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A Political Economy Analysis

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Abstract
This study considers public education policy and its impact on growth and welfare across generations. In particular, the study compares two fiscal perspectives—tax finance and debt finance—and shows that in a competitive equilibrium context, the growth and utility in the debt-finance case could be higher than those in the tax-finance case in the long run. However, the result is reversed when the policy is shaped by politics. Voters choose debt finance, despite its worse performance, in each period because a current generation can pass the cost of debt repayment to future generations.

• Keywords: Economic growth, Human capital, Public debt, Political equilibrium
• JEL Classification: D70, E24, H63,
1 Introduction

This paper considers the political determinants of fiscal policy and its impact on growth and welfare across generations, particularly related to public education expenditures, which work as both an intergenerational transfer from parents to children and an engine of economic growth. Parents with an altruistic concern for their children are likely to support public education at the cost of a tax burden because they can benefit from highly educated children. However, when the government has access to debt financing, parents may prefer debt to taxes because debt enables the parents to shift the fiscal burden to future generations.

The discussion above leads to the following questions: how do debt and tax financing differ in terms of growth and welfare across generations, and which fiscal stance is adopted via voting. To consider these questions, this study presents a three-period overlapping-generations model with physical and human capital accumulation (see, e.g., Lambrecht, Michel, and Vidal, 2005; Kunze, 2014). Parents care about their children’s human capital. Public education spending and parental human capital are inputs in the process of human capital formation. Government spending is financed by tax on labor income as well as public bond issues if the government has access to debt financing.

Within this framework, this study first investigates an economic equilibrium where fiscal policy is exogenously given. In particular, we assume a fixed education expenditure-GDP ratio, and compare the case of financing solely through a labor income tax to the case of financing through tax and public bond issues. The analysis shows that under a certain condition, the growth and utility in the debt-finance case are higher than those in the tax-finance case in the long run. Debt financing passes a part of the burden of the expenditure to future generations. However, the costs for future generations are outweighed by the benefits they will enjoy through human capital accumulation. The result suggests that the debt-financed public education spending may be desirable from the perspective of growth and utility.

The result above depends critically on the assumption of a fixed education expenditure-GDP ratio. In the real world, however, the expenditure is determined through political competition. To demonstrate endogenous determinants of the expenditure and their impacts, we present the politics of fiscal policy formation. In each period, parents are assumed to participate in voting, and the government chooses fiscal policy to maximize their utility. Given this process, we provide a characterization of political equilibrium in both the tax- and debt-finance cases, and compare them in terms of growth rates. We find that the growth rate in the debt-finance case is lower than that in the tax-finance case. The debt overrides capital accumulation and adds the cost of debt repayment, reducing the government’s available resources, and thus decreases public education expenditure as
an engine of economic growth. The peculiar negative impact on the expenditure in the
debt-finance case is the key to the result.

We undertake further analysis of fiscal policy formation by introducing voting on fiscal
rules. Specifically, we assume that in each period, the government proposes two fiscal
rules, the tax-finance and the debt-finance rules, and one is chosen based on maximizing
parental utility. Then, for a given fiscal rule, the parents vote on fiscal policy. In this
setting, we show that debt finance is chosen in each period because parents aiming to
maximize their utility do not care about debt repayment costs for future generations.
Following the same reasoning, every generation finds it optimal to choose debt financing.
However, this choice is not optimal in the long run. As described above, debt reduces
public education expenditures and thus retards human capital formation. Because of this
negative effect, future generations are worse off than in the tax-finance case. The result
suggests that the current generation’s myopic choice in fiscal rules results in loss of welfare
for future generations.

The rest of the paper is structured as follows. We first present a literature survey
in Subsection 1.1. Thereafter, Section 2 presents the model. Section 3 characterizes the
economic equilibrium and Section 4 describes the political equilibrium and investigates
the choice of fiscal rules and the impact on growth and welfare. Section 5 offers concluding
remarks.

1.1 Literature Review

Our study is related to research into public education and economic growth in the pres-
ence of parental altruism toward children. Examples include studies from Glomm and
competitive equilibrium context assuming that the central government can control policy
processes and outcomes across periods. Under this assumption, these researchers investi-
gate how changes in educational policy affect growth and welfare across generations and
the optimal policy in terms of long-term growth and/or welfare.

Several studies attempted to relax the assumption by introducing voting into the
process of policymaking. These studies tend to focus on factors affecting policymaking,
such as aging (Zhang, Zhang, and Lee, 2003; Gradstein and Kaganovich, 2004; Kunze,
2014), inequality (Saint-Paul and Verdier, 1993), expectations of future policy (Glomm
and Ravikumar, 1995), private education as an alternative to public education (Gradstein
and Justman, 1996), social cohesion (Gradstein and Justman, 2002), and social security
(Kemnitz, 2000; Poutvaara, 2006; Soares, 2006; Gonzalez-Eiras and Niepelt, 2012; Iturbe-
Ormaetxe and Valera, 2012; Kaganovich and Meier, 2012; Naito, 2012; Lancia and Russo,
2015; Ono, 2015; Ono and Uchida, 2015). However, all of these studies assume financing
from taxes where the government budget constraint is balanced each period. In other words, debt financing of public education is abstracted away from their analyses.

Zhang (2003) and Greiner (2008) analyzed debt financing. Zhang (2003) demonstrates the optimal policy for public education when the government can issue public bonds, and Greiner (2008) investigates the impact of debt-financed public education on economic growth. These studies are thus based in the competitive equilibrium context, and say nothing about the political determinants of debt finance policy. Our study instead demonstrates endogenous determinants of debt-financed public education expenditures and its impact on growth and welfare across generations. With this analysis, we can evaluate the relative performance of the two fiscal stances and explain why debt finance has lingered through the past decades in many developed countries despite its poorer performance.

2 Model

The discrete time economy starts at period 0 and consists of overlapping generations. Individuals are identical within a generation, and live for three periods: youth, middle, and elderly ages. Each middle-aged individual gives birth to $1 + n$ children. The middle-aged population for the period $t$ is $N_t$, and the population grows at a constant rate of $n(-1): N_{t+1} = (1 + n)N_t$.

2.1 Individuals

Individuals display the following economic behavior over their life cycles. During youth, individuals make no economic decisions and receive public education financed by the government. In middle age, individuals work, receive market wages, and make tax payments. They use after-tax income for consumption and savings. Individuals retire in their elderly years and receive and consume returns from savings.

Consider an individual born in period $t - 1$. In period $t$, the individual is middle aged and endowed with $h_t$ units of human capital. The individual supplies them inelastically in the labor market, and obtains labor income $w_t h_t$, where $w_t$ is the wage rate per efficient unit of labor in period $t$. After paying tax $\tau_t w_t h_t$, where $\tau_t \in (0, 1)$ is the period $t$ income tax rate, the individual distributes the after-tax income between consumption $c_t$ and savings invested in physical capital $s_t$. Therefore, the period $t$ budget constraint for the middle age becomes:

$$c_t + s_t \leq (1 - \tau_t)w_t h_t.$$

The period $t + 1$ budget constraint in the elderly age is

$$d_{t+1} \leq R_{t+1} s_t,$$
where \(d_{t+1}\) is consumption, \(R_{t+1}(>0)\) is the gross return from investment in capital, and \(R_{t+1}s_t\) is the return from savings.

Period \(t\) middle-aged individuals care about their children’s income, \(w_{t+1}h_{t+1}\). The children’s human capital in period \(t + 1\), \(h_{t+1}\), is a function of government spending on public education, \(x_t\), and the parent’s human capital, \(h_t\). In particular, \(h_{t+1}\) is formulated by the following equation:

\[
h_{t+1} = D(x_t)^\eta (h_t)^{1-\eta},
\]

where \(D(>0)\) is a scale factor, and \(\eta \in (0, 1)\) denotes the elasticity of education technology with respect to public education spending.

We note that private investment in education may also contribute to human capital formation. For example, parents’ time (Glomm and Ravikumar, 1995, 2001, 2003; Glomm and Kaganovich, 2008) or spending (Glomm, 2004; Lambrecht, Michel, and Vidal, 2005; Kunze, 2014) devoted to education may complement public education. In the present study, we abstract private education from the analysis to simplify the presentation of the model and to focus on the effect of public education on growth and utility.

We assume that parents are altruistic toward their children and are concerned about their income in middle age, \(w_{t+1}h_{t+1}\). The preferences of an individual born in period \(t - 1\) are specified by the following expected utility function of the logarithmic form:

\[
U_t = \ln c_t + \ln \left(1 + \frac{1}{1+\beta} \cdot (1 - \tau_t)w_t h_t \right),
\]

where \(\beta \in (0, 1)\) is a discount factor and \(\gamma(>0)\) denotes the intergenerational degree of altruism. We substitute the budget constraints and human capital production function into the utility function to form the following unconstrained maximization problem:

\[
\max \ln \left[\left(1 - \tau_t\right)w_t h_t - s_t\right] + \beta \ln R_{t+1}s_t + \gamma \ln w_{t+1}D(x_t)^\eta (h_t)^{1-\eta}.
\]

By solving the problem, we obtain the following savings and consumption functions:

\[
s_t = \frac{\beta}{1+\beta} \cdot (1 - \tau_t)w_t h_t, \quad \tag{1}
\]

\[
c_t = \frac{1}{1+\beta} \cdot (1 - \tau_t)w_t h_t \quad \text{and} \quad d_{t+1} = \frac{\beta R_{t+1}}{1+\beta} \cdot (1 - \tau_t)w_t h_t. \quad \tag{2}
\]

### 2.2 Firms

Each period contains a continuum of identical firms that are perfectly competitive profit maximizers. According to Cobb–Douglas technology, they produce a final good \(Y_t\) using two inputs, aggregate physical capital \(K_t\) and aggregate human capital \(H_t \equiv N_t h_t\). The aggregate output is given by:

\[
Y_t = A(K_t)^\alpha (H_t)^{1-\alpha},
\]
where $A(>0)$ is a scale parameter and $\alpha \in (0,1)$ denotes the capital share.

Let $k_t \equiv K_t/H_t$ denote the ratio of physical to human capital. The first-order conditions for profit maximization with respect to $H_t$ and $K_t$ are:

$$w_t = (1 - \alpha)A(k_t)^\alpha, \quad \rho_t = \alpha A(k_t)^{\alpha-1},$$

(3)

where $w_t$ and $\rho_t$ are labor wages and the rental price of capital, respectively. The conditions state that firms hire human and physical capital until the marginal products are equal to the factor prices.

### 2.3 Government Budget Constraint

Public education expenditures are financed by both tax on labor income and public bond issues. Let $B_t$ denote the aggregate inherited debt. A government budget constraint in period $t$ is:

$$B_{t+1} + \tau_t w_t h_t N_t = N_{t+1} x_t + R_t B_t,$$

where $B_{t+1}$ is newly issued public bonds, $\tau_t w_t h_t N_t$ is the aggregate labor income tax revenue, $N_{t+1} x_t$ is the aggregate expenditure for public education, and $R_t B_t$ is debt repayment. We assume a one-period debt structure to derive analytical solutions from the model, and assume that the government in each period is committed to not repudiating the debt.

By dividing both sides of the above expression, we obtain a per-capita from of the constraint:

$$(1 + n)\hat{b}_{t+1} + \tau_t w_t h_t = (1 + n)x_t + R_t \hat{b}_t,$$

(4)

where $\hat{b}_t \equiv B_t/N_t$ is the per-capita public debt. We use the notation $\hat{b}_t$ rather than $b_t$ to distinguish the per-capita public debt, $\hat{b}_t \equiv B_t/N_t$, from the public debt per human capital, $b_t \equiv B_t/H_t$, which we introduce in the next section.

In an alternative case, the government imposes a rule of keeping a balanced budget in each period. That is, the government satisfies the following constraint:

$$\begin{cases} \tau_0 w_0 h_0 = (1 + n)x_0 + R_0 \hat{b}_0, \\ \tau_t w_t h_t = (1 + n)x_t, \quad t \geq 1. \end{cases}$$

(5)

This rule is stricter than the rule where debt cannot increase across periods (see, e.g., Azzimonti, Battaglini, and Coate, 2010). Our study thus adopts a stricter rule to demonstrate the impact of debt financing public expenditure simply.

### 3 Economic Equilibrium

Public bonds are traded in a domestic capital market. The market clearing condition for capital is $B_{t+1} + K_{t+1} = N_t s_t$, which expresses the equality of total savings by the
middle-aged population in period $t$, $N_{t} s_{t}$, to sum of the stocks of aggregate public debt and aggregate physical capital at the beginning of period $t + 1$, $B_{t+1} + K_{t+1}$. Using $k_{t+1} \equiv K_{t+1} / H_{t+1}$, $h_{t+1} = H_{t+1} / N_{t+1}$, and the savings function in (1), we can rewrite the condition as:

$$(1 + n) \cdot \left( k_{t+1} h_{t+1} + \hat{b}_{t+1} \right) = \frac{\beta}{1 + \beta} \cdot (1 - \tau_{t}) w_{t} h_{t}. \quad (6)$$

The following defines the economic equilibrium in the present model.

**Definition 1.** Given a sequence of policies, $\{\tau_{t}, x_{t}\}_{t=0}^{\infty}$, an economic equilibrium is a sequence of allocations $\{c_{t}, d_{t}, s_{t}, k_{t+1}, \hat{b}_{t+1}, h_{t+1}\}_{t=0}^{\infty}$ and prices $\{\rho_{t}, w_{t}, R_{t}\}_{t=0}^{\infty}$ with the initial conditions $k_{0} (>0)$, $\hat{b}_{0} (\geq 0)$ and $h_{0} (>0)$ such that (i) given $(w_{t}, R_{t+1}, \tau_{t}, x_{t})$, $(c_{t}, c_{t+1}^{o}, s_{t})$ solves the utility maximization problem; (ii) given $(w_{t}, \rho_{t})$, $k_{t}$ solves a firm’s profit maximization problem; (iii) given $(w_{t}, h_{t}, k_{t}, \hat{b}_{t})$, $(\tau_{t}, x_{t}, \hat{b}_{t+1})$ satisfies the government budget constraint; (iv) $\rho_{t} = R_{t}$ holds; and (v) the capital market clears: $(1 + n) \cdot \left( k_{t+1} h_{t+1} + \hat{b}_{t+1} \right) = s_{t}$.

