a cyclic group of prime power order.*

(d) *There exists a partition $\{\pi_j | j \in J\}$ of $\pi(\mathfrak{K})$ such that $\mathfrak{K} = \times_{j \in J}\mathfrak{S}_{\pi_j}$ and $|\pi_j| > 1$ for all $j \in J$.*

(e) $\mathfrak{L} \subseteq \mathfrak{N}_{\pi(\mathfrak{L})}$.

References

1. Shemetkov L.A. *Formations of finite groups.* Moscow: Nauka, 1978.

2. Mazurov V.D., Khukhro E.I. *Unsolved problems in group theory: The Kourovka Notebook.* Novosibirsk: Russian Academy of Sciences, Siberian Branch, Institute of Mathematics, 2010.

3. Ballester-Bolinches A., Doerk K., Perez-Ramos M.D. *On the lattice of $\mathfrak{F}$-subnormal subgroups* // J. Algebra. 1992. V. 148. No. 1. P. 42–52.

4. Vasil'ev A.F., Kamornikov S.F., Semenchuk V.N. *On lattices of subgroups of finite groups* // Infinite groups and related algebraic structures. Kiev: Institute of Mathematics of National Academy of Sciences of Ukraine. 1993. P. 27–54.

5. Vasil'ev A.F., Kamornikov S.F. *The Kegel-Shemetkov problem on lattices of generalized subnormal subgroups of finite groups* // Algebra and Logic. 2002. V. 41. No. 4. P. 228–236.

6. Kegel O.H. *Subgroup lattices of finite groups which properly contain the lattice of subnormal subgroups* // Arch. Math. 1978. V. 30. No. 3. P. 225–228.