WEAK LAW OF LARGE NUMBERS FOR ARRAYS OF RANDOM ELEMENTS IN P – UNIFORMLY SMOOTH BANACH SPACES

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We estabish a weak law of large numbers for weighted sums of the form $b_n^{-1}\sum_{j=1}^{n} a_j(V_{nj}-c_{nj})$, where $\{V_{nj},\ n\ge 1,\ 1\le j\le n\}$ be array of random elements with values in *p*-uniformly smooth Banach space, $0<b_n\uparrow\infty$, $\{a_n\}$ and $\{c_{nj},\ n\ge 1,\ 1\le j\le n\}$ are suitable sequences.

Keywords: weak law of large numbers, martingale, *p*-uniformly smooth Banach space.

I. INTRODUCTION

For an array $\{V_{nj},\ j\ge 1,\ n\ge 1\}$ of rowwise independent Banach space valued random elements, $\{a_n,\ n\ge 1\}$, $\{b_n\ne 0,\ n\ge 1\}$ be sequences of constants with $0<b_n\to\infty$, $\{c_{ni},\ n\ge 1,\ i\ge 1\}$ be a centering array consisting of element in Banach space *X*. The weak law of large numbers (WLLN) will be established for weighted sum forms

$$\frac{\sum_{i=1}^{n} a_i(V_{ni}-c_{ni})}{b_n}\xrightarrow{P}0,\ \text{as}\ n\to\infty.$$

In [1], A. Adler, A. Rosalsy and A.I. Volodin have considered following WLLN

$$\frac{1}{b_n}\sum_{i=1}^{n} a_i\left[V_{ni}-EV_{ni}I(\|V_{ni}\|\le\frac{b_n}{|a_n|})\right]\xrightarrow{P}0,\quad n\to\infty,$$

with $\{V_{ni},\ i\ge 1,\ n\ge 1\}$ be an array of rowwise independent random elements in Rademacher type p $(1\le p\le 2)$ Banach space *X*, $\{V_{ni},\ i\ge 1,\ n\ge 1\}$ is stochastically dominated by a random element *V*, it means for some finite constant *D* then

$$P(\|V_{ni}\|>t)\le DP(\|V\|>t),\quad t\ge 0,\ n\ge 1,\ i\ge 1,$$

with *V* be a random element in *X* satisfies

$$nP(\|V\|>\frac{b_n}{|a_n|})=o(1),$$

$\{a_n,\ n\geq 1\}$ and $\{b_n,\ n\geq 1\}$ be sequences of constants with $a_n\neq 0,\ b_n>0,\ n\geq 1,$ satisfy the following conditions

$$\frac{b_n}{n\,|\,a_n\,|}\to\infty,\ \frac{b_n}{a_n}\uparrow,\ \sum_{i=1}^{n}|\,a_i\,|^p=o(b_n^p),\ \sum_{i=1}^{n}|\,a_i\,|^p=O(n\,|\,a_n\,|^p),\ \text{and}$$

$$\sum_{i=1}^{n}\frac{b_n^p}{i^2\,|\,a_i\,|^p}=O\left(\frac{b_n^p}{\sum_{i=1}^{n}|\,a_i\,|^p}\right).$$

In this page, we will consider the WLLN form (1.2) with $\{V_{ni},\ i\geq 1,\ n\geq 1\}$ be an array of rowwise adapted in p-uniformly smooth Banach space X $(1\leq p\leq 2)$ (it means for all $n=1,\ 2,\ \ldots,\ \{V_{ni},F_{n,i}\}_{i=1}^{n}$ be an adapted sequence) and with other conditions on sequences $\{a_n\}$ and $\{b_n\}$.

II. PRELIMINARIES

A real separable Banach space X is said to be p-uniformly smooth $(1\leq p\leq 2)$ if

$$\ell(\tau)=\sup\{\frac{\|\,x+y\,\|+\|\,x-y\,\|}{2}-1;\ \forall x,\ y\in X;\ \|\,x\,\|=1,\ \|\,y\,\|=\tau\}\leq C\tau^p,$$

for some constant C.

Theorem 2.1. (P. Assouad, Hoffmann Jorgensen) A real Banach space X is p-uniformly smooth $(1\leq p\leq 2)$ if and only if there exists a positive K such that for all $x,\ y\in X$ we have

$$\|\,x+y\,\|^p+\|\,x-y\,\|^p\leq 2\,\|\,x\,\|^p+K\,\|\,y\,\|^p\ . \tag{2.1}$$

Theorem 2.2. (P. Assouad - 1975) A real separable Banach space X is a p-uniformly smooth $(1\leq p\leq 2)$ if and only if for all $q\geq 1,$ there exists a positive constant C such that for all X valued martingale $\{M_n,\ F_n,\ n\geq 1\}$ we have

$$E\,\|\,M_n\,\|^q\leq CE(\sum_{i=1}^{n}\|\,dM_i\,\|^p)^{q/p},\ \text{(with } dM_i=M_i-M_{i-1}) \tag{2.2}$$

(*Marcinkiewicz – Zygmund inequality*).