In the following, we consider two cases: a ”tax-finance” case where public education expenditures are financed solely through labor income tax, and a ”debt-finance” case where the expenditures are financed by both labor income taxes and public bond issues. We characterize an economic equilibrium for each case, and then compare the two cases in terms of steady-state growth rates and utility. In particular, we identify the condition where shifting from tax to debt finance improves steady-state growth and utility.

### 3.1 Tax-finance Economic Equilibrium

In the tax-finance case, the government budget constraint is given by (5). Note that here we impose the following expenditure rule:

$$\frac{N_{t+1} x_{t}}{Y_{t}} = \frac{(1 + n) x_{t}}{A(k_{t})^{\alpha} h_{t}} = X_{\text{econ}} \in (0, 1 - \alpha), \quad (7)$$

where $X_{\text{econ}}$ is a constant parameter. The rule indicates that the government fixes the ratio of public education spending to GDP at a constant rate. The upper limit of $X_{\text{econ}}$, $1 - \alpha$, implies that the government can use a part of labor income for public education expenditures through taxation. We introduce this rule to compare the tax- and debt-finance cases under the same expenditure structure.

Given $X_{\text{econ}}$, a tax rate is determined satisfying the constraint (5). Thus, the tax rate under the expenditure rule in (7) is given by:

$$\begin{align*}
\tau_{0} &= \frac{X_{\text{econ}}}{1 - \alpha} + \frac{\alpha}{1 - \alpha} \cdot \frac{k_{0}}{k_{0} h_{0}}, \\
\tau_{t} &= \frac{X_{\text{econ}}}{1 - \alpha} \quad \text{for } t \geq 1.
\end{align*} \quad (8)$$
Substituting the tax rate and the human capital production function into the capital market clearing condition in (6) leads to the following law of the motion of capital:

\[
\begin{align*}
    k_t &= \frac{\beta}{1+\beta} \left( (1 - X_{\text{econ}}) \right) (A(k_0)^{\alpha})^{1-\eta}, \\
    \frac{\beta}{1+\beta} \left( (1 - X_{\text{econ}}) \right) &\left( A(k_t)^{\alpha} \right)^{1-\eta} \text{ for } t \geq 1
\end{align*}
\]

The expression suggests that there is a unique, stable, steady-state equilibrium described by:

\[
k = k|_{t=\text{tax}} = \left[ \frac{\beta}{1+\beta} \left( (1 - X_{\text{econ}})(1 - \alpha) \right) \left( (1 + n)D \left( X_{\text{econ}} \right) \right)^{\eta} \right]^{1/(1-\alpha(\eta))}.
\]

The following proposition summarizes the results up to this point.

**Proposition 1.** Assume the expenditure rule in (7). There is a unique, stable steady-state tax-finance economic equilibrium.

### 3.2 Debt-finance Economic Equilibrium

We next consider the debt-finance case, where the government budget constraint is given by (4). We follow the expenditure rule in the tax finance case to compare the two cases using the same expenditure structure. We also assume that the tax rate is constant across periods, \( \tau = \tau_{\text{debt}} \) for all \( t \), and that \( \tau_{\text{debt}} \) satisfies:

\[
\tau_{\text{debt}} \leq \frac{X_{\text{econ}}}{1 - \alpha}.
\]

The assumption in (10) indicates that the tax revenue is not sufficient to finance public education expenditures, and the government covers the revenue shortfall by issuing public bonds.

Adopting the profit maximization conditions in (3), we can reformulate the government budget constraint in the debt-finance case as:

\[
(1 + n)\hat{b}_{t+1} + \tau_{\text{deb}}(1 - \alpha)A(k_t)^{\alpha} h_t = (1 + n)x_t + \alpha A(k_t)^{\alpha-1} \hat{b}_t.
\]

Let \( b_t \equiv \hat{b}_t / h_t = B_t / H_t \) denote public debt per human capital. The above expression is thus rewritten as:

\[
(1 + n)b_{t+1} \frac{h_{t+1}}{h_t} + \tau_{\text{deb}}(1 - \alpha)A(k_t)^{\alpha} = X_{\text{econ}}A(k_t)^{\alpha} + \alpha A(k_t)^{\alpha-1} b_t.
\]

Using the human capital production function, the government budget constraint in (11) is reformulated as:

\[
(1 + n)b_{t+1} D \left( \frac{X_{\text{econ}}}{1 + n} A(k_t)^{\alpha} \right)^{\eta} = [X_{\text{econ}} - \tau_{\text{deb}}(1 - \alpha)] A(k_t)^{\alpha} + \alpha A(k_t)^{\alpha-1} b_t.
\]
This is the first key equation in the debt-finance economic equilibrium. The other key equation is the capital market clearing condition, which can be derived by substituting (12) into the capital market clearing condition in (6):

\[
k_{t+1} = \frac{1}{1+n}D \left( \frac{X_{\text{econ}}}{1+n} \right)^\eta \\
\times \left[ \left\{ \frac{\beta}{1+\beta} (1 - \tau_{\text{debt}})(1 - \alpha) - (X_{\text{econ}} - \tau_{\text{debt}}(1 - \alpha)) \right\} (A(k_t)\alpha)^{1-\eta} - \alpha (A(k_t)\alpha)^{1-\eta} \frac{b_t}{k_t} \right]. \tag{13}
\]

Therefore, given an initial condition, \(k_0 > 0\) and \(b_0 \geq 0\), the sequence of physical capital and public debt per human capital, \(\{k_t, b_t\}\), in a debt-finance economic equilibrium is characterized by (12) and (13). The following proposition establishes the condition for the existence and stability of a steady-state debt-finance economic equilibrium.

**Proposition 2.** Assume the expenditure rule in (7) and that the following conditions hold:

\[\tau_{\text{debt}}(1 - \alpha) < X_{\text{econ}} < \bar{X}_{\text{econ}},\]

\[0 < \tau_{\text{debt}} \leq 1 - \frac{1+\beta}{\beta} \cdot \frac{\alpha}{1-\alpha}; \text{ and } \alpha < \frac{\beta}{1+2\beta},\]

where

\[\bar{X}_{\text{econ}} \equiv \alpha + \frac{\beta}{1+\beta} (1 - \tau_{\text{debt}})(1 - \alpha) + \tau_{\text{debt}}(1 - \alpha) - 2\alpha \left( \frac{\beta}{1+\beta} (1 - \tau_{\text{debt}}) \frac{1-\alpha}{\alpha} \right)^{1/2} .\]

There are two steady-state debt-finance economic equilibria with \(b > 0\), denoted by \(e_l\) and \(e_h\), where \(e_l\) (\(e_h\)) is distinguished by a lower (higher) \(k\). Then, \(e_l\) is a saddle, and \(e_h\) is a sink.

**Proof.** See Appendix A.1.

Figure 1 illustrates an example phase diagram for the planner system (12) and (13). The phase line denoted by \(KK\) is a set of points \((k, b)\) for which there is no change in \(k\) in the capital market clearing condition in (13). The phase line denoted by \(BB\) is a set of points \((k, b)\) for which there is no change in \(b\) in the government budget constraint in (12). The vertical line at \(k = \hat{k}\) indicates that financial balance state varies around this point, where \(\hat{k}\) is defined by:

\[\hat{k} \equiv \left[ \frac{\alpha A}{(1+n)D \left( \frac{X_{\text{econ}}}{1+n} A \right)^\eta} \right]^{1/(1-\alpha(1-\eta))} .\]
Below (above) $k = \hat{k}$, the government budget has a negative (positive) debt in the steady state. The two intersections in phase lines $KK$ and $BB$ correspond to the two steady-state equilibria with positive debt, which are common in the literature (see, e.g., Azariadis, 1993; de la Croix and Michel, 2002).

Proposition 2 demonstrates the three conditions for the existence of multiple steady-state equilibria with positive debt. The first condition determines the range of $X_{econ}$ for the two steady-state equilibria with $b > 0$. In particular, the first inequality condition, $\tau_{debt}(1 - \alpha) < X_{econ}$ implies that the government must finance part of the expenditure by borrowing in the capital market. If the condition fails, tax revenue is sufficient to cover the expenditure, so the steady-state equilibrium would have a negative public debt, which is out of scope for this study.

The second inequality condition, $X_{econ} < \tilde{X}_{econ}$ ensures that the two steady-state equilibria exist. If the condition fails, the government must finance large expenditures for public education by issuing public bonds. However, this induces high debt repayment burdens on households, which creates a negative income effect on saving and thus implies a low debt level that clears the capital market. Therefore, given $\hat{k}$, the government budget constraint is satisfied for a high level of public debt, while the capital market clears for a low level of public debt. That is, there is no debt level that satisfies both the government budget constraint and the capital market clearing condition if $X_{econ}$ is above the critical level.

The second condition requires a tax rate below the critical value of $1 - \frac{1 + \beta}{\beta} \cdot \frac{\alpha}{1 - \alpha}$. This condition determines the upper bound of $X_{econ}$ given in the first condition. The third condition ensures that the tax rate in the second condition is non-empty.

### 3.3 Growth and Utility in the Steady States

The previous two subsections describe the steady-state equilibrium for the tax- and debt-finance cases. To evaluate their relative performance, we here compare the two cases in terms of growth rate and utility in the steady state, and show the results.

**Proposition 3.** (i) The steady-state growth rate in the debt-finance economic equilibrium is higher than that in the tax-finance economic equilibrium if $X_{econ} > (1 - \alpha)(1 - \alpha(1 + \beta)/(1 - \alpha)\beta)$; (ii) the steady-state utility in the debt-finance economic equilibrium is higher than that in the tax-finance economic equilibrium if $X_{econ} > (1 - \alpha)(1 - \alpha(1 + \beta)/(1 - \alpha)\beta)$ and $\gamma > (1 - \alpha)\beta/\alpha$.

**Proof.** See Appendix A.3.

To compare the two cases, we here denote by $h'/h_{tax}$ and $U|_{tax}$ the steady-state growth rate and utility for the tax-finance case, respectively. $z|_{tax}$ and $h_{t, tax}$ denote the
steady-state level of $z(=k, b)$ and the period-$t$ human capital for the tax-finance case, respectively. The corresponding notations are $h'/h|_{\text{debt}}, U|_{\text{debt}}, z|_{\text{debt}}$ and $h_t|_{\text{debt}}$ for the debt-finance case. The variable $h'$ in the expressions $h'/h|_{\text{tax}}$ and $h'/h|_{\text{debt}}$ denotes the next-period $h$.

To understand the results in Proposition 3, let us first compare the steady-state growth rates. Given the expenditure rule in (7), the public education expenditures are $x = X_{\text{econ}}A(k)\alpha h/(1 + n)$ for both cases. We substitute this expression into the human capital production function, $h' = D(x)^{\eta}(h)^{1-\eta}$, to obtain:

$$
\frac{h'}{h} = D \left( \frac{X_{\text{econ}}}{1 + n} A(k)^{\alpha} \right)^{\eta}.
$$

This is common for both cases.

Substituting the steady-state level of capital in (9) into (14) yields the steady-state growth rate in the tax-finance economic equilibrium:

$$
\frac{h'}{h} \bigg|_{\text{tax}} = \left[ D \left( \frac{X_{\text{econ}}}{1 + n} A \right)^{\eta} \right]^{\frac{1-\alpha}{1-\eta}} \cdot \left[ \frac{\beta}{1 + \beta} \left( (1 - \alpha) - X_{\text{econ}} \right) A \right]^{\frac{\alpha - \eta}{1 - \eta}}. \tag{15}
$$

In the debt-finance case, the steady-state capital level is higher than $\hat{k}$, as illustrated in Figure 1. Therefore, the growth rate in the debt-finance economic equilibrium satisfies $h'/h|_{\text{debt}} > \left( X_{\text{econ}} A(\hat{k})^\alpha/(1 + n) \right)^{\eta}$, that is,

$$
\frac{h'}{h} \bigg|_{\text{debt}} > \left[ D \left( \frac{X_{\text{econ}}}{1 + n} A \right)^{\eta} \right]^{\frac{1-\alpha}{1-\eta}} \cdot \left[ \frac{\alpha A}{1 + n} \right]^{\frac{\alpha - \eta}{1 - \eta}}. \tag{16}
$$

The expressions in (15) and (16) suggest that the ratio of public education expenditure to GDP, represented by $X_{\text{econ}}$, affects the steady-state growth rates in several ways. First, a higher expenditure ratio increases the growth rate for both cases, as shown by the term (#1) in (15) and (16). However, the tax-finance case also contains a negative effect of public education expenditures as shown by the term (#2) in (15). A higher ratio implies a larger household tax burden. This discourages savings and capital accumulation, thereby reducing public education expenditures and economic growth in the next period. This negative effect does not appear in the debt-finance case, as demonstrated by the term (#3) in (16) because the public bond issues in the debt-finance case compensates for the tax burden. Therefore, the growth rate in the debt-finance case is higher than that in the tax-finance case if the term (#3) outweighs the term (#2); that is, if $X_{\text{econ}} > (1 - \alpha) (1 - \alpha (1 + \beta)/(1 - \alpha) \beta)$. 

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Next, we compare the steady-state utility in both cases. Given \( k \) and \( b \) in the steady state, \( U|_{\text{tax}} \) and \( U|_{\text{debt}} \) are computed as follows:

\[
U|_{\text{tax}} = (1 + \beta) \ln \left( (1 - \alpha) - X_{\text{econ}} \right) A (k|_{\text{tax}})^\alpha + \{ \beta (\alpha - 1) + \gamma \alpha \} \ln k|_{\text{tax}} \\
+ \gamma \eta \ln \frac{X_{\text{econ}}}{1 + n} A (k|_{\text{tax}})^\alpha + (1 + \beta + \gamma) \ln h_t,\text{tax} + C_0, \tag{17}
\]

\[
U|_{\text{debt}} = (1 + \beta) \ln \left[ (1 - \alpha) - X_{\text{econ}} \right] A (k|_{\text{debt}})^\alpha + \left\{ -\alpha A (k|_{\text{tax}})^{\alpha - 1} + (1 + n) \frac{h_t}{h_{\text{debt}}} \right\} \cdot b|_{\text{debt}} \\
+ \{ \beta (\alpha - 1) + \gamma \alpha \} \ln k|_{\text{debt}} + \gamma \eta \ln \frac{X_{\text{econ}}}{1 + n} A (k|_{\text{debt}})^\alpha + (1 + \beta + \gamma) \ln h_t,\text{debt} + C_0, \tag{18}
\]

where \( C_0 \) includes constant terms. Appendix A.3 provides the derivation of (17) and (18) and the definition of \( C_0 \).

