DOMINATION TRIANGLE, IRREDUNDANCE TRIANGLE AND 1-TRIANGLE GRAPHS

P.A. Irzhavski, Y.A. Kartynnik, Y.L. Orlovich

Belarusian State University, Faculty of Applied Mathematics and Computer Science
4 Nezavisimosti Ave, 220030, Minsk, Belarus
irzhavski@bsu.by, kartynnik@bsu.by, orlovich@bsu.by

1. Introduction. Orlovich et al. [1, 2] have used the non-hereditary class of *triangle graphs* as a model to establish computational complexity results for several independence and domination related parameters. We introduce some its non-hereditary subclasses and use them to study computational complexity for other similar parameters.

Triangle graphs arise from the study of general partition graphs by DeTemple et. al. [3]. A *general partition graph* G is an intersection graph on a set S such that every maximal independent set of G corresponds to a partition of S. General partition graphs have been studied in connection to the lattice polygon triangulations and in a more general setting [4].

It has been noted by McAvaney et. al. [5] that for a graph G to be a general partition graph, the following *triangle condition* is sufficient: For every maximal independent set $I \subseteq V(G)$ and every edge $uv \in G - I$ there exists a vertex $w \in I$ such that uvw is a triangle in G. Such graphs have been called *triangle graphs* by Orlovich et. al. [1].

According to Sampathkumar and Neeralagi [6], a vertex set $S \subseteq V(G)$ is called a *neighbourhood set* if

$$G = \bigcup_{v \in S} G(N[v]).$$

A neighbourhood set S is thus a dominating set having a property that every edge not covered by the vertices of S has both its ends adjacent to the same arbitrary vertex in S.

This definition obviously leads to the way to specify the triangle graphs equivalently as the graphs having the property of every maximal independent set being a neighbourhood set (actually an *independent neighbourhood set* [7]). As a result of this coincidence, Orlovich et. al. have established that the minimum independent neighbourhood set cardinality, $n_i(G)$, which coincides with the independent domination number $i(G)$ in triangle graphs, is polynomially inapproximable in a triangle graph G up to the factor of $|V(G)|^{1-\varepsilon}$ for arbitrary $\varepsilon > 0$ unless $P = NP$ [1]; and for the maximum minimal independent neighbourhood set cardinality ($N_i(G)$, coinciding with $\alpha(G)$ in triangle graphs) they have shown NP-hardness of computation [2].

2. Domination and irredundance triangle graphs. We introduce *domination* and *irredundance triangle* graphs as the graphs having every dominating and maximal irredundant set, respectively, being a neighbourhood set. A set $S \subseteq V(G)$ is called *irredundant* [8] if $N[S \setminus \{s\}] \neq N[S]$ for every $s \in S$. Anbeek et. al. have noted that a sufficient condition for the graph to be a general partition graph is edge simpliciality (i.e. having every edge belong to a simplicial clique, a clique having a vertex adjacent only to the other vertices in that clique), which they have called the *edge condition* [9]. We give the following characterization:

Theorem 1. *The classes of domination triangle graphs and irredundance triangle graphs coincide and are precisely defined by the edge condition.*

It is also interesting to note that the edge condition in fact characterizes the class of upper-bound graphs known from the intersection graph theory and defined in terms of partially ordered sets [10].

We notice that the minimum-cardinality parameters for dominating, neighbourhood and maximal irredundant sets – respectively the domination number $\gamma(G)$, the neighbourhood number $\bar{n}(G)$ and the irredundance number $ir(G)$ for a graph G – coincide in such graphs. This observation is then used to prove the following result.

Theorem 2. *Computing the parameters $\gamma(G) = \bar{n}(G) = ir(G)$ for an arbitrary edge-simplicial graph is $c \log |V(G)|$-inapproximable in polynomial time for some fixed $c > 0$ unless $P = NP$.*

It is worth noting that we are not aware of any prior approximation bounds for $ir(G)$ and computational complexity results for $\bar{n}(G)$.

3. 1-triangle graphs. We then introduce the class of 1-*triangle* graphs which are triangle graphs with the restriction that every edge $uv \in G - I$ for every maximal independent set I forms a triangle with exactly one vertex in I. We provide the following characterization of 1-triangle graphs:

Theorem 3. *Every connected* 1*-triangle graph is either a complete bipartite graph, a graph isomorphic to $K_m \times K_m$, or a $(K_4 - e)$-free domination triangle graph.*

We show that every maximal independent set in 1-triangle graphs is a perfect neighbourhood set, which is defined by Sampathkumar and Neeralagi [7] as a neighbourhood set in which the closed neighbourhoods of every two vertices are edge disjoint. This is used in the proof of the following

Theorem 4. *Computing the minimum and maximum cardinalities $n_p(G)$ and $N_p(G)$ of perfect neighbourhood sets in a* 1*-triangle graph G is NP-hard.*

To the best of our knowledge, this is the first account of computational complexity for n_p.

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