ON THE ISOMORPHISM PROBLEM FOR GENERALIZED BAUMSLAG–SOLITAR GROUPS

F.A. Dudkin

Sobolev Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences
4 Acad. Koptyug avenue, 630090, Novosibirsk, Russia
Novosibirsk State University, Ministry of Education and Science of the Russian Federation
2 Pirogova Str., 630090, Novosibirsk, Russia
DudkinF@ngs.ru

Call a finitely generated group G a *generalized Baumslag-Solitar group* or a *GBS* group if G can act on a tree so that the stabilizers of vertices and edges are infinite cyclic groups. By the Bass-Serre theorem, G is representable as $\pi_1(\mathbb{A})$, the fundamental group of a graph of groups $\mathbb{A}$ (see [1]).

Given a GBS group G, we can present the corresponding graph of groups $\mathbb{A}$ by a labeled graph (A, λ), where A is a finite connected graph and $\lambda\colon E(A) \to \mathbb{Z} \setminus \{0\}$ labels the edges of A. The label λ_e of an edge e with the source vertex v defines an embedding $\alpha_e\colon e \to v^{\lambda_e}$ of the cyclic edge group $\langle e \rangle$ into the cyclic vertex group $\langle v \rangle$. Using the notion of expansion for labeled graphs, we can easily see that every GBS group can be presented by infinitely many labeled graphs.

Recently GBS groups have been quite actively studied [2–4]. In particular, the isomorphism problem for GBS groups has been discussed: to determine algorithmically when two given labeled graphs define isomorphic GBS groups. Despite that, the isomorphism problem is solved only in several special cases [5–7], the general solution is not established.

If two labeled graphs $\mathbb{A}$ and $\mathbb{B}$ define isomorphic GBS groups $\pi_1(\mathbb{A}) \cong \pi_1(\mathbb{B})$ and $\pi_1(\mathbb{A})$ is not isomorphic to $\mathbb{Z}, \mathbb{Z}^2$ or Klein bottle group then there exists a finite sequence of *expansion* and *collapse* (see fig.1) moves connecting $\mathbb{A}$ and $\mathbb{B}$ [8]. A labeled graph is called *reduced* if it admits no collapse move (equivalently, the labeled graph contains no edges with distinct endpoints and labels ± 1).

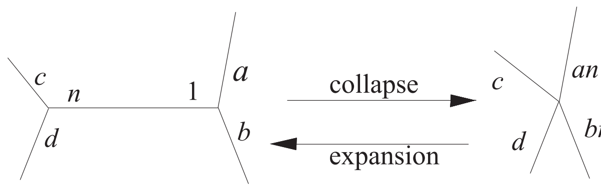

Fig. 1: Expansion and collapse moves.

Given a labeled graph $\mathbb{A}$ (a GBS group G), denote the set of reduced labeled graphs with the fundamental group isomorphic to $\pi_1(\mathbb{A})$ (resp. G) by $R(\mathbb{A})$ (resp. $R(G)$).

Three types of transformations of labeled graphs plays an important role in studing GBS groups: slide (see fig. 2), induction, $\mathscr{A}^{\pm}$–moves.

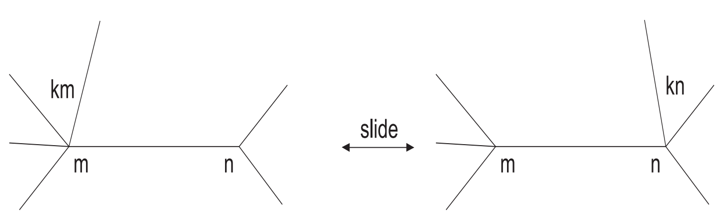

Fig. 2: Slide.

Theorem (Clay M., Forester M. [2]). *Given GBS group G and $\mathbb{A}, \mathbb{B} \in R(G)$, then $\mathbb{A}$ and $\mathbb{B}$ related by a finite sequence of slides, inductions and $\mathscr{A}^{\pm 1}$–moves, with all intermediate labeled graphs reduced.*

An edge e of a labeled graph $\mathbb{A}$ is called *mobile* (see [6]), if there exists $t \in \pi_1(\mathbb{A})$ such that $G_e^t \subset G_e$. There G_e is an edge cyclic group, corresponding to the edge e. In [6] it is proved that there is an algorithm to decide if given edge e mobile or not.

The main result of this paper is another piece of the isomorphism problem for GBS groups:

Theorem. *Given labeled graphs $\mathbb{A}$ and $\mathbb{B}$. Suppose that $\mathbb{A}$ has no more than one mobile edge. Then there is an algorithm to decide if groups $\pi_1(\mathbb{A})$ and $\pi_1(\mathbb{B})$ are isomorphic.*

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