For both cases, the first term represents disposable income, the second term includes the rate of return from savings and the next generation’s wage, the third term shows the public education expenditures, and the fourth term summarizes the human capital accumulation effect. By comparing (17) and (18), we find that \( U|_{\text{debt}} > U|_{\text{tax}} \) holds if each term in Eq. (18) is larger than the corresponding term in Eq. (17).

After some calculation, we find that (i) in the debt-finance case, the debt repayment \( -\alpha A (k|_{\text{tax}})^{\alpha - 1} b|_{\text{debt}} \) is equal to the debt issue revenue \( (1 + n) \cdot (h'/h|_{\text{debt}}) \cdot b|_{\text{debt}} \) in the steady state, and (ii) if \( h'/h|_{\text{debt}} > h'/h|_{\text{tax}} \) holds, then \( k|_{\text{debt}} > k|_{\text{tax}} \) holds; that is, a higher growth rate results in higher steady-state capital. These observations suggest that the first, second, third, and fourth terms in Eq. (18) are larger than those in Eq. (17) if \( h'/h|_{\text{debt}} > h'/h|_{\text{tax}} \) and \( \beta (\alpha - 1) + \gamma \alpha > 0 \). Therefore, the debt-finance case attains a higher steady-state utility if \( X_{\text{econ}} > (1 - \alpha) (1 - \alpha(1 + \beta)/(1 - \alpha)\beta) \) and \( \gamma > (1 - \alpha)\beta/\alpha \).

The result in Proposition 3 suggests that under the condition \( X_{\text{econ}} \), debt-financed spending is desirable in terms of steady-state growth and utility. In fact, debt financing enables us to pass a part of the burden of the expenditure to the future generations who benefit from it through human capital accumulation. However, in the real world, the expenditure-GDP ratio does not necessarily satisfy the condition \( X_{\text{econ}} \) in Proposition 3 because it is determined through political competition influenced by economic conditions such as physical and human capital accumulation. Therefore, the following questions arise: in what condition will voters support debt finance, and which financing system is desirable when fiscal policy is endogenous? We address these questions in the next section.
4 Political Equilibrium

This section demonstrates the politics of fiscal policy formation for the tax- and debt-finance cases. In each period, the young, the middle-aged, and the elderly can participate in voting. However, the young and the elderly are indifferent between any two policies because they owe no burden and obtain no benefit. Therefore, the government’s aim in each period is to maximize the utility of the middle-aged, given by:

\[ V_t = (1 + \beta) \ln \left\{ (1 - \alpha)A(k_t) - \tau_t w_t h_t \right\} + \beta(n - 1) + \gamma \alpha \ln k_{t+1} + \gamma \eta \ln x_t + \gamma(1 - \eta) \ln h_t + C_0. \] (19)

The term \( (1 - \alpha)A(k_t) - \tau_t w_t h_t \) shows the after-tax income, and thus represents utility from consumption. The second term includes the utility from the return on savings, \( \beta(n - 1) \ln k_{t+1} \), and the utility from the next generation’s wage income, \( \gamma \alpha \ln k_{t+1} \). The third and fourth terms correspond to the utility from the next generation’s human capital, \( h_{t+1} \).

Given the state variables \( k_t, h_t, \) and \( \hat{b}_t \), the period-\( t \) government chooses fiscal policy to maximize \( V_t \) subject to the capital-market-clearing condition and the government budget constraint. Formally, Definition 2 describes the political equilibrium in this framework.

**Definition 2.** A political equilibrium is a sequence of policies \( \{\tau_t, x_t\}_{t=0}^\infty \), allocations \( \{c_t, d_t, s_t, k_{t+1}, \hat{b}_{t+1}, h_{t+1}\}_{t=0}^\infty \), and prices \( \{p_t, w_t, R_t\}_{t=0}^\infty \) with the initial conditions \( k_0(>0), h_0(>0) \) and \( \hat{b}_0(\geq 0) \) such that (i) the conditions in Definition 1 (Economic Equilibrium) are satisfied; and (ii) in period \( t(\geq 0) \), the government chooses \( x_t \) and \( \tau_t \) (or \( \hat{b}_{t+1} \)) to maximize \( V_t \) subject to the capital market clearing condition and the government budget constraint, given by \( k_t, h_t, \) and \( \hat{b}_t \).

In the following analysis, we first characterize a political equilibrium in the tax-finance case in Subsection 4.1 and that in the debt-finance case in Subsection 4.2. Then, in Subsection 4.3, we compare the two cases in terms of government expenditure-GDP ratio, the tax rate, and the steady-state growth rate. In Subsection 4.4, we demonstrate the endogenous choice of the two fiscal rules and its long-term consequence in terms of utility.

4.1 Tax-finance Political Equilibrium

In the tax-finance case, (5) gives the government budget constraint. Substituting this into \( V_t \) in (19) leads to:

\[ V_t = (1 + \beta) \ln (i_t - (1 + n)x_t) + \beta(n - 1) + \gamma \alpha \ln k_{t+1} + \gamma \eta \ln x_t + \gamma(1 - \eta) \ln h_t + C_0. \] (20)

where \( i_t \) is defined by

\[ i_t \equiv (1 - \alpha)A(k_t)^{\alpha} h_t - \alpha A(k_t)^{\alpha-1} \hat{b}_t. \]
The term \((1 - \alpha)A(k_t)^\alpha h_t\) denotes labor income, and the term \(\alpha A(k_t)^{\alpha - 1} \hat{b}_t\) denotes debt repayment in period 0 only. Therefore, \(i_t\) represents the government’s available resources for its expenditures. Using the definition of \(i_t\) and the human capital formation function, \(h_{t+1} = D(x_t)^\eta (h_t)^{1 - \eta}\), we can reformulate the capital-market-clearing condition in (6) as:

\[
(1 + n)k_{t+1}D(x_t)^\eta (h_t)^{1 - \eta} = \frac{\beta}{1 + \beta} (i_t - (1 + n)x_t).
\]

The problem of the period \(t\) government is to maximize \(V_t\) in (20) subject to (21) given \(k_t, h_t, \hat{b}_t\). The first-order condition with respect to \(x_t\) is:

\[
(1 + n) \frac{1 + (\beta + \gamma)\alpha}{i_t - (1 + n)x_t} = \frac{\eta(\beta + \gamma)(1 - \alpha)}{x_t},
\]

where the left-hand side is the marginal cost of public education expenditure, and the right-hand side is its marginal benefit. The government chooses \(x_t\) to balance the marginal cost and benefit, so \(x_t\) is given by:

\[
x_t = X_0 \cdot i_t,
\]

where

\[
X_0 = \frac{\eta(\beta + \gamma)(1 - \alpha)}{(1 + n) [1 + (\beta + \gamma) \left\{ \alpha + (1 - \alpha)\eta \right\}]}.
\]

To understand the property of the policy function of \(x_t\), let us focus on the numerator of \(X_0, \eta(\beta + \gamma)(1 - \alpha)\). This term is decomposed as \(\eta(\beta + \gamma)(1 - \alpha) = \eta \left[ -\left\{ \beta(\alpha - 1) + \gamma \alpha \right\} + \gamma \right]\) where the terms \(-\left\{ \beta(\alpha - 1) + \gamma \alpha \right\}\) and \(\gamma\) in the square brackets correspond to the coefficients of the second and third terms of \(V_t\) in (20), respectively. The term \(-\left\{ \beta(\alpha - 1) + \gamma \alpha \right\}\) implies that an increase in public education expenditures reduces the physical to human capital ratio, and this in turn increases the return on saving but decreases the children’s wage income. The term \(\gamma\) suggests that public education expenditures enhance the children’s human capital formation and thus improves parents’ utility. In sum, the net effect of these terms is positive.

Substituting the policy function in (22) into the government budget constraint, we obtain the corresponding tax rate:

\[
\tau_t = \begin{cases} 
(1 + n)X_0 + (1 - (1 + n)X_0) \frac{\alpha \cdot \hat{b}_n}{1 - \alpha \cdot k_0} & \text{for } t = 0, \\
(1 + n)X_0 & \text{for } t \geq 1.
\end{cases}
\]

The term \((1 + n)X_0\) indicates that an increase in education expenditures is financed by raising the tax rate. In addition, in the initial period, the government must further increase the tax rate for debt repayment, as represented by the term \(1 \cdot (\alpha/(1 - \alpha)) \cdot (\hat{b}_0/k_0)\). However, increasing budget pressure is partially offset by a reduction in public education expenditures, as represented by the term \((-1) \cdot (1 + n)X_0 \cdot (\alpha/(1 - \alpha)) \cdot (\hat{b}_0/k_0)\).
We can use the policy functions of $x_t$ and $\tau_t$ to reformulate the capital market clearing condition in (6) as follows:

$$k_{t+1} = \begin{cases} \Phi_K \cdot \left\{ \frac{(1 - \alpha)A(k_0)^\alpha}{\Phi_K} - \alpha A(k_0)^{\alpha-1}b_0 \right\}^{1-\eta} & \text{for } t = 0, \\ \Phi_K \cdot \left\{ (1 - \alpha)A(k_t)^\alpha \right\}^{1-\eta} & \text{for } t \geq 1, \end{cases}$$

where

$$\Phi_K = \frac{\beta}{1 + \beta} \cdot \frac{1 - (1+n)X_0}{(1+n)D(X_0)^\eta}.$$ 

Given $k_0(>0)$ and $b_0(\geq 0)$, a sequence of $\{k_t\}$ is uniquely determined to satisfy the above condition. The condition for $t \geq 1$, $k_{t+1} = \Phi_K \cdot \left\{ (1 - \alpha)A(k_t)^\alpha \right\}^{1-\eta}$ suggests that the sequence stably converges to the unique steady state. The following proposition summarizes the result established thus far.

**Proposition 4.** There is a unique, stable, steady-state tax-financed political equilibrium with $k_t > 0$ for all $t \geq 0$.

### 4.2 Debt-financed Political Equilibrium

In the debt-finance case, (4) gives the government budget constraint. Substituting this into $V_t$ in (19) leads to:

$$V_t = (1 + \beta) \ln \left( i_t - (1 + n)x_t + (1 + n)b_{t+1} \right) + \beta(\alpha - 1) + \gamma \alpha \ln k_{t+1} + \gamma \eta \ln x_t + \gamma(1 - \eta) \ln h_t + C_0.$$ 

(23)

A main departure from the tax-finance case is that the term $(1 + n)b_{t+1}$ appears in the debt-finance case. This term represents the positive income effect arising from public bond issues that enable the government to shift the burden of the expenditure from the current to future generations. Therefore, it produces a positive income effect on households.

The government’s problem is to maximize $V_t$ in (23) subject to the government budget constraint in (4) and the capital-market-clearing condition in (6). To more conveniently reformulate the problem, we substitute the government budget constraint and the human capital formation function into the capital-market-clearing condition and obtain:

$$k_{t+1} = \frac{1}{(1+n)D(x_t)^\eta(h_t)^{1-\eta}} \cdot \frac{\beta}{1 + \beta} \cdot \left[ i_t - (1 + n)x_t - \frac{(1 + n)b_{t+1}}{\beta} \right].$$ 

(24)

We substitute this into $V_t$ in (23) and obtain:

$$V_t = (1 + \beta) \ln \left[ i_t - (1 + n)x_t + (1 + n)b_{t+1} \right] + \beta(\alpha - 1) + \gamma \alpha \ln \left[ i_t - (1 + n)x_t - \frac{(1 + n)b_{t+1}}{\beta} \right]$$

$$+ \eta(\beta + \gamma)(1 - \alpha) \ln x_t + (1 - \gamma)(\beta + \gamma)(1 - \alpha) \ln h_t + C_1,$$

(25)
where $C_1$ includes the following constant terms:

$$C_1 \equiv C_0 + \{\beta(\alpha - 1) + \gamma \alpha\} \ln \frac{\beta}{(1 + \beta)(1 + n)D}.$$  

The first three terms in (25) are relevant in policymaking. The first term represents the utility of lifetime consumption. The second term shows the utility of the next generation’s wage income affected by physical capital accumulation. The third term includes the three factors affected by public education: the utility of the return on savings, the utility of the next generation’s wage income, and the utility of the next generation’s human capital. The government’s problem is to choose a pair of $(x_t, \hat{b}_{t+1})$ that maximizes $V_t$ in (25). Solving the problem yields the following policy functions.

**Lemma 1.** The policy functions of $x_t, \hat{b}_{t+1},$ and $\tau_t$ in the debt-finance political equilibrium are

$$x_t = X_0 \cdot i_t,$$

$$\hat{b}_{t+1} = B_0 \cdot i_t,$$

$$\tau_t = (1 + n) (X_0 - B_0) + \frac{\alpha}{1 - \alpha} \cdot \{1 - (1 + n) (X_0 - B_0)\} \cdot \frac{b_t}{k_t},$$

where

$$B_0 \equiv \frac{(1 + \beta) \beta - \{\beta(\alpha - 1) + \gamma \alpha\}}{(1 + n) [1 + (\beta + \gamma) \{\alpha + \eta(1 - \alpha)\}]}.$$

**Proof.** See Appendix A.4.

The policy function of $x_t$ is identical to that in the tax-finance political equilibrium. Public debt produces debt repayment costs, and thus reduces the government’s available resources $i_t$. However, it does not affect the fraction of the resources devoted to public education expenditures.

The policy function of $\hat{b}_{t+1}$ indicates that the two factors, $(1 + \beta) \beta$ and $\{\beta(\alpha - 1) + \gamma \alpha\}$, are crucial to the government’s state of financial balance. The term $(1 + \beta) \beta$ corresponds to the coefficient of the first term of $V_t$ in (25). A greater weight on the utility of consumption incentivizes the government to issue more public bonds to raise consumption levels in the middle and elderly ages. The term $\{\beta(\alpha - 1) + \gamma \alpha\}$ corresponds to the coefficient of the second term of $V_t$ in (25). An increase in public bond issues overwhelms physical capital accumulation, which in turn creates two opposing effects: an increase in the interest rate represented by $(1 + \beta) \beta$, and a decrease in the children’s wage income represented by $\gamma \alpha$. Therefore, the government borrows in the capital market if the effect of the first term, $(1 + \beta) \beta$, outweighs the effect of the second term, $\{\beta(\alpha - 1) + \gamma \alpha\}$.