In this paper, we assume that X is a p-uniformly smooth Banach space $(1\leq p\leq 2)$, $\{V_{ni},F_{n,j};\ n=1,2,\ldots;1\leq j\leq n\}$ be an array of rowwise adapted random elements in X, $\{F_{n,j}\}$ are sub σ-algebras of σ -algebras F, $F_{n,1}\subset F_{n,2}\subset\ldots\subset F_{n,n},\ \ \forall\ n=1,2,\ldots$

III. MAIN RESULTS

Lemma 3.1. (see [4]) Assume $f_n:R\to R^+$ satisfy: $0\leq f_n\leq 1,\ n=1,2,\ldots$ and $\sup_{n\in N}(xf_n(x))\to 0$ as $x\to\infty.$ Then

$$\sup_{n\in N}\left(\frac{1}{y}\int_0^y xf_n(x)dx\right)\to 0,\ \text{as}\ y\to\infty.$$

Theorem 3.2. Let $\{V_{ni},F_{n,j};\ n=1,2,\ldots;1\leq j\leq n\}$ be an array of rowwise adapted random elements in X, $\{V_{ni},\ n=1,2,\ldots;1\leq j\leq n\}$ is stochastically dominated by random element V,

$E\|V\|^{p/2}<\infty$, $\{a_n\},\{b_n\}$ be sequences of constants with $a_n\neq 0$, $b_n>0$, $n\geq 1$, satisfy the following conditions:

$$\frac{b_n}{|a_n|}\uparrow\infty,\quad \sum_{j=1}^{n}|a_j|^p=O(b_n^{p/2}|a_n|^{p/2})\quad (or\ \sum_{j=1}^{n}|a_j|^p=o(b_n^{p/2}|a_n|^{p/2})).\qquad (3.1)$$

If

$$nP(\|V\|>\frac{b_n}{|a_n|})\to 0,\ \text{as}\ n\to\infty \qquad (3.2)$$

then

$$\frac{1}{b_n}\sum_{j=1}^{n}a_j[V_{nj}-E(V_{nj}I(\|V_{nj}\leq\frac{b_n}{|a_n|})/F_{n,j-1})]\xrightarrow{P}0,\ \text{as}\ n\to\infty. \qquad (3.3)$$

Proof. Put $c_n=b_n/|a_n|$, $n\geq 1$, $c_0=0$, and $U_{nj}=V_{nj}I(\|V_{nj}\|\leq c_n)$. Clearly $E\|U_{nj}\|<+\infty$, for all $n=1,2,\ldots;1\leq j\leq n$.

For arbitrary $\varepsilon>0$ we have

$$P(\|\sum_{j=1}^{n}a_j(V_{nj}-U_{nj})\|>b_n\varepsilon)\leq P(\sum_{j=1}^{n}a_jV_{nj}\neq\sum_{j=1}^{n}a_jU_{nj})$$
$$\leq P(\bigcup_{j=1}^{n}\{\|V_{nj}\|>c_n\})$$
$$\leq\sum_{j=1}^{n}P(\|V_{nj}\|>c_n)$$
$$\leq DnP(\|V\|>c_n)=o(1).\ \text{(by (3.2))}$$

So that it suffices to prove that

$$\frac{1}{b_n}\sum_{j=1}^{n}a_j[U_{nj}-E(U_{nj}/F_{n,j-1})]\xrightarrow{P}0,\ \text{as}\ n\to\infty. \qquad (3.4)$$

For arbitrary $\varepsilon>0$

$$P(\|\sum_{j=1}^{n}a_j[U_{nj}-E(U_{nj}/F_{n,j-1})]\|>b_n\varepsilon)\leq\frac{1}{\varepsilon^p b_n^p}E(\|\sum_{j=1}^{n}a_j[U_{nj}-E(U_{nj}/F_{n,j-1})]\|^p). \qquad (3.5)$$

Note that, with all $n=1,2,\ldots,$ $\sum_{j=1}^{k}a_j[U_{nj}-E(U_{nj}/F_{n,j-1})];F_{n,k}\}_{k=1}^{n}$ be a martingale. So, applies (2.2) with $q=p$ we have

$$\frac{1}{\varepsilon^p b_n^p}E(\|\sum_{j=1}^{n}a_j[U_{nj}-E(U_{nj}/F_{n,j-1})]\|^p)$$
$$\leq\frac{C}{\varepsilon^p b_n^p}E(\sum_{j=1}^{n}\|a_j[U_{nj}-E(U_{nj}/F_{n,j-1})]\|^p)$$
$$=\frac{C}{\varepsilon^p b_n^p}\sum_{j=1}^{n}|a_j|^p\ E\|[U_{nj}-E(U_{nj}/F_{n,j-1})]\|^p$$
$$\leq\frac{C}{\varepsilon^p b_n^p}\sum_{j=1}^{n}|a_j|^p\ E[2\|U_{nj}\|^p+K\|E(U_{nj}/F_{n,j-1})\|^p]\quad\text{(by (2.1))}$$
$$\leq\frac{C}{\varepsilon^p b_n^p}\sum_{j=1}^{n}|a_j|^p\ E[2\|U_{nj}\|^p+KE(\|U_{nj}\|^p/F_{n,j-1})]$$