The policy function of $\tau_t$ suggests that it is affected by both public education expenditures and public bond issues. The first term $(1 + n) (X_0 - B_0)$ indicates that the
government attempts to increase the tax rate to finance public education expenditures, but can cut the tax rate and finance a part of the expenditure by issuing public bonds. The second term shows the effect of debt repayment on the tax rate. In particular, the term “1” of \(1 - (1 + n) (X_0 - B_0)\) indicates that the government raises the tax rate to finance debt repayment. However, debt repayment pressure incentivizes the government to cut public education expenditures and issue more public bonds. The former effect, represented by \((1 + n) X_0\), works to decrease the tax rate, while the latter, represented by \((1 + n) B_0\), works in the opposite direction.

Having established the policy functions, we are now ready to demonstrate the accumulation of physical and human capital and public debt. We substitute the policy functions in Lemma 1 into the capital accumulation equation in (24), the government budget constraint in (4), and the human capital formation function given by \(h_{t+1} = D (x_t)^\eta (h_t)^{1-\eta}\), and obtain:

\[
\begin{align*}
k_{t+1} &= \Psi_K \cdot \left[ (1 - \alpha) A (k_t)^{\alpha} - \alpha A (k_t)^{\alpha-1} b_t \right]^{1-\eta}, \\
b_{t+1} &= \Psi_B \cdot \left[ (1 - \alpha) A (k_t)^{\alpha} - \alpha A (k_t)^{\alpha-1} b_t \right]^{1-\eta}, \\
h_{t+1} &= D (X_0)^\eta \cdot \left[ (1 - \alpha) A (k_t)^{\alpha} - \alpha A (k_t)^{\alpha-1} b_t \right]^{\eta},
\end{align*}
\]

where \(\Psi_K\) and \(\Psi_B\) are defined by:

\[
\begin{align*}
\Psi_K &\equiv \frac{\beta \left\{ 1 - (1 + n) X_0 - \frac{1+n}{\beta} B_0 \right\}}{(1 + n) D (X_0)^\eta (1 + \beta)}, \\
\Psi_B &\equiv \frac{B_0}{D (X_0)^\eta},
\end{align*}
\]

respectively.

Given \(\{k_0, b_0, h_0\}\), a sequence of \(\{k_t, b_t, h_t\}\) in the debt-finance political equilibrium contains the three equations above. We use these equations to obtain the following proposition.

**Proposition 5.** Assume the following conditions:

\[
\max \left\{ \frac{\beta \left\{ (1 + \beta) + (1 - \alpha)(1-\eta) \right\}}{\eta (1 - \alpha) + \alpha}, \frac{\beta (1 + \alpha \beta)}{\alpha} \right\} < \gamma < \frac{\beta \left\{ (1 + \beta) + (1 - \alpha) \right\}}{\alpha},
\]

\[
\frac{b_0}{k_0} \in \left( 0, \frac{1 - \alpha}{\alpha} \right).
\]

There is a unique, stable, steady-state debt-finance political equilibrium with \(k_{t+1} > 0, b_{t+1} > 0\) and \(\tau_t \in (0, 1)\) along the equilibrium path.

**Proof.** See Appendix A.5.

A noteworthy feature of the result in Proposition 5 is that it provides a unique debt-finance political equilibrium different from the result in the previous section demonstrating
multiple economic equilibria: the two economies sharing the same structural parameter values but different initial conditions may converge to different steady states. In other words, the initial condition influences long-term outcomes when fiscal policy is exogenous (see, e.g., Azariadis, 1993; de la Croix and Michel, 2002). However, Proposition 5 shows that the economy stably converges to a unique steady state for any initial condition. Therefore, the initial condition is of little importance when fiscal policy is endogenous.

The \( \gamma \) and \( b_0/k_0 \) assumptions in Proposition 5 outline the equilibrium path with \( k > 0, b > 0, \) and \( \tau \in (0,1) \). To gain intuition for these assumptions, let us first consider the \( \gamma \) assumption that represents the degree of altruism. First, suppose that \( \gamma \) is above the upper bound level given by \( \beta \{(1 + \beta) + (1 - \alpha)(1 - \eta)\}\). Greater altruism encourages adults to leave higher wage rates to their children, and will do so by decreasing public bond issues and thus weakening its overpowering effect on capital accumulation. Given this adult preference, the government finds it optimal to lend rather than to borrow in the capital market. Therefore, we can avoid the case of government lending with the upper bound of \( \gamma \).

Next, consider the role of the lower bound of \( \gamma \). When \( \gamma \) is below the first lower bound given by \( \beta \{(1 + \beta) + (1 - \alpha)\} / \{\eta(1 - \alpha) + \alpha\} \), the degree of altruism is too low to incentivize the adults to provide higher levels of education, and they would thus rather support more public bond issues to finance public education expenditures. In this case, the government gains a surplus in revenue, and can thus refund it by subsidizing households. We can rule out this possibility using the first lower bound of \( \gamma \). In addition, more public bond issues increases the debt repayment cost, and thus results in a higher tax rate of more than 100%. The second lower bound of \( \gamma \) guarantees that the tax rate is set below 100\% for period \( t \geq 1 \). The upper bound of \( b_0/k_0 \) guarantees the period 0 tax rate below 100\%.

### 4.3 Fiscal Rule Comparison

Public bond issues enable the current generation to pass costs on to future generations, which in turn influences the government’s decisions representing the current middle. To consider the impact of debt finance on policy, we here compare the aggregate expenditure-GDP ratio and the tax rate in the debt-finance case with those in the tax-finance case, and obtain the following result.

**Proposition 6.** Consider the aggregate expenditure-GDP ratio, \( N_{t+1}x_t/Y_t \) and the tax rate, \( \tau_t \). (i) For \( t = 0 \), \( N_1x_0/Y_0|_{\text{tax}} = N_1x_0/Y_0|_{\text{debt}} \). For \( t \geq 1 \), \( N_{t+1}x_t/Y_t|_{\text{tax}} > N_{t+1}x_t/Y_t|_{\text{debt}} \). (ii) For \( t = 0 \), \( \tau_0|_{\text{tax}} > \tau_0|_{\text{debt}} \). For \( t \geq 1 \), \( \tau_t|_{\text{tax}} \leq \tau_t|_{\text{debt}} \) if and only if \( \gamma \leq \beta(1 - \alpha)/\alpha + (1 + \beta)^2/(1 - \alpha) \).

**Proof.** See Appendix A.6.
Figure 2 plots a numerical example of the evolution of the aggregate expenditure-GDP ratio (in Panel (a)) and the tax rate (in Panel (b)) from period 0. The first part of Proposition 6 states that both cases have equal aggregate expenditure-GDP ratios in the initial period, but they differ from period 1. To understand this result, recall the policy function $x_t$, which indicates that the government uses a fraction, $X_0$, of its available resources, $i_t$, for public education expenditures for both cases. The available resources are equal in period 0 and given by $i_0 = \{(1 - \alpha)A(k_0)^{\alpha} - \alpha A(k_0)^{\alpha-1} b_0\} h_0$ for both cases. Therefore, the expenditure-GDP ratios are equal in period 0. However, they differ from period 1. In the debt-finance case, the government must manage debt repayment, which reduces its available resources. Therefore, the government attains a lower expenditure-GDP ratio in the debt-finance case than in the tax-finance case.

The second part of Proposition 6 states that the tax rate in the tax-finance case is higher than that in the debt-finance case in the initial period, but this relationship reverses in the next period. In the debt-finance case, the government can implement a tax cut financed by public bond issues. Because of this tax-cut effect, the period 0 tax rate is lower in the debt-finance case than in the tax-finance case. However, from period 1, the government has debt repayment costs in the debt-finance case, and must then raise the tax rate. The second part of Proposition 6 shows that this tax-hike effect is greater (less) than the tax-cut effect if $\gamma$ is below (above) the critical value given by $\beta(1 - \alpha)/\alpha + (1 + \beta)^2/(1 - \alpha)$.

To check the empirical plausibility of the $\gamma \leq \beta(1 - \alpha)/\alpha + (1 + \beta)^2/(1 - \alpha)$ condition, we assume $\alpha = 0.3$ and $\beta = (0.997)^{30}$. The $\beta = (0.997)^{30}$ assumption implies that each generation lasts for 30 years, and a single-period discount factor is 0.997. In this setting, the critical value of $\gamma$ is larger than the upper limit of $\gamma$ presented in Proposition 5. Therefore, we can conclude that under empirically plausible parameter conditions, the tax-hike effect outweighs the tax-cut effect, and this suggests that debt finance enables the government to cut tax in the initial period, but results in a permanent tax hike from the next period.

We next compare the steady-state growth rates for the two cases.

**Proposition 7.** The steady-state growth rate in the tax-finance political equilibrium is higher than in the debt-finance political equilibrium; $h'/h|_{\text{tax}} > h'/h|_{\text{debt}}$.

**Proof.** See Appendix A.7.
The steady-state growth rates in the tax-finance and debt-finance cases are given by:

\[
\frac{h'}{h_{\text{tax}}} = D \cdot \left[ X_0 \left( \frac{\beta}{1 + \beta} \cdot \frac{1 - (1 + n)X_0}{(1 + n)D(X_0)^\eta} \right)^{-\frac{\alpha}{\gamma(1-\eta)}} \{(1 - \alpha)A\}^{-\frac{1}{\gamma(1-\eta)}} \right]^{\eta},
\]

\[
\frac{h'}{h_{\text{debt}}} = D \cdot \left[ X_0 \left( \frac{\beta}{1 + \beta} \cdot \frac{1 - (1 + n)X_0 - \frac{1+n}{\beta}B_0}{(1 + n)D(X_0)^\eta} \right)^{-\frac{\alpha}{\gamma(1-\eta)}} \left((1 - \alpha)A - \alpha \frac{\Psi_B}{\Psi_K}\right)^{-\frac{1}{\gamma(1-\eta)}} \right]^{\eta},
\]

respectively. The equation \(h'/h\)\_\text{debt} suggests that there are two negative effects on the growth rate peculiar to the debt-finance case. First, the cost of debt repayment reduces the government’s resources, which in turn decreases the public education expenditures as an engine of economic growth. This effect is observed in the term \(-(1 + n)B_0/\beta\). Second, the cost of debt repayment overwhelms capital accumulation and thus lowers the steady-state level of capital, resulting in a higher interest rate that then further increases the debt repayment cost. This effect is observed in the term \(\alpha A\Psi_B/\Psi_K\). These two negative effects from public debt result in a lower steady-state growth rate in the debt-finance case than the tax-finance case. Figure 3 plots a numerical example of the evolution of the growth rate.

4.4 Voting on Fiscal Rules

The analysis has thus far assumed a given fiscal rule, namely either a rule where the government must balance its budget in each period (Subsection 4.1) or where the government can borrow in the capital market (Subsection 4.2). However, in the real world, the rule is also established through the political process; some countries and states adopt a balanced-budget rule or something similar and others do not. For example, the US federal government has no balanced budget requirement in its Constitution (Poterba, 1995), while some European countries have another form of the balanced budget rule, such as the Maastricht Treaty criteria (Corsetti and Roubini, 1996). Therefore, a question naturally arises: under what conditions will the government adopt debt finance rather than tax finance?

To address this question, we consider the following vote on the rule. In each period, the government proposes the two fiscal rules for a given set of state variables, \(k\), \(b\), and \(h\). One is chosen through voting with the aim of maximizing the value of the political objective function, that is, the utility of the middle-aged. Second, for a given fiscal rule, the middle-aged vote on fiscal policy. We solve the model by backward induction. We have already demonstrated the vote on fiscal policy for a given rule in the previous two subsections, and based on the result thus far, we can compare the utility of the middle-aged under the tax-finance rule with that under the debt-finance rule for a given set of \(k\), \(b\), and \(h\), and obtain the following result.
Proposition 8. For a given \( k, b, \) and \( h, \) debt finance is chosen through voting.

Proof. See Appendix A.8.

To understand the results in Proposition 8, consider period 0. Debt finance enables the government to cut the tax rate (Proposition 6(ii)), which increases the disposable income and thus lifetime consumption for the middle-aged in period-0, while the public education expenditures remain equal in both cases (Proposition 6(i)). Therefore, the period-0 middle-aged agents find it optimal to choose debt rather than the tax financing to maximize utility. Because this result holds for any initial conditions of \( k > 0, b > 0, \) and \( h > 0, \) successive generations also adopt debt finance.

Debt finance is optimal for the currently living generation because it decreases its tax burden and shifts it to future generations. This implies that future generations are worse off due to debt repayments. To examine the welfare implication of debt finance across generations, we here compare the two fiscal rules in terms of utility. That is, we plot the utility of successive generations from period 0 under the tax-finance rule and that under the debt-finance rule in Figure 4, and summarize the result in Table 1. The figure demonstrates that the period 0 generation is better off, while successive generations are worse off when the tax-finance rule is replaced by the debt-finance rule. The result suggests that the current generation’s myopic choice of fiscal rule decreases welfare in future generations.

5 Conclusion

This paper developed an overlapping-generations model with physical and human capital accumulation using public education and parental human capital as inputs in the process of human capital formation. Public education spending is financed through tax on labor income in the tax-finance case, and by both tax and public bond issues in the debt-finance case. Within this framework, we compared and evaluated the two fiscal choices in terms of growth and utility, and showed that the debt-finance case could be superior to the tax-finance case when the education expenditure-GDP ratio is fixed.

Studies examining growth often assume a fixed expenditure-GDP ratio, which enables us to evaluate the effect of a change in the ratio on growth and welfare. However, in the real world, the ratio is endogenous since the expenditure itself is determined by a political process of voting by agents who recognize the impact of their actions. Given this background, we presented endogenous policy formation and showed that the current generation will prefer debt finance since it passes the debt repayment cost on to future

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generations. However, the debt repayment cost induces successive generations to cut the expenditure, making them worse off than in the tax-finance case. Our result provides one possible explanation for why debt finance continues predominantly in many developed countries despite its expected worse performance.
A  Proofs

A.1  Proof of Proposition 2

We first show the existence of two steady-state equilibria with \( b > 0 \). Recall that given an initial condition \((k_0, b_0)\), the equilibrium sequence of \( \{k_t, b_t\} \) has the government budget constraint in (12) and the capital-market-clearing condition in (13). In the steady state, they are:

\[
\begin{align*}
  b &= M \cdot \left[ (X_{econ} - \tau_{debt}(1 - \alpha)) + \alpha \frac{b}{k} \right] \cdot (A(k)^\alpha)^{1 - \eta}, \\
  k &= M \cdot \left[ \frac{\beta}{1 + \beta} (1 - \tau_{debt}) (1 - \alpha) - (X_{econ} - \tau_{debt}(1 - \alpha)) - \alpha \frac{b}{k} \right] \cdot (A(k)^\alpha)^{1 - \eta},
\end{align*}
\]

where \( M \) is defined by:

\[
M = \frac{1}{(1 + n)D (\frac{X_{econ}}{1 + n})^{\eta}}.
\]

These expressions are summarized as:

\[
\frac{b}{k} = \frac{(X_{econ} - \tau_{debt}(1 - \alpha)) + \alpha \frac{b}{k}}{\frac{\beta}{1 + \beta} (1 - \tau_{debt}) (1 - \alpha) - (X_{econ} - \tau_{debt}(1 - \alpha)) - \alpha \frac{b}{k}},
\]

or

\[
f \left( \frac{b}{k} \right) = \alpha \cdot \left( \frac{b}{k} \right)^2 - x \cdot \frac{b}{k} + (X_{econ} - \tau_{debt}(1 - \alpha)) = 0, \quad \text{(26)}
\]

where \( f(0) = X_{econ} - \tau_{debt}(1 - \alpha) > 0 \) under the assumption in Proposition 2, and \( x \) is defined by:

\[
x = \frac{\beta}{1 + \beta} (1 - \tau_{debt}) (1 - \alpha) - (X_{econ} - \tau_{debt}(1 - \alpha)) - \alpha.
\]

Our task is to show that (26) has two distinct solutions with \( b/k > 0 \). There are two distinct solutions for \( f(b/k) = 0 \) if \( x^2 - 4\alpha (X_{econ} - \tau_{debt}(1 - \alpha)) \geq 0 \), i.e.,

\[
\left[ \frac{\beta}{1 + \beta} (1 - \tau_{debt}) (1 - \alpha) - (X_{econ} - \tau_{debt}(1 - \alpha)) + \alpha \right]^2 > 4 \alpha^2 \frac{\beta}{1 + \beta} (1 - \tau_{debt}) \frac{1 - \alpha}{\alpha}.
\]

In addition, given that \( f(0) > 0 \), the two distinct solutions are both positive if \( f'(0) < 0 \), that is:

\[
X_{econ} < \frac{\beta}{1 + \beta} (1 - \tau_{debt}) (1 - \alpha) + \tau_{debt}(1 - \alpha) - \alpha. \quad \text{(28)}
\]

Under the condition in (28), we have \( \frac{\beta}{1 + \beta} (1 - \tau_{debt}) (1 - \alpha) - (X_{econ} - \tau_{debt}(1 - \alpha)) + \alpha > 0 \). (27) is reformulated as

\[
X_{econ} < \frac{\beta}{1 + \beta} (1 - \tau_{debt}) (1 - \alpha) + \tau_{debt}(1 - \alpha) + \alpha - 2\alpha \left( \frac{\beta}{1 + \beta} (1 - \tau_{debt}) \frac{1 - \alpha}{\alpha} \right)^{1/2}. \quad \text{(29)}
\]
Therefore, there are two distinct solutions with \( b > 0 \) if (28) and (29) hold assuming 
\[
\tau_{\text{debt}} < 1 - \alpha \quad \text{in (7) and } \tau_{\text{debt}}(1 - \alpha) \leq X_{\text{econ}} \quad \text{in (13)}.
\]

That is:
\[
\tau_{\text{debt}}(1 - \alpha) \leq X_{\text{econ}} < \min \left\{ 1 - \alpha, \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) + \tau_{\text{debt}}(1 - \alpha) - \alpha, \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) + \tau_{\text{debt}}(1 - \alpha) + \alpha - 2\alpha \left( \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) \right)^{1/2} \right\}.
\]

To determine the upper limit of \( X_{\text{econ}} \) in (30), we first compare the first two terms in parentheses in (30), and find:
\[
1 - \alpha > \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) + \tau_{\text{debt}}(1 - \alpha) - \alpha.
\]

We next compare the last two terms and find that the upper limit of \( X_{\text{econ}} \) is given by:

(i) \[
\frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) + \tau_{\text{debt}}(1 - \alpha) - \alpha \quad \text{if } \tau_{\text{debt}} \geq 1 - \frac{1 + \beta}{\beta} \frac{\alpha}{1 - \alpha},
\]

(ii) \[
\frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) + \tau_{\text{debt}}(1 - \alpha) + \alpha - 2\alpha \left( \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) \right)^{1/2} \quad \text{if } \tau_{\text{debt}} < 1 - \frac{1 + \beta}{\beta} \frac{\alpha}{1 - \alpha}.
\]

In case (i), the range of \( X_{\text{econ}} \) is given by:
\[
\tau_{\text{debt}}(1 - \alpha) \leq X_{\text{econ}} < \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) + \tau_{\text{debt}}(1 - \alpha) - \alpha.
\]

The set of \( X_{\text{econ}} \) is nonempty if \( \tau_{\text{debt}} < 1 - \frac{1 + \beta}{\beta} \frac{\alpha}{1 - \alpha} \), which contradicts the presumption in case (i).

In case (ii), the range of \( X_{\text{econ}} \) is given by:
\[
\tau_{\text{debt}}(1 - \alpha) \leq X_{\text{econ}} < \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) + \tau_{\text{debt}}(1 - \alpha) + \alpha - 2\alpha \left( \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) \right)^{1/2}.
\]

The set of \( X_{\text{econ}} \) is nonempty for any \( \alpha, \beta, \) and \( \tau_{\text{debt}} \) because
\[
\tau_{\text{debt}}(1 - \alpha) \leq 0 \leq \left[ \left( \frac{\beta}{1 + \beta} (1 - \tau_{\text{debt}})(1 - \alpha) \right)^{1/2} - 1 \right]^2.
\]

Therefore, there are two distinct solutions with \( b/k > 0 \) if (31) and \( 0 < \tau_{\text{debt}} \leq 1 - \frac{1 + \beta}{\beta} \frac{\alpha}{1 - \alpha} \) hold. The range of \( \tau_{\text{debt}} \) is nonempty if \( \alpha < \beta/(1 + 2\beta) \).
Next, we show the stability of the two steady-state equilibria. We differentiate (12) and (13) with respect to \( k \) and \( b \) and evaluate them at the steady state to obtain:

\[
dk = M \cdot \left[ \mu_2 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) - \mu_1 \right] dk - M \mu_2 db, \tag{32}
\]
\[
\db = M \mu_1 dk + M \mu_2 db, \tag{33}
\]

where

\[
\mu_1 \equiv \left( X_{\text{econ}} - \tau_{\text{debt}}(1 - \alpha) \right) \alpha(1 - \eta) + \alpha \left\{ \alpha(1 - \eta) - 1 \right\} \frac{b}{k} \cdot (A(k)^\alpha)^{1-\eta} \cdot \frac{1}{k},
\]
\[
\mu_2 \equiv \alpha \cdot (A(k)^\alpha)^{1-\eta} \cdot \frac{1}{k}. \tag{34}
\]

Let \( b = \Psi_k(k) \) denote the phase line of \( \Delta k = 0 \) in Eq. (13), and \( b = \Psi_b(k) \) the phase line of \( \Delta b = 0 \) in Eq. (12). Using (32) and (33), we can compute the slope of \( \Psi_k(k) \) and \( \Psi_b(k) \) as follows:

\[
\frac{\partial \Psi_k(k)}{\partial k} = \frac{1}{M \mu_2} \cdot \left[ M \left\{ \mu_2 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) - \mu_1 \right\} - 1 \right], \tag{35}
\]
\[
\frac{\partial \Psi_b(k)}{\partial k} = \frac{M \mu_1}{1 - M \mu_2}. \tag{36}
\]

The Jacobian matrix at any steady state of the planner system in (12) and (13) is given by:

\[
J = \begin{bmatrix}
M \left\{ \mu_2 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) - \mu_1 \right\} & -M \mu_2 \\
M \mu_1 & M \mu_2 
\end{bmatrix}.
\]

The term \( M \mu_2 \) satisfies

\[
0 < M \mu_2 < 1. \tag{37}
\]

The first inequality condition holds because \( M > 0 \) and \( \mu_2 > 0 \). To show that the second inequality condition holds, rewrite \( M \mu_2 < 1 \) as:

\[
M \mu_2 < 1 \Leftrightarrow \frac{1}{(1 + n)D \left( \frac{X_{\text{econ}}}{1+n} \right)^\eta} \cdot \alpha \cdot (A(k)^\alpha)^{1-\eta} \cdot \frac{1}{k} < 1
\]
\[
\Leftrightarrow \hat{k} \equiv \frac{\alpha A}{(1 + n)D \left( \frac{X_{\text{econ}}}{1+n} A \right)^\eta}^{1/(1-\alpha(1-\eta))} < k.
\]

The last condition, \( \hat{k} < k \), holds, as illustrated in Figure 1.

For any \((k, b) >> 0\), the trace and the determinant of \( J \) are

\[
\text{tr}J = M \cdot \left[ \mu_2 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) - \mu_1 + \mu_2 \right],
\]
\[
\det J = (M \mu_2)^2 \cdot \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) \in (0, 1),
\]
where $\det J \in (0, 1)$ holds because $M_2 \in (0, 1)$ holds from (37). Moreover,

$$
\Delta \equiv (\text{tr} J)^2 - 4 \det J
$$

$$
= (M)^2 \cdot \left\{ \mu_2 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) - \mu_1 + \mu_2 \right\}^2 - 4 \mu_2 \mu_1 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta)
$$

$$
= (M)^2 \cdot \left\{ \mu_2 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) - \mu_1 - \mu_2 \right\}^2 - 4 \mu_2 \mu_1
\right].
$$

Appendix A.2 shows that $\mu_1 < 0$ if the condition in Proposition 2 applies. Therefore, $\Delta > 0$ holds and thus $J$ has two distinct eigenvalues at each steady state if the condition in Proposition 2 holds.

We derive the eigenvalues of $J$ by solving the following equation:

$$
p(\lambda) = (\lambda)^2 - \text{tr} J \cdot \lambda + \det J = 0,
$$

where $\lambda$ denotes an eigenvalue. At the $e = e_l$ steady state, the slope of $\Psi_k$ is larger than that of $\Psi_b : \partial \Psi_k/\partial k > \partial \Psi_b/\partial k$. Applying (35) and (36), we obtain:

$$
\frac{\partial \Psi_k(k)}{\partial k} > \frac{\partial \Psi_b(k)}{\partial k}
$$

$$
\iff M \mu_2 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) - M \mu_1 + M \mu_2
$$

$$
- (M \mu_2)^2 \frac{\beta}{1 + \beta} (1 - \tau)(1 - \alpha)(1 - \eta) - 1 > 0
$$

$$
\iff \text{tr} J - \det J - 1 > 0
$$

$$
\iff p(1) = 1 - \text{tr} J + \det J < 0.
$$

Moreover,

$$
p(-1) = 1 + \text{tr} J + \det J
$$

$$
> 1 + (1 + \det J) + \det J
$$

$$
> 0,
$$

where we obtain the second line from $\text{tr} J > 1 + \det J$ in (38) and the third line from $\det J > 0$. Therefore, $p(1) < 0$ and $0 < p(-1)$ imply that the $e = e_l$ steady-state equilibrium exhibits saddle point stability.

At the $e = e_h$ steady state, $\partial \Psi_k/\partial k < \partial \Psi_b/\partial k$ holds. Using (35) and (36), we obtain:

$$
\frac{\partial \Psi_k(k)}{\partial k} < \frac{\partial \Psi_b(k)}{\partial k} \iff p(1) = 1 - \text{tr} J + \det J > 0.
$$

(39)

In addition,

$$
p(0) = \det J > 0,
$$

$$
p(\lambda) = \left( \lambda - \frac{\text{tr} J}{2} \right)^2 - \left( \frac{\text{tr} J}{2} \right)^2 + \det J.
$$

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Therefore, if $0 < \text{tr} J < 2$, then there are two distinct eigenvalues, $\lambda_1$ and $\lambda_2$, satisfying $\lambda_1, \lambda_2 \in (0, 1)$.

To show that the condition $0 < \text{tr} J < 2$ holds, we first use the condition in (39), which implies

$\text{tr} J < 1 + \det J < 2$,

where we obtain the second inequality from $\det J < 1$ in (37). To show $\text{tr} J > 0$, recall $\text{tr} J = M \cdot \left( \mu_2 \frac{\beta}{1+\beta} (1-\tau)(1-\alpha)(1-\eta) - \mu_1 + \mu_2 \right)$, which is rewritten as:

$$\text{tr} J = M \cdot \left[ \mu_2 \frac{\beta}{1+\beta} (1-\tau_{\text{debt}})(1-\alpha)(1-\eta) \right.$$

$$\left. - \left\{ \left( X_{\text{econ}} - \tau_{\text{debt}}(1-\alpha) \right)(1-\eta) + \{ \alpha(1-\eta) - 1 \} \frac{b}{k} \right\} \mu_2 + \mu_2 \right].$$

or

$$\frac{\text{tr} J}{M\mu_2} = \frac{\beta}{1+\beta} (1-\tau_{\text{debt}})(1-\alpha)(1-\eta) - \left( X_{\text{econ}} - \tau_{\text{debt}}(1-\alpha) \right)(1-\eta) - \left\{ \alpha(1-\eta) - 1 \right\} \frac{b}{k} + 1.$$

Applying the upper limit of $X_{\text{econ}}$ in (31), we can reformulate the above expression as:

$$\frac{\text{tr} J}{M\mu_2} > \frac{\beta}{1+\beta} (1-\tau_{\text{debt}})(1-\alpha)(1-\eta)$$

$$- \left\{ \left( \frac{\beta}{1+\beta} (1-\tau_{\text{debt}})(1-\alpha) + \tau_{\text{debt}}(1-\alpha) + \alpha - 2\alpha \left( \frac{\beta}{1+\beta} (1-\tau_{\text{debt}}) \frac{1-\alpha}{\alpha} \right)^{1/2} \right) \cdot (1-\eta) \right\}$$

$$+ \tau_{\text{debt}}(1-\alpha)(1-\eta) + \{ 1 - \alpha(1-\eta) \} \frac{b}{k} + 1$$

$$= \alpha(1-\eta) \cdot 2 \left( \frac{\beta}{1+\beta} (1-\tau_{\text{debt}}) \frac{1-\alpha}{\alpha} \right)^{1/2} + \{ 1 - \alpha(1-\eta) \} \frac{b}{k} + \{ 1 - \alpha(1-\eta) \}$$

$$> 0,$$

indicating that $\text{tr} J > 0$. Therefore, there are two distinct eigenvalues, $\lambda_1$ and $\lambda_2$, with $\lambda_1, \lambda_2 \in (0, 1)$, implying that the $e = e_h$ steady-state equilibrium exhibits a sink.

\[\blacksquare\]

### A.2 Proof of $\mu_1 < 0$

Recall the definition of $\mu_1$ in (34):

$$\mu_1 \equiv \left[ \left( X_{\text{econ}} - \tau_{\text{debt}}(1-\alpha) \right)(1-\eta) + \{ \alpha(1-\eta) - 1 \} \frac{b}{k} \right] \cdot \mu_2.$$
Given that $\mu_2 > 0$, we have

$$\mu_1 < 0 \iff (X_{econ} - \tau_{debt}(1 - \alpha))(1 - \eta) < \{1 - \alpha(1 - \eta)\} \frac{b}{k}. \quad (40)$$

Let denote by $b/k$ a smaller solution of $f(b/k) = 0$ in (26). (40) indicates that $\mu_1 < 0$ if

$$(X_{econ} - \tau_{debt}(1 - \alpha))(1 - \eta) < \{1 - \alpha(1 - \eta)\} \cdot \left(\frac{b}{k}\right). \quad (41)$$

Our task is to show that (41) holds under the condition provided in Proposition 2.

Solving $f(b/k) = 0$, we obtain:

$$\frac{b}{k} = \frac{x - \sqrt{(x)^2 - 4\alpha \cdot (X_{econ} - \tau_{debt}(1 - \alpha))}}{2\alpha}. \quad (42)$$

We substitute this solution into (41) and obtain:

$$\sqrt{(x)^2 - 4\alpha \cdot (X_{econ} - \tau_{debt}(1 - \alpha))} < x - \frac{2\alpha(1 - \eta) \cdot (X_{econ} - \tau_{debt}(1 - \alpha))}{1 - \alpha(1 - \eta)}. \quad (42)$$

The right-hand side in (42) is positive if

$$X_{econ} < \frac{1 - \alpha(1 - \eta)}{1 + \alpha(1 - \eta)} \cdot \left\{ \frac{\beta}{1 + \beta}(1 - \tau_{debt})(1 - \alpha) - \alpha \right\} + \tau_{debt}(1 - \alpha).$$

This is satisfied since the right-hand side in the above expression is greater than the upper limit of $X_{econ}$ provided in Proposition 2.

Given that both sides of (42) are positive, (42) is reformulated as follows:

$$(x)^2 - 4\alpha \cdot (X_{econ} - \tau_{debt}(1 - \alpha)) < (x)^2 - 2x \cdot \frac{2\alpha(1 - \eta) \cdot (X_{econ} - \tau_{debt}(1 - \alpha))}{1 - \alpha(1 - \eta)}$$

$$+ \frac{4(\alpha)^2(1 - \eta)^2 (X_{econ} - \tau_{debt}(1 - \alpha))^2}{\{1 - \alpha(1 - \eta)\}^2},$$

or

$$0 < \{1 - \alpha(1 - \eta)\}^2 - x(1 - \eta) \{1 - \alpha(1 - \eta)\} + \alpha(1 - \eta)^2 (X_{econ} - \tau_{debt}(1 - \alpha)).$$

Using the definition of $x$, we reformulate the above expression as:

$$\tau_{debt}(1 - \alpha) + \{1 - \alpha(1 - \eta)\} \left[ -\frac{1}{1 - \eta} + \frac{\beta}{1 + \beta}(1 - \tau_{debt})(1 - \alpha) \right] < X_{econ}. \quad (43)$$

The left-hand side is less than $\tau_{debt}(1 - \alpha)$ because $-\frac{1}{1 - \eta} < -1$ and $\frac{\beta}{1 + \beta}(1 - \tau_{debt})(1 - \alpha) \in (0, 1)$. This implies that (43) holds if $\tau_{debt}(1 - \alpha) \leq X_{econ}$: this is the condition for $X_{econ}$ provided in Proposition 2. Therefore, $\mu_1 < 0$ holds under the condition in Proposition 2. 

$\blacksquare$
A.3 Proof of Proposition 3

(i) The proof of the first part is given in the text.

(ii) The utility function for the middle-aged in period-\( t \) is:

\[
U_t = \ln c_t + \beta \ln d_{t+1} + \gamma \ln w_{t+1} h_{t+1}
\]

\[
= \ln \frac{1}{1+\beta} (1-\tau_t) w_t h_t + \beta \ln \frac{\beta R_{t+1}}{1+\beta} (1-\tau_t) w_t h_t + \gamma \ln w_{t+1} + \gamma \ln D \left( x_t \right)^\eta (h_t)^{1-\eta}
\]

\[
= (1+\beta) \ln (1-\tau_t) w_t h_t + \beta \ln R_{t+1} + \gamma \ln w_{t+1} + \gamma \eta \ln x_t + \gamma (1-\eta) \ln h_t
\]

\[
+ \left( \ln \frac{1}{1+\beta} + \beta \ln \frac{\beta}{1+\beta} + \gamma \ln D \right),
\]

where we derive the second line by substituting the consumption and human capital production functions.

We substitute the profit maximization conditions in (3) into the above expression and obtain:

\[
U_t = (1+\beta) \ln \left\{ (1-\alpha) A (k_t)^\alpha h_t - \tau_t w_t h_t \right\} + \left\{ \beta (\alpha - 1) + \gamma \alpha \right\} \ln k_{t+1} + \gamma \ln(1-\eta) \ln h_t + C_0,
\]

where

\[
C_0 \equiv \ln \frac{1}{1+\beta} + \beta \ln \frac{\beta}{1+\beta} + \gamma \ln D + \beta \ln \alpha A + \gamma \ln(1-\alpha) A.
\]

We rewrite this expression using the expenditure rule in (7), as follows:

\[
U_t = (1+\beta) \ln \left\{ (1-\alpha) A (k_t)^\alpha h_t - \tau_t w_t h_t \right\} + \left\{ \beta (\alpha - 1) + \gamma \alpha \right\} \ln k_{t+1}
\]

\[
+ \gamma \eta \ln \frac{X_{econ}}{1+\eta} A (k_t)^\alpha h_t + \gamma (1-\eta) \ln h_t + C_0. \tag{44}
\]

In the tax-finance case, the government budget constraint is reduced to \( \tau_t = X_{econ}/(1-\alpha) \). Plugging this into (44), we obtain:

\[
U_{t,tax} = (1+\beta) \ln \left\{ (1-\alpha - X_{econ}) A (k_t)^\alpha \right\} + \left\{ \beta (\alpha - 1) + \gamma \alpha \right\} \ln k_{t+1} + \gamma \eta \ln \frac{X_{econ}}{1+\eta} A (k_t)^\alpha
\]

\[
+ (1+\beta + \gamma) \ln h_t + C_0. \tag{45}
\]

In the debt-finance case, the government budget constraint is:

\[
(1+n) \hat{b}_{t+1} + \tau_{debt}(1-\alpha)A (k_t)^\alpha h_t = (1+n) x_t + \alpha A (k_t)^{\alpha-1} \hat{b}_t,
\]

that is,

\[
(1+n) \frac{b_{t+1}}{h_t} h_t + \tau_{debt}(1-\alpha)A (k_t)^\alpha h_t = X_{econ} A (k_t)^\alpha h_t + \alpha A (k_t)^{\alpha-1} b_t h_t.
\]

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We substitute this expression into (44) and obtain:

\[
U_{t,\text{debt}} = \left(1 + \beta\right) \ln \left[\left(1 - \alpha\right)A(k_t)^{\alpha}h_t - X_{\text{econ}}A(k_t)^{\alpha}h_t - \alpha A(k_t)^{\alpha-1}b_t h_t + \left(1 + n\right)b_{t+1} \frac{h_{t+1}}{h_t}\right]
\]

\[
+ \{\beta (\alpha - 1) + \gamma \alpha\} \ln k_{t+1} + \gamma \eta \ln \frac{X_{\text{econ}}}{1 + n} A(k_t)^{\alpha}h_t + \gamma (1 - \eta) \ln h_t + C_0,
\]

that is,

\[
U_{t,\text{debt}} = \left(1 + \beta\right) \ln \left\{\left(1 - \alpha\right) - X_{\text{econ}}\right\} A(k_t)^{\alpha} + \left\{-\alpha A(k_t)^{\alpha-1}b_t + \left(1 + n\right)b_{t+1} \frac{h_{t+1}}{h_t}\right\}
\]

\[
+ \{\beta (\alpha - 1) + \gamma \alpha\} \ln k_{t+1} + \gamma \eta \ln \frac{X_{\text{econ}}}{1 + n} A(k_t)^{\alpha} + \left(1 + \beta + \gamma\right) \ln h_t + C_0.
\]  

In the steady state, \(k\) and \(b\) are constant. Therefore, by comparing (45) and (46), we find that \(U_{t,\text{debt}} > U_{t,\text{tax}}\) holds if each term in (46) is larger than the corresponding term in (45), that is, if the following four conditions hold in the steady state:

\[
\begin{align*}
-\alpha A\left(k_{|\text{debt}}\right)^{\alpha-1} + \left(1 + n\right)\left(\frac{h'}{h}_{|\text{debt}}\right) & \geq 0, \\
k_{|\text{debt}} & > k_{|\text{tax}}, \\
h_{t,\text{debt}} & > h_{t,\text{tax}}, \\
\beta (\alpha - 1) + \gamma \alpha & \Leftrightarrow \gamma > \frac{(1 - \alpha)\beta}{\alpha}.
\end{align*}
\]  

To show the first condition in (47) holds, we rewrite it as:

\[
(1 + n)\left(\frac{h'}{h}_{|\text{debt}}\right) \geq \alpha A \frac{1}{\left(k_{|\text{debt}}\right)^{1-\alpha}},
\]

Given that \(k_{|\text{debt}} > \hat{k}\), the above condition holds if

\[
(1 + n)\left(\frac{h'}{h}_{|\text{debt}}\right) \geq \alpha A \frac{1}{\left(\hat{k}\right)^{1-\alpha}},
\]

that is,

\[
(1 + n) \cdot \left[D \left(\frac{X_{\text{econ}}}{1 + n} A\right)^{\eta} \frac{1-\alpha}{1-\alpha(1-\eta)} \right] \cdot \left(\frac{\alpha A}{1 + n}\right)^{\frac{\alpha \eta}{1-\alpha(1-\eta)}} \geq \alpha A \cdot \left[D \left(\frac{X_{\text{econ}}}{1 + n} A\right)^{\eta} \frac{1-\alpha}{1-\alpha(1-\eta)} \right] \cdot \left(\frac{\alpha A}{1 + n}\right)^{\frac{\alpha \eta}{1-\alpha(1-\eta)}}.
\]

This expression is rewritten as:

\[
\left[D \left(\frac{X_{\text{econ}}}{1 + n} A\right)^{\eta} \frac{1-\alpha}{1-\alpha(1-\eta)} \right] \cdot \left(\frac{\alpha A}{1 + n}\right)^{\frac{\alpha \eta}{1-\alpha(1-\eta)}} \geq \left[D \left(\frac{X_{\text{econ}}}{1 + n} A\right)^{\eta} \frac{1-\alpha}{1-\alpha(1-\eta)} \right] \cdot \left(\frac{\alpha A}{1 + n}\right)^{\frac{\alpha \eta}{1-\alpha(1-\eta)}},
\]

which holds with an equality. Therefore, the first condition in (47) is satisfied in the steady state.
The second condition in (47) holds if \( \hat{k} > k_{\text{tax}} \) because \( k_{\text{debt}} > \hat{k} \) holds in the steady state. Given the definition of \( \hat{k} \) and \( k_{\text{tax}} \) in (9), we can reformulate the second condition in (47) as:

\[
\left[ \frac{\alpha A}{(1 + n)(1 + n)A} \right]^{1 - \alpha(1 - \eta)} > \left[ \frac{\beta}{1 + \beta} \left( 1 - \frac{X_{\text{econ}}}{1 - \alpha} \right) (1 - \alpha) A \right]^{1 - \alpha(1 - \eta)},
\]

that is,

\[
X_{\text{econ}} > (1 - \alpha) - \alpha \frac{1 + \beta}{\beta}.
\]

The third condition in (47) holds if \( h'/h_{\text{debt}} > h'/h_{\text{tax}} \), i.e., \( X_{\text{econ}} > (1 - \alpha) - \alpha \frac{1 + \beta}{\beta} \) as shown in the first part of Proposition 3. Therefore, the steady-state utility in the debt-finance case is higher than that in the tax-finance case if \( X_{\text{econ}} > (1 - \alpha) - \alpha \frac{1 + \beta}{\beta} \) and \( \gamma > \frac{(1 - \alpha)\beta}{\alpha} \).

\[\blacksquare\]

### A.4 Proof of Lemma 1

The first-order conditions with respect to \( x_t \) and \( \hat{b}_{t+1} \) are:

\[
x_t : \quad \frac{(1 + \beta)(1 + n)}{i_t - (1 + n)x_t + (1 + n)\hat{b}_{t+1}} + \frac{\beta (\alpha - 1) + \gamma \alpha}{i_t - (1 + n)x_t - \frac{1 + n}{\beta} \hat{b}_{t+1}} = \frac{\eta(\beta + \gamma)(1 - \alpha)}{x_t}, \quad (48)
\]

\[
\hat{b}_{t+1} : \quad \frac{(1 + \beta)(1 + n)}{i_t - (1 + n)x_t + (1 + n)\hat{b}_{t+1}} = \frac{\beta (\alpha - 1) + \gamma \alpha}{i_t - (1 + n)x_t - \frac{1 + n}{\beta} \hat{b}_{t+1}}. \quad (49)
\]

We substitute (49) into (48), and obtain:

\[
\frac{\beta (\alpha - 1) + \gamma \alpha}{i_t - (1 + n)x_t - \frac{1 + n}{\beta} \hat{b}_{t+1}} + \frac{\beta (\alpha - 1) + \gamma \alpha}{i_t - (1 + n)x_t - \frac{1 + n}{\beta} \hat{b}_{t+1}} = \frac{\eta(\beta + \gamma)(1 - \alpha)}{x_t},
\]

or

\[
x_t = \frac{\eta(\beta + \gamma)(1 - \alpha)}{(1 + n)\left\{ \beta (\alpha - 1) + \gamma \alpha \right\}(1 + \beta) + \eta \beta (\beta + \gamma)(1 - \alpha)} \cdot \left[ \beta i_t - (1 + n)\hat{b}_{t+1} \right]. \quad (50)
\]

We substitute (50) into (49). After rearranging the terms, we obtain the policy function of \( \hat{b}_{t+1} \). We then substitute this into (50) to obtain the policy function of \( x_t \).

To obtain the policy function of \( \tau_t \), recall the government budget constraint in the debt-finance case, which is reformulated as \( (1 + n)\hat{b}_{t+1} - \tau_t (1 - \alpha)A(k_t)^{\alpha}h_t = (1 + n)x_t + \alpha A(k_t)^{\alpha - 1}b_t \). Plugging the policy function of \( x_t \) and \( \hat{b}_{t+1} \) into this constraint and using \( b_t = \hat{b}_1 h_t \), we have:

\[
\tau_t (1 - \alpha)A(k_t)^{\alpha}h_t = (1 + n) X_0 i_t + \alpha A(k_t)^{\alpha - 1}b_t h_t - (1 + n) B_0 o_t
\]

\[
= (1 + n)(X_0 - B_0) \{ (1 - \alpha)A(k_t)^{\alpha}h_t - \alpha A(k_t)^{\alpha - 1}b_t h_t \} + A(k_t)^{\alpha - 1}b_t h_t
\]

\[
= (1 + n)(X_0 - B_0)(1 - \alpha)A(k_t)^{\alpha}h_t + \{ 1 - (1 + n)(X_0 - B_0) \alpha A(k_t)^{\alpha - 1}b_t h_t.
\]


Dividing both sides by \((1 - \alpha)A(k_t)\alpha h_t\), we obtain:

\[
\tau_t = (1 + n)(X_0 - B_0) + \frac{\alpha}{1 - \alpha} \{1 - (1 + n)(X_0 - B_0)\} \frac{b_t}{k_t}.
\]

\[\blacksquare\]

### A.5 Proof of Proposition 5

In the following proof, we first derive the conditions for which \(k_{t+1} > 0\), \(b_{t+1} > 0\), and \(\tau_t \in (0, 1)\) hold along the equilibrium path. Second, we show the existence and uniqueness of the steady-state equilibrium. Finally, we show that the unique steady-state equilibrium is a sink.

**Step 1.**

Recall the capital and debt accumulation equations given by:

\[
k_{t+1} = \Psi_K \cdot [(1 - \alpha)A(k_t)^\alpha - \alpha A(k_t)^{\alpha-1} b_t]^{1-\eta} = \Psi_K \cdot \left(\frac{i_t}{h_t}\right)^{1-\eta}, \quad (51)
\]

\[
b_{t+1} = \Psi_B \cdot [(1 - \alpha)A(k_t)^\alpha - \alpha A(k_t)^{\alpha-1} b_t]^{1-\eta} = \Psi_B \cdot \left(\frac{i_t}{h_t}\right)^{1-\eta}, \quad (52)
\]

respectively. \(k_{t+1} > 0\) and \(b_{t+1} > 0\) hold along the equilibrium path if \(i_t/h_t > 0\) for all \(t\), \(\Psi_K > 0\), and \(\Psi_B > 0\).

In period 0, we reformulate the term \(i_0/h_0\) as:

\[
\frac{i_0}{h_0} = (1 - \alpha)A(k_0)^\alpha \cdot \left[1 - \frac{\alpha}{1 - \alpha} \cdot \frac{b_0}{k_0}\right].
\]

Thus, \(i_0/h_0 > 0\) holds if

\[
\frac{b_0}{k_0} < \frac{1 - \alpha}{\alpha}. \quad (53)
\]

In period \(t \geq 1\), the term \(i_t/h_t\) is rewritten as

\[
\frac{i_t}{h_t} = (1 - \alpha)A(k_t)^\alpha \cdot \left[1 - \frac{\alpha}{1 - \alpha} \cdot \frac{b_t}{k_t}\right]
\]

\[
= (1 - \alpha)A(k_t)^\alpha \cdot \left[1 - \frac{\alpha}{1 - \alpha} \cdot \frac{\Psi_B}{\Psi_K}\right]
\]

\[
= (1 - \alpha)A(k_t)^\alpha \cdot \left\{\frac{\left(1 - (1 + n)X_0 - \frac{1 + n}{\beta} B_0\right) - (1 + n)\frac{\alpha(1 + \beta)}{(1 - \alpha)\beta} B_0}{1 - (1 + n)X_0 - \frac{1 + n}{\beta} B_0}\right\},
\]

where we obtain the second line using (51) and (52), and the third line using the definition of \(\Psi_K\) and \(\Psi_B\).

The expression above suggests that \(i_t/h_t > 0\) holds for \(t \geq 1\) if the following condition holds:

\[
B_0 > 0 \text{ and } \left\{1 - (1 + n)X_0 - \frac{1 + n}{\beta} B_0\right\} - (1 + n)\frac{\alpha(1 + \beta)}{(1 - \alpha)\beta} B_0 > 0. \quad (54)
\]

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In addition, the definitions of $\Psi_K$ and $\Psi_B$ suggest that $\Psi_K > 0$ and $\Psi_B > 0$ hold if (54) holds. After some manipulation, we have:

$$B_0 > 0 \iff \gamma < \frac{\beta}{\alpha} \left\{ (1 + \beta) + (1 - \alpha) \right\},$$  \hspace{1cm} (55)

$$\left\{ 1 - (1 + n)X_0 - \frac{1 + n}{\beta} \frac{(1 + \beta)}{(1 - \alpha)\beta} B_0 \right\} - (1 + n) \frac{\alpha(1 + \beta)}{(1 - \alpha)\beta} B_0 > 0 \iff \frac{\beta}{\alpha} (1 + \alpha \beta) < \gamma.$$  \hspace{1cm} (56)

Therefore, $k_{t+1} > 0$ and $b_{t+1} > 0$ for $t \geq 0$ if (53), (55), and (56) hold.

Next, consider the tax rate:

$$\tau_t = (1 + n) (X_0 - B_0) + \frac{\alpha}{1 - \alpha} \cdot \left\{ 1 - (1 + n) (X_0 - B_0) \right\} \cdot \frac{b_t}{k_t},$$

where

$$1 - (1 + n) (X_0 - B_0) > 0.$$

Given $b_0/k_0 > 0$, $\tau_0 > 0$ holds if $X_0 - B_0 > 0$, that is, if

$$\frac{\beta \left\{ (1 + \beta) + (1 - \eta)(1 - \alpha) \right\}}{\eta(1 - \alpha) + \alpha} < \gamma.$$  \hspace{1cm} (57)

For $t \geq 1$, $\tau_t > 0$ holds if $X_0 - B_0 > 0$ and $b_t/k_t > 0$, that is, if (55), (56), and (57) hold.

To find the condition for $\tau_t < 1$, let us first consider the period-0 tax rate. By direct calculation, we have:

$$\tau_0 < 1 \iff \frac{b_0}{k_0} < \frac{1 - \alpha}{\alpha} \iff (53).$$

For period $t \geq 1$, $b_t/k_t = \Psi_B/\Psi_K$ holds. We thus have:

$$\tau_t < 1 \iff \frac{\alpha}{1 - \alpha} \cdot \frac{\Psi_B}{\Psi_K} < 1 \iff \frac{\beta (1 + \alpha \beta)}{\alpha} < \gamma \iff (56).$$

The results established thus far suggest that $k_{t+1} > 0$, $b_{t+1} > 0$, and $\tau_t \in (0, 1)$ hold if (53), (55), (56), and (57) hold, that is, if:

$$\max \left\{ \frac{\beta \left\{ (1 + \beta) + (1 - \alpha)(1 - \eta) \right\}}{\eta(1 - \alpha) + \alpha}, \frac{\beta (1 + \alpha \beta)}{\alpha} \right\} < \gamma < \frac{\beta \left\{ (1 + \beta) + (1 - \alpha) \right\}}{\alpha},$$

and

$$\frac{b_0}{k_0} \in \left( 0, \frac{1 - \alpha}{\alpha} \right).$$

**Step 2.**

The steady-state pair of $(k, b)$ satisfies:

$$k = \Psi_K \cdot \left[ (1 - \alpha)A (k)^{\alpha - \alpha} A (k)^{\alpha - 1} b \right]^{1 - \eta},$$

$$b = \Psi_B \cdot \left[ (1 - \alpha)A (k)^{\alpha - \alpha} A (k)^{\alpha - 1} b \right]^{1 - \eta}.$$
These equations lead to \( k/b = \Psi_K/\Psi_B \), that is, \( b = (\Psi_B/\Psi_K) \cdot k \). Plugging this into \( k = \Psi_K \cdot [(1 - \alpha)A(\kappa)^\alpha - \alpha A(\kappa)^{\alpha - 1} \, b]^{1-\eta} \), and rearranging the terms, we obtain a unique value of \( k \) given by:

\[
k = \left[ \Psi_K \cdot \left\{ (1 - \alpha)A - \alpha A \left( \frac{\Psi_B}{\Psi_K} \right)^{1-\eta} \right\} \right]^{1/(1-\alpha(1-\eta))}.
\] (58)

The corresponding value of \( b \) is also uniquely determined using \( b = (\Psi_B/\Psi_K) \cdot k \).

**Step 3.**
Recall the law of motions of capital and debt:

\[
k_{t+1} = \Psi_K \cdot [(1 - \alpha)A(k_t)^\alpha - \alpha A(k_t)^{\alpha - 1} b_t]^{1-\eta}, \quad b_{t+1} = \Psi_B \cdot [(1 - \alpha)A(k_t)^\alpha - \alpha A(k_t)^{\alpha - 1} b_t]^{1-\eta}.
\]

Differentiating these with respect to \( k \) and \( b \) and evaluating them at the steady state, we obtain:

\[
\begin{bmatrix}
dk_{t+1} \\
\db_{t+1}
\end{bmatrix}
= J
\begin{bmatrix}
dk_t \\
\db_t
\end{bmatrix},
\]

where

\[
J \equiv \begin{bmatrix}
\Psi_K \cdot \hat{i}(k, b) \cdot \alpha(1 - \alpha)A(k)^{\alpha - 2}(k + b) - \Psi_K \cdot \hat{i}(k, b) \cdot \alpha A(k)^{\alpha - 1} \\
\Psi_B \cdot \hat{i}(k, b) \cdot \alpha(1 - \alpha)A(k)^{\alpha - 2}(k + b) - \Psi_B \cdot \hat{i}(k, b) \cdot \alpha A(k)^{\alpha - 1}
\end{bmatrix},
\]

and

\[
\hat{i}(k, b) \equiv [(1 - \alpha)A(k)^\alpha - \alpha A(k)^{\alpha - 1} b]^{-\eta} \cdot (1 - \eta).
\]

For any \((k, b) \gg 0\), the trace and determinant of \( J \) are

\[
\begin{cases}
\text{tr} J = \alpha A(k)^{\alpha - 1} \cdot \hat{i}(k, b) \cdot [(1 - \alpha)\Psi_K - \alpha \Psi_B] > 0, \\
\text{det} J = 0.
\end{cases}
\]

Here, the sign of the term \((1 - \alpha)\Psi_K - \alpha \Psi_B\) is

\[
(1 - \alpha)\Psi_K - \alpha \Psi_B > 0 \iff \gamma > \frac{\beta}{\alpha}(1 + \alpha \beta),
\]

where \( \gamma > \frac{\beta}{\alpha}(1 + \alpha \beta) \) is satisfied with the assumption in Proposition 5. Moreover, it is true that:

\[
\Delta \equiv (\text{tr} J)^2 - 4\text{det} J = (\text{tr} J)^2 > 0.
\]

Therefore, \( J \) has two distinct positive eigenvalues, denoted by \( \lambda_1 \) and \( \lambda_2 (> \lambda_1) \), at the steady state.

We obtain the eigenvalues of \( J \) by solving the following equation:

\[
p(\lambda) \equiv (\lambda)^2 - (\text{tr} J) \lambda + \text{det} J = 0.
\]
The solution of $p(\lambda) = 0$ is

\[ \lambda_1 = 0 \text{ and } \lambda_2 = \text{tr}J (>0). \]

The remaining task is to show $\lambda_2 < 1$. For this purpose, recall the law of motion of capital in the steady state, $k = \Psi_K \cdot [(1 - \alpha)A(k) - \alpha A(k)^{\alpha-1} b]^{1-\eta}$. This is reformulated as:

\[ i(k,b) = (1 - \eta) \left( \frac{k}{\Psi_K} \right)^{1-\eta}. \]  

Using (58) and (59), we rewrite $\lambda_2 = \text{tr}J$ as follows:

\[ \lambda_2 = \text{tr}J = \alpha A(k)^{\alpha-1}(1 - \eta) \left( \frac{k}{\Psi_K} \right)^{1-\eta} \left[(1 - \alpha)\Psi_K - \alpha \Psi_B\right]. \]  

\[ = \alpha A(\Psi_K)^{1-\eta} \frac{1}{(1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K}} (1 - \eta) \left( \frac{1}{\Psi_K} \right)^{1-\eta} \left[(1 - \alpha)\Psi_K - \alpha \Psi_B\right], \]  

\[ = \alpha(1 - \eta) \in (0, 1), \]

where the first line comes from (59) and the second line is derived using (58).

\[ \blacksquare \]

A.6 Proof of Proposition 6.

(i) In both cases, the policy function of $x$ is $x_t = X_0 i_t$, where $i_t \equiv \{(1 - \alpha)A(k_t) - \alpha A(k_t)^{\alpha-1}\} h_t$. Given that $Y_t = A(k_t)^{\alpha} h_t N_t$, we compute the ratio $N_{t+1}/y_t$ as follows:

\[ N_{t+1}/y_t \bigg|_{\text{tax}} = \frac{(1 + n)X_0 \{(1 - \alpha)A(k_t) - \alpha A(k_t)^{\alpha-1} b_t\}}{A(k_t)^{\alpha} h_t} \]

\[ = \begin{cases} 
(1 + n)X_0 \{(1 - \alpha) - \alpha b_0/k_0\} & \text{for } t = 0, \\
(1 + n)X_0(1 - \alpha) & \text{for } t \geq 1, 
\end{cases} \]

and

\[ N_{t+1}/y_t \bigg|_{\text{deb}} = \frac{(1 + n)X_0 \{(1 - \alpha) - \alpha b_t/k_{t|\text{deb}}\}}{A(k_t)^{\alpha} h_t} \]

\[ = (1 + n)X_0 \{(1 - \alpha) - \alpha b_t/k_t\}, \]

where $b_t > k_t$ holds for any equilibrium path. Therefore, we obtain the first part of Proposition 2 and Proposition 6.

(ii) Recall that the tax rates in the tax-finance and debt-finance cases are given by:

\[ \tau_{t|\text{tax}} = \begin{cases} 
(1 + n)X_0 + (1 - (1 + n)X_0) \frac{\alpha k_t}{1 - \alpha k_0} & \text{for } t = 0, \\
(1 + n)X_0 & \text{for } t \geq 1, 
\end{cases} \]
\[ \tau_t |_{\text{debt}} = \begin{cases} (1 + n) (X_0 - B_0) + \frac{\alpha}{1 - \alpha} \cdot \{1 - (1 + n) (X_0 - B_0)\} \cdot \frac{b_0}{k_0} & \text{for } t = 0, \\ (1 + n) (X_0 - B_0) + \frac{\alpha}{1 - \alpha} \cdot \{1 - (1 + n) (X_0 - B_0)\} \cdot \frac{b_0}{k_0} & \text{for } t \geq 1. \end{cases} \]

For \( t = 0 \), \( \tau_0 |_{\text{tax}} \) and \( \tau_0 |_{\text{debt}} \) are compared as follows:

\[ \tau_0 |_{\text{tax}} \geq \tau_0 |_{\text{debt}} \iff \frac{b_0}{k_0} \geq \frac{1 - \alpha}{\alpha}. \]

Assuming \( \frac{b_0}{k_0} < \frac{1 - \alpha}{\alpha} \) in Proposition 5, we obtain \( \tau_0 |_{\text{tax}} > \tau_0 |_{\text{debt}} \).

For \( t \geq 1 \), \( \tau_t |_{\text{tax}} \) and \( \tau_t |_{\text{debt}} \) are compared as follows:

\[ \tau_t |_{\text{tax}} \geq \tau_t |_{\text{debt}} \iff 1 \geq \frac{\alpha}{1 - \alpha} \left\{1 - (1 + n) (X_0 - B_0)\right\} \cdot \frac{1 + \beta}{\beta \left\{1 - (1 + n) X_0 - \frac{1 + n}{\beta} B_0\right\}} \]

\[ \iff \beta \left\{1 - (1 + n) X_0 - \frac{1 + n}{\beta} B_0\right\} \geq \frac{\alpha}{1 - \alpha} (1 + \beta) \left\{1 - (1 + n) (X_0 - B_0)\right\} \]

\[ \iff \gamma \geq \frac{\beta (1 - \alpha)}{\alpha} + \frac{(1 + \beta)^2}{1 - \alpha}, \]

where the second line comes from \( 1 - (1 + n) X_0 - \frac{1 + n}{\beta} B_0 > 0 \), and the third line comes from \( (\beta + \gamma) \frac{\alpha}{\beta} - 1 > 0 \), which holds under the assumption in Proposition 5. Therefore, we obtain the second part of Proposition 6.

\[ \blacksquare \]

### A.7 Proof of Proposition 7

Recall the policy function of \( x \) and the capital accumulation equation in the steady state presented in Subsection 4.1:

\[ x = X_0 \cdot (1 - \alpha) A(k)^\alpha h, \]

\[ k = \tilde{\Psi}_K \cdot \{(1 - \alpha) A(k)^\alpha \}^{1 - \eta}. \]

Combining these equations, we obtain:

\[ x = X_0 (1 - \alpha) A \left(\tilde{\Psi}_K\right)^{\frac{\alpha}{1 - \alpha(1 - \eta)}} \{(1 - \alpha) A\}^{\frac{\alpha(1 - \eta)}{1 - \alpha(1 - \eta)}} h. \]

Substituting this into \( h' = D(x)^\eta (h)^{1 - \eta} \) leads to:

\[ h' = D \cdot \left[X_0 (1 - \alpha) A \left(\tilde{\Psi}_K\right)^{\frac{\alpha}{1 - \alpha(1 - \eta)}} \{(1 - \alpha) A\}^{\frac{\alpha(1 - \eta)}{1 - \alpha(1 - \eta)}} h\right]^\eta (h)^{1 - \eta}, \]

or

\[ \frac{h'}{h} |_{\text{tax}} = D \cdot \left[X_0 \{(1 - \alpha) A\}^{\frac{1}{1 - \alpha(1 - \eta)}} \left(\tilde{\Psi}_K\right)^{\frac{\alpha}{1 - \alpha(1 - \eta)}}\right]^\eta. \quad (60) \]
Next, recall the set of three equations presented in Subsection 4.2 and evaluate them at the steady state:

\[ k = \Psi_K \cdot \left[ (1 - \alpha)A (k)^{\alpha - 1} b \right]^{1-\eta}, \]

\[ b = \Psi_B \cdot \left[ (1 - \alpha)A (k)^\alpha - \alpha A (k)^{\alpha - 1} b \right]^{1-\eta}, \]

\[ \frac{h'}{h} = D (X_0)^\eta \cdot \left[ (1 - \alpha)A (k)^\alpha - \alpha A (k)^{\alpha - 1} b \right]^\eta. \]

The first two equations imply

\[ k = (\Psi_K)^\frac{1}{1-\alpha(1-\eta)} \left[ (1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K} \right]^{\frac{1-\eta}{1-\alpha(1-\eta)}}. \]

Plugging this into the first equation, we obtain:

\[ \left[ (1 - \alpha)A (k)^\alpha - \alpha A (k)^{\alpha - 1} b \right]^{1-\eta} = (\Psi_K)^\frac{\alpha(1-\eta)}{1-\alpha(1-\eta)} \left[ (1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K} \right]^{\frac{1-\eta}{1-\alpha(1-\eta)}}, \]

or

\[ \left[ (1 - \alpha)A (k)^\alpha - \alpha A (k)^{\alpha - 1} b \right]^\eta = (\Psi_K)^\frac{\alpha\eta}{1-\alpha(1-\eta)} \left[ (1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K} \right]^{\frac{\eta}{1-\alpha(1-\eta)}}. \]

We substitute this into the third equation and obtain:

\[ \left. \frac{h'}{h} \right|_{\text{tax}} = D (X_0)^\eta (\Psi_K)^\frac{\alpha\eta}{1-\alpha(1-\eta)} \left[ (1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K} \right]^{\frac{\eta}{1-\alpha(1-\eta)}}. \] (61)

A direct comparison of (60) and (61) leads to:

\[ \left. \frac{h'}{h} \right|_{\text{tax}} \geq \left. \frac{h'}{h} \right|_{\text{debt}} \iff \left( \Psi_K \right)^\alpha (1 - \alpha)A \geq (\Psi_K)^\alpha \left[ (1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K} \right], \]

\[ \iff \left[ \frac{\beta}{1 + \beta} \cdot \frac{1 - (1 + n)X_0}{(1 + n)D (X_0)^\eta} \right]^\alpha (1 - \alpha)A \]

\[ \geq \left[ \frac{\beta}{1 + \beta} \cdot \frac{1 - (1 + n)X_0 - \frac{1+n}{\beta} B_0}{(1 + n)D (X_0)^\eta} \right]^\alpha \left[ (1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K} \right], \]

\[ \iff [1 - (1 + n)X_0]^\alpha (1 - \alpha)A \geq \left[ 1 - (1 + n)X_0 - \frac{1+n}{\beta} B_0 \right]^\alpha \left[ (1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K} \right], \]

where we obtain the second line from the definitions of $\tilde{\Psi}_K$ and $\Psi_K$.

Given that $B_0 > 0$ and $X_0 > 0$, we have:

\[ 1 - (1 + n)X_0 > 1 - (1 + n)X_0 - \frac{1+n}{\beta} B_0, \]

\[ (1 - \alpha)A > (1 - \alpha)A - \alpha A \frac{\Psi_B}{\Psi_K}. \]

Therefore, we find that $\left. \frac{h'}{h} \right|_{\text{tax}} > \left. \frac{h'}{h} \right|_{\text{debt}}$.
A.8 Proof of Proposition 8

Recall the indirect utility function of the middle-aged in the tax-finance case, (20), and that in the debt-finance case, (23). Substituting the corresponding policy functions leads to:

\[
V_{\text{tax},t} = \{1 + (\beta + \gamma)(\alpha + \eta(1 - \alpha))\} \ln \left\{ (1 - \alpha)A(k_t)^\alpha - \alpha A(k_t)^{\alpha-1}b_t \right\} + (1 - \eta)(\beta + \gamma)(1 - \alpha) \ln h_t + \{1 + (\beta + \gamma)\alpha \} \ln (1 - (1 + n)X_0) + \eta(\beta + \gamma)(1 - \alpha) \ln X_0 + C_1,
\]

\[
V_{\text{debt},t} = \{1 + (\beta + \gamma)(\alpha + \eta(1 - \alpha))\} \ln \left\{ (1 - \alpha)A(k_t)^\alpha - \alpha A(k_t)^{\alpha-1}b_t \right\} + (1 - \eta)(\beta + \gamma)(1 - \alpha) \ln h_t + (1 + \beta) \ln [1 - (1 + n)X_0 + (1 + n)B_0] + \{\beta(\alpha - 1) + \gamma\alpha \} \ln \left\{ (1 - (1 + n)X_0 - \frac{1 + n}{\beta}B_0 \right\} + \eta(\beta + \gamma)(1 - \alpha) \ln X_0 + C_1,
\]

where \(k_t, b_t, \) and \(h_t\) are given.

By comparing \(V_{\text{tax},t}\) and \(V_{\text{debt},t}\), we obtain:

\[
V_{\text{tax},t} \geq V_{\text{debt},t} \iff 1 \geq \left[ \frac{1 - (1 + n)X_0 + (1 + n)B_0}{1 - (1 + n)X_0} \right]^{1+\beta} \cdot \left[ \frac{1 - (1 + n)X_0 - \frac{1+n}{\beta}B_0}{1 - (1 + n)X_0} \right]^{\beta(\alpha-1) + \gamma\alpha}.
\]

Define \(\delta(\gamma) \equiv (\beta + \gamma)\alpha\). Then, we have:

\[
V_{\text{tax},t} \geq V_{\text{debt},t} \iff 1 \geq f(\delta) = \left[ \frac{(1 + \beta)^2}{1 + \delta} \right]^{1+\beta} \cdot \left[ \frac{(1 + \beta)\left(\frac{\delta - \alpha}{\beta} - 1\right)}{1 + (\beta + \gamma)\alpha} \right]^{\beta(\alpha-1) + \gamma\alpha},
\]

where the upper limit of \(\delta\), denoted by \(\bar{\delta}\), is:

\[
\bar{\delta} \equiv \delta|_{\gamma = \frac{\beta(1 + \beta) + (1 - \alpha))}{\alpha}} = \beta(2 + \beta),
\]

and \(f(\bar{\delta}) = 1\) holds. Therefore, \(V_{\text{tax},t} < V_{\text{debt},t}\) holds if \(f'(\cdot) < 0\).

To show \(f'(\cdot) < 0\), we take the logarithm of \(f\):

\[
\ln f(\cdot) = (1 + \beta) \ln \left(\frac{(1 + \beta)^2}{1 + \delta} \right) + (\delta - \beta) \ln \left(\frac{(1 + \beta)\left(\frac{\delta - \alpha}{\beta} - 1\right)}{1 + \delta} \right)
\]

\[
= \left\{2(1 + \beta) + (\delta - \beta)\right\} \ln (1 + \beta) + (\delta - \beta) \ln \left(\frac{\delta}{\beta} - 1\right) - (1 + \delta) \ln (1 + \delta).
\]

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Differentiating $\ln f(\cdot)$ with respect to $\delta$ yields:

$$\frac{\partial \ln f(\cdot)}{\partial \delta} = \ln \frac{(1 + \beta) \left( \frac{1}{\delta} - 1 \right)}{1 + \delta} = \ln \frac{(1 + \beta) \left( \frac{1}{\beta} - \frac{1}{\delta} \right)}{\frac{1}{\delta} + 1},$$

showing that $\partial \ln f(\cdot)/\partial \delta$ is increasing in $\delta$.

Given that $\delta < \tilde{\delta} \equiv \beta(2 + \beta)$, $\partial \ln f(\cdot)/\partial \delta < 0$ holds if $\partial \ln f(\cdot)/\partial \delta \leq 0$ at $\delta = \tilde{\delta}$. Direct calculation results in:

$$\left. \frac{\partial \ln f(\cdot)}{\partial \delta} \right|_{\delta = \tilde{\delta}} = \ln 1 = 0.$$

Therefore, we obtain $\partial \ln f(\cdot)/\partial \delta < 0$ for $\delta < \tilde{\delta}$ and thus $V_{\text{tax},t} < V_{\text{debt},t}$ for given $k_t, b_t$, and $h_t$. 

\[\blacksquare\]
References


Figure 1: Numerical example of the phase diagram for the planner system, (12) and (13). The parameters values are set at \(\alpha = 0.3, \beta = (0.997)^{30}, \eta = 0.154, n = (1.006)^{30} - 1, A = 1, D = 3.347, \tau_{\text{debt}} = 0.076, X_{\text{econ}} = 0.05323\). The values of \(\beta, \eta, A, D,\) and \(n\) follow those in Lancia and Russo (2015). The value of \(X_{\text{econ}}\) is based on the average public education expenditure-GDP ratios in OECD countries in 2011 (OECD, education database: https://data.oecd.org/eduresource/public-spending-on-education.htm, December 28, 2015). The value of \(\tau_{\text{debt}}\) is set to satisfy the condition in Proposition 2.
Figure 2: Numerical example of the evolution of the aggregate expenditure-GDP ratio (Panel (a)) and the tax rate (Panel (b)) for the tax-finance and the debt cases from period 0. The value of $\gamma$ is set at $\gamma = 5.5$, and other parameter values follow those in Figure 1. The initial conditions are $k_0 = 0.005$, $h_0 = 0.005$, and $b_0 = 0.0001$. All values apply in the following examples.
Figure 3: Numerical example of the evolution of the growth rate for the tax-finance and the debt-finance cases.
Figure 4: Numerical example of the evolution of utility for the tax-finance and the debt-finance cases.
Table 1: Summary: Numerical results from period 0 to period 2.

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<th>Tax-finance case</th>
<th>Debt-finance case</th>
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<td>-48.0236</td>
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<td>Period-1 utility</td>
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<td>-38.6928</td>